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Integrable and Continuous Solutions of the Nonlinear Delayed Abel Fractal Integral Equation of the Second Kind

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Abstract. In this paper, we define the nonlinear delayed Abel fractal integral equation of the second kind. The existence of solutions in the two classes, of continuous $C[0,T]$ and integrable $L_1[0,T]$ functions, will be proved. The continuous dependence of the unique solution on the parameters will be proved. The Hyers-Ulam stability of the problem itself will be studied.

1. Introduction

Fractal calculus operators (fractal integral and fractal derivative) are extensions of ordinary (integral and differential) calculus. This calculus has some applications in physics (like quantum mechanics on fractal space-time), biology (e.g., modeling blood vessels), finance (fractal market theory), and image analysis ([1], [2], [4], [13], [17], [20], [21], [22]).

Let $\beta \in (0,1)$. Let $C[0,T] = C(I)$ and $L_1[0,T] = L_1(I)$ be the two classes of continuous and integrable functions defined on the interval $I = [0,T]$.

The fractal integral I_β and derivative D_β operators are defined as follows

Definition 1.1. *Let $x : I \to R$ be differentiable. The fractal derivative of the function x is defined by ([7], [8])*

$$D_\beta\, x(t) = \frac{d}{dt^\beta}\, x(t) = \frac{t^{1-\beta}}{\beta}\, \frac{d}{dt}\, x(t),\ t > 0.$$

Definition 1.2. *Let $\beta\, t^{\beta-1}\, x(t) \in L_1(I)$. The fractal integral of the function x is defined by [8]*

$$I_\beta\, x(t) = \int_0^t \beta\, s^{\beta-1}\, x(s)\, ds$$

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with the following property

$$\frac{d}{dt}\int_0^t \beta\, s^{\beta-1}\, x(s)\, ds \;=\; \beta\, t^{\beta-1}\, x(t) \textit{ and } D_\beta\, I_\beta\, x(t) \;=\; x(t),\ a.e,\ t \in I.$$

1.1. **Formulations of the problem.** The Abel integral equations of the first and second kind are generally defined by ([11], [14])

$$\int_0^t \frac{x(s)}{(t-s)^\alpha}\, ds \;=\; g(t)$$

and

$$x(t) \;+\; \lambda \int_0^t \frac{x(s)}{(t-s)^\alpha}\, ds \;=\; g(t),\ t \in I$$

respectively, where $\alpha \in (0,1)$ and λ is a positive parameter.

Moreover, The linear Abel fractal integral equations of the first and second kind are defined by [18]

$$\int_0^t \beta\, s^{\beta-1}\, x(s)\, ds \;=\; g(t)$$

and

$$x(t) + \lambda \int_0^t \beta\, s^{\beta-1}\, x(s)\, ds \;=\; g(t),\ t \in I$$

respectively, where $\beta \in (0,1)$ and λ is a positive parameter.

Now, we define the nonlinear Abel fractal integral equation of the second kind by

$$x(t) + f_1(t, \lambda_1\, I_\beta\, x(t)) = f_2(t, x(t))$$

and the nonlinear delayed Abel fractal integral equation of the second kind by

$$x(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds) = f_2(t, \lambda_2\, x(t)),\ t \in I \tag{1.1}$$

where $\beta \in (0,1)$ and the parameters $\lambda_i > 0,\ i = 1,2$.

Our aim, here is to study the existence of integrable and continuous solutions of (1.1). The continuous dependence of the solution on the parameters λ_i and on the functions f_i, ϕ and the Hyers - Ulam stability of the problem itself will be studied.

2. Integrable solution

2.1. **Existence of solution.** Consider equation (1.1) under the following conditions:

(i) $f_i : I \times R \to R,\ i = 1,2$ are measurable in $t \in I$ for any $x \in R$ and continuous in $x \in R$ for $t \in I$.

(ii) There exist $a_i : I \to R$ such that $\beta\, t^{\beta-1}|a_i(t)| \in L_1(I)$, $\int_0^T \beta\, t^{\beta-1}|a_i(t)|\, dt \le a$ and a positive constants b_i such that

$$|f_i(t,x)| \;\le\; |a_i(t)| + b_i\, |x|,\ i = 1,2. \tag{2.1}$$

(iii) $\phi : I \to I$ is a continuous function and $\phi(t) \le t$.

(iv) $b\, \lambda\, (1 + T^\beta) < 1$, where $b = \max\{b_i\}$ and $\lambda = \max\{\lambda_i, \lambda_i^*\}$, $i = 1,2$.

Theorem 2.1. *Let the conditions $(i)-(iv)$ be satisfied, then equation (1.1) has at least one solution $x \in L_1(I)$.*

Proof. The integral equation (1.1) can be written as

$$x(t) \;=\; -f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds) + f_2(t, \lambda_2\, x(t))$$

which can be given by

$$\beta\, t^{\beta-1} x(t) \;=\; -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} \beta s^{\beta-1} x(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} \beta\, t^{\beta-1} x(t)).$$

Let $y(t) = \beta\, t^{\beta-1} x(t)$, then

$$y(t) = -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \tag{2.2}$$

and

$$x(t) = \frac{t^{1-\beta}}{\beta} y(t). \tag{2.3}$$

Define the operator F by

$$Fy(t) = -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t))$$

and the non-empty, bounded, convex and closed ball

$$Q_r = \{y \in L_1(I) : \|y\|_1 \leq r\}, \text{ where } r = \frac{2\, a}{1 - b\, \lambda\, (1 + T^{\beta})}.$$

Now, let $y \in Q_r$, then

$$\begin{aligned}
|Fy(t)| \;&=\; |-\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t))\, | \\
&\leq\; \beta\, t^{\beta-1}\, |\, f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds)\, | \;+ \beta\, t^{\beta-1}\, |\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t))\, | \\
&\leq\; \beta\, t^{\beta-1}\, |a_1(t)| + b_1\, \beta\, t^{\beta-1}\, \lambda_1 \int_0^{\phi(t)} |y(s)|\, ds + \beta\, t^{\beta-1}\, |a_2(t)| + \beta\, t^{\beta-1}\, b_2\, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, |y(t)| \\
&\leq\; \beta\, t^{\beta-1}\, |a_1(t)| + b_1\, \beta\, t^{\beta-1}\, \lambda_1 \int_0^{t} |y(s)|\, ds + \beta\, t^{\beta-1}\, |a_2(t)| + b_2\, \lambda_2\, |y(t)| \\
&\leq\; \beta\, t^{\beta-1}\, |a_1(t)| + b\, \beta\, t^{\beta-1}\, \lambda\, \|y\|_1 + \beta\, t^{\beta-1}\, |a_2(t)| + b\, \lambda\, |y(t)|
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T |Fy(t)|\, dt \;&\leq\; \int_0^T \beta\, t^{\beta-1} |a_1(t)|\, dt + b\lambda\, \|y\|_1 \int_0^T \beta\, t^{\beta-1} dt + \int_0^T \beta\, t^{\beta-1} |a_2(t)|\, dt + b\lambda \int_0^T |y(t)|\, dt \\
&\leq\; a + b\, \lambda\, T^{\beta}\, r + a + b\, \lambda\, \|y\|_1 \\
&\leq\; 2\, a + b\, \lambda\, T^{\beta}\, r + b\, \lambda\, r \;=\; r.
\end{aligned}$$

This proves that the operator F maps Q_r into itself, and the class $\{Fy\}$ is uniformly bounded in Q_r.

Let $\Omega \in Q_r$ and $y \in \Omega$, then

$$\begin{aligned}\int_0^T |\ (Fy(s))_h - (Fy(s))\ |\ ds &= \int_0^T |\ \frac{1}{h}\int_t^{t+h} Fy(\theta)\ d\theta - Fy(s)\ |\ ds\\ &= \int_0^T \frac{1}{h}\ |\int_t^{t+h} (\ Fy(\theta) - Fy(s)\ )\ d\theta\ |\ ds\\ &\leq \int_0^T \frac{1}{h}\ \int_t^{t+h} |\ Fy(\theta) - Fy(s)\ |\ d\theta\ ds,\end{aligned}$$

then

$$\|(Fy)_h - (Fy)\|_1 \leq \int_0^T \frac{1}{h}\ \int_t^{t+h} |\ Fy(\theta) - Fy(s)\ |\ d\theta\ ds.$$

Since $Fy(t) = -\beta\ t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\ ds) + \beta\ t^{\beta-1}\ f_2(t, \lambda_2\ \frac{t^{1-\beta}}{\beta} y(t))\ \in\ Q_r\ \subset\ L_1(I)$, from Lebesgue point theorem [16], it follows that

$$\frac{1}{h}\ \int_t^{t+h} |\ Fy(\theta) - Fy(s)\ |\ d\theta \to 0,\ \text{as}\ h \to 0,$$

then

$$\|(Fy)_h - (Fy)\|_1 \to 0,\ \text{as}\ h \to 0.$$

Hence the class $\{Fy\}$ is relatively compact [12]. Then F is compact operator ([3], [16]).

Now, let $\{y_n\} \subset Q_r$, $y_n \to y$ and

$$\begin{aligned}|y_n(t)| &= |-\beta\ t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_n(s)\ ds) + \beta\ t^{\beta-1}\ f_2(t, \lambda_2\ \frac{t^{1-\beta}}{\beta} y_n(t))\ |\\ &\leq \beta\ t^{\beta-1}\ |\ f_1(t, \lambda_1 \int_0^{\phi(t)} y_n(s)\ ds)\ |\ + \beta\ t^{\beta-1}\ |\ f_2(t, \lambda_2\ \frac{t^{1-\beta}}{\beta} y_n(t))\ |\\ &\leq \beta\ t^{\beta-1}\ |a_1(t)| + b_1\ \beta\ t^{\beta-1}\ \lambda_1 \int_0^t |y_n(s)|\ ds + \beta\ t^{\beta-1}\ |a_2(t)| + b_2\ \lambda_2\ |y_n(t)|\\ &\leq \beta\ t^{\beta-1}\ |a_1(t)| + b\ \beta\ t^{\beta-1}\ \lambda\ \|y_n\|_1 + \beta\ t^{\beta-1}\ |a_2(t)| + b\ \lambda\ |y_n(t)|\end{aligned}$$

and

$$(1 - b\ \lambda)\ |y_n(t)|\ \leq\ \beta\ t^{\beta-1}\ |a_1(t)| + b\ \beta\ t^{\beta-1}\ \lambda\ \|y_n\|_1 + \beta\ t^{\beta-1}\ |a_2(t)|,$$

$$|y_n(t)|\ \leq\ \frac{1}{1 - b\ \lambda}\ [\ \beta\ t^{\beta-1}\ |a_1(t)| + b\ \beta\ t^{\beta-1}\ \lambda\ \|y_n\|_1 + \beta\ t^{\beta-1}\ |a_2(t)|\ ]\ =\ \omega(t),$$

where $\omega(t) \in L_1(I)$.

Now, we have

$$Fy_n(t) = -\beta\ t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_n(s)\ ds) + \beta\ t^{\beta-1}\ f_2(t, \lambda_2\ \frac{t^{1-\beta}}{\beta} y_n(t)),$$

then we can apply Lebesgue dominated convergence theorem ([3], [16]) and obtain

$$\begin{aligned}
\lim_{n\to\infty} Fy_n(t) &= -\beta\, t^{\beta-1} f_1(t, \lambda_1 \lim_{n\to\infty} \int_0^{\phi(t)} y_n(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} \lim_{n\to\infty} y_n(t)) \\
&= -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} \lim_{n\to\infty} y_n(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \\
&= -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \\
&= Fy(t).
\end{aligned}$$

This means that $Fy_n(t) \to Fy(t)$. Then the operator F is continuous [5].
Now, by Schauder fixed point Theorem [10], there exist at least one integrable solution $y \in Q_r \subset L_1(I)$ of the integral equation (2.2).
Consequently, there exist at least one solution $x(t) = \frac{t^{1-\beta}}{\beta} y(t) \in L_1(I)$ of the fractal integral equation (1.1).

2.2. **Uniqueness of the solution.** Now, consider the following condition
$(i)^*$ $f_i : I \times R \to R,\ i = 1, 2$ is measurable in $t \in I$ for every $x \in R$ and satisfies the Lipschitz condition

$$|f_i(t, x) - f_i(t, y)| \le b_i\, |x - y|,\ b_i > 0. \tag{2.4}$$

Remark 1. From the Lipschitz condition (2.4), we can get the growth condition (2.1) as $f_i(t, 0) = a_i(t),\ i = 1, 2.$

Theorem 2.2. *Let the conditions $(i)^*$, (iii) and (iv) be satisfied, then the solution $y \in L_1(I)$ of the equation (2.2) is unique. Consequently, the solution of (1.1) is unique.*

Proof. Let the conditions of Theorem 2.1 be satisfied, and the solution of the integral equation (2.2) exists. Let y_1 and y_2 be two solutions of equation (2.2), then

$$\begin{aligned}
|y_1(t) - y_2(t)| &= |\, -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_1(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_1(t)) \\
&+ \beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_2(s)\, ds) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_2(t))\, | \\
&\le \beta\, t^{\beta-1}\, |\, f_1(t, \lambda_1 \int_0^{\phi(t)} y_2(s)\, ds) - f_1(t, \lambda_1 \int_0^{\phi(t)} y_1(s)\, ds)\, | \\
&+ \beta\, t^{\beta-1}\, |\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_1(t)) - f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_2(t))\, | \\
&\le \beta\, t^{\beta-1}\, b_1\, \lambda_1\, |\int_0^{\phi(t)} y_2(s)\, ds - \int_0^{\phi(t)} y_1(s)\, ds\, | \\
&+ \beta\, t^{\beta-1}\, b_2\, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, |\, y_1(t) - y_2(t)\, |
\end{aligned}$$

$$
\begin{aligned}
&\leq b_1\,\lambda_1\,\beta\,t^{\beta-1}\int_0^{\phi(t)} |\,y_2(s)-y_1(s)\,|\,ds + b_2\,\lambda_2\,|\,y_1(t)-y_2(t)\,| \\
&\leq b\,\lambda\,\beta\,t^{\beta-1}\int_0^{t} |\,y_1(s)-y_2(s)\,|\,ds + b\,\lambda\,|\,y_1(t)-y_2(t)\,| \\
&\leq b\,\lambda\,\beta\,t^{\beta-1}\,\|y_1-y_2\|_1 + b\,\lambda\,|\,y_1(t)-y_2(t)\,|
\end{aligned}
$$

and

$$
\begin{aligned}
\int_0^T |y_1(t)-y_2(t)|\,dt &\leq b\,\lambda\int_0^T \beta\,t^{\beta-1}dt\,\|y_1-y_2\|_1 + b\,\lambda\int_0^T |\,y_1(t)-y_2(t)\,|\,dt, \\
\|y_1-y_2\|_1 &\leq b\,\lambda\,T^{\beta}\,\|y_1-y_2\|_1 + b\,\lambda\,\|y_1-y_2\|_1,
\end{aligned}
$$

then

$$[1-b\,\lambda\,(1+T^{\beta})]\,\|y_1-y_2\|_1 \leq 0,$$

which implies that

$$\|y_1-y_2\|_1 = 0.$$

This means that $y_1 = y_2$, and the solution of the integral equation (2.2) is unique.
Consequently, for any two solutions $x_1, x_2 \in L_1[0,T]$ of the equation (2.3), we get

$$
\begin{aligned}
|x_1(t)-x_2(t)| &= |\,\frac{t^{1-\beta}}{\beta}y_1(t) - \frac{t^{1-\beta}}{\beta}y_2(t)\,| \\
&= \frac{t^{1-\beta}}{\beta}\,|\,y_1(t)-y_2(t)\,| \\
&\leq \frac{T^{1-\beta}}{\beta}\,|\,y_1(t)-y_2(t)\,|,
\end{aligned}
$$

then

$$\int_0^T |x_1(t)-x_2(t)|\,dt \leq \frac{T^{1-\beta}}{\beta}\int_0^T |\,y_1(t)-y_2(t)\,|\,dt$$

and

$$\|x_1-x_2\|_1 \leq \frac{T^{1-\beta}}{\beta}\,\|y_1-y_2\|_1 = 0,$$

which means that $x_1 = x_2$ and the solution of the fractal integral equation (1.1) is unique.

2.3. **Continuous Dependence.**

Definition 2.1. *The solution $x \in L_1(I)$ of equation (1.1) depends continuously on the parameters λ_i and on the functions f_i [9], if $\forall \epsilon > 0$, $\exists \delta > 0$ such that*

$$\max\{\,|\lambda_i-\lambda_i^*|,\ |f_i-f_i^*|\,\} < \delta,\ i = 1,2$$

implies

$$\|x-x^*\|_1 < \epsilon,$$

where x^* is the unique solution of

$$x^*(t) = -f_1^*(t, \lambda_1^* \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) + f_2^*(t, \lambda_2^*\, x^*(t)) \tag{2.5}$$

which can be given by

$$x^*(t) = \frac{t^{1-\beta}}{\beta} y^*(t), \tag{2.6}$$

where

$$y^*(t) = -\beta\, t^{\beta-1} f_1^*(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) + \beta\, t^{\beta-1}\, f_2^*(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)). \tag{2.7}$$

Theorem 2.3. *Let the assumptions of Theorem 2.2 be satisfied. Then the unique solution of equation (1.1) depends continuously on the parameters λ_i and on the functions $f_i,\ i = 1, 2$.*

Proof. Let x and x^* be two solutions of (2.3) and (2.6) respectively, where y and y^* be two solutions of (2.2) and (2.7) respectively, then

$$\begin{aligned}
|y(t) - y^*(t)| &= \Big| -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \\
&+ \beta\, t^{\beta-1} f_1^*(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) - \beta\, t^{\beta-1}\, f_2^*(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) \Big| \\
&= \Big| -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1} f_1(t, \lambda_1^* \int_0^{\phi(t)} y(s)\, ds) \\
&- \beta\, t^{\beta-1} f_1(t, \lambda_1^* \int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1} f_1(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) \\
&- \beta\, t^{\beta-1} f_1(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) + \beta\, t^{\beta-1} f_1^*(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) \\
&+ \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y(t)) \\
&+ \beta\, t^{\beta-1}\, f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y(t)) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) \\
&+ \beta\, t^{\beta-1}\, f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) - \beta\, t^{\beta-1}\, f_2^*(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) \Big| \\
&\le \beta\, t^{\beta-1} \Big| f_1(t, \lambda_1^* \int_0^{\phi(t)} y(s)\, ds) - f_1(t, \lambda_1 \int_0^{\phi(t)} y(s)\, ds) \Big| \\
&+ \beta\, t^{\beta-1} \Big| f_1(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) - f_1(t, \lambda_1^* \int_0^{\phi(t)} y(s)\, ds) \Big| \\
&+ \beta\, t^{\beta-1} \Big| f_1^*(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) - f_1(t, \lambda_1^* \int_0^{\phi(t)} y^*(s)\, ds) \Big| \\
&+ \beta\, t^{\beta-1} \Big| f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) - f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y(t)) \Big| \\
&+ \beta\, t^{\beta-1} \Big| f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y(t)) - f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) \Big|
\end{aligned}$$

$$\begin{aligned}
&+ \quad \beta\, t^{\beta-1}\, |\, f_2(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t)) - f_2^*(t, \lambda_2^*\, \frac{t^{1-\beta}}{\beta} y^*(t))\, | \\
&\leq \quad \beta\, t^{\beta-1}\, b_1\, |\, \lambda_1^* - \lambda_1\, | \int_0^{\phi(t)} |y(s)|\, ds \\
&+ \quad \beta\, t^{\beta-1}\, b_1\, \lambda_1^* \int_0^{\phi(t)} |\, y^*(s) - y(s)\, |\, ds \\
&+ \quad \beta\, t^{\beta-1}\, \delta \, + \, \beta\, t^{\beta-1}\, b_2\, |\, \lambda_2 - \lambda_2^*\, |\, \frac{t^{1-\beta}}{\beta}\, |y(t)| \\
&+ \quad \beta\, t^{\beta-1}\, b_2\, \lambda_2^*\, \frac{t^{1-\beta}}{\beta}\, |\, y(t) - y^*(t)\, | \, + \, \beta\, t^{\beta-1}\, \delta \\
&\leq \quad \beta\, t^{\beta-1}\, b\, \delta \int_0^t |y(s)|\, ds \, + \, \beta\, t^{\beta-1}\, b\, \lambda_1^* \int_0^t |\, y(s) - y^*(s)\, |\, ds \\
&+ \quad 2\, \beta\, t^{\beta-1}\, \delta \, + \, b\, \delta\, |y(t)| \, + \, b\, \lambda_2^*\, |\, y(t) - y^*(t)\, | \\
&\leq \quad b\, \delta\, \|y\|_1\, \beta\, t^{\beta-1} \, + \, b\, \lambda\, \|y - y^*\|_1\, \beta\, t^{\beta-1} \\
&+ \quad 2\, \delta\, \beta\, t^{\beta-1} \, + \, b\, \delta\, |y(t)| \, + \, b\, \lambda\, |\, y(t) - y^*(t)\, |,
\end{aligned}$$

then

$$\begin{aligned}
\|y - y^*\|_1 \quad &\leq \quad b\, \delta\, \|y\|_1 \int_0^T \beta\, t^{\beta-1}\, dt \, + \, b\, \lambda\, \|y - y^*\|_1 \int_0^T \beta\, t^{\beta-1}\, dt \\
&+ \quad 2\, \delta \int_0^T \beta\, t^{\beta-1}\, dt \, + \, b\, \delta \int_0^T |y(t)|\, dt \, + \, b\, \lambda \int_0^T |\, y(t) - y^*(t)\, |\, dt \\
&= \quad b\, \delta\, T^\beta\, \|y\|_1 \, + \, b\, \lambda\, T^\beta\, \|y - y^*\|_1 \\
&+ \quad 2\, \delta\, T^\beta \, + \, b\, \delta\, \|y\|_1 \, + \, b\, \lambda\, \|y - y^*\|_1
\end{aligned}$$

and

$$[1 - b\, \lambda\, (1 + T^\beta)]\, \|y - y^*\|_1 \, \leq \, 2\, \delta\, T^\beta + b\, \delta\, (1 + T^\beta)\, \|y\|_1,$$

thus

$$\|y - y^*\|_1 \, \leq \, \frac{2\, \delta\, T^\beta + b\, \delta\, (1 + T^\beta)\, \|y\|_1}{1 - b\, \lambda\, (1 + T^\beta)} \, = \, \epsilon_1.$$

Now,

$$\begin{aligned}
|x(t) - x^*(t)| \quad &= \quad |\frac{t^{1-\beta}}{\beta} y(t) - \frac{t^{1-\beta}}{\beta} y^*(t)| \\
&= \quad \frac{t^{1-\beta}}{\beta} |y(t) - y^*(t)| \\
&\leq \quad \frac{T^{1-\beta}}{\beta} |y(t) - y^*(t)|
\end{aligned}$$

and

$$\begin{aligned}\int_0^T |x(t)-x^*(t)|\,dt &\le \frac{T^{1-\beta}}{\beta}\int_0^T |y(t)-y^*(t)|\,dt\\ &= \frac{T^{1-\beta}}{\beta}\,\|y-y^*\|_1\\ &\le \frac{T^{1-\beta}}{\beta}\,\epsilon_1 \;=\; \epsilon,\end{aligned}$$

thus

$$\|x-x^*\|_1 \;\le\; \epsilon.$$

Definition 2.2. *Let $\phi^* : I \to I$ be a continuous function and $\phi^*(t) \le t$. The solution $x \in L_1(I)$ of equation (1.1) depends continuously on the delay function ϕ, if $\forall \epsilon > 0,\ \exists \delta > 0$ such that*

$$|\phi-\phi^*| \;<\; \delta,$$

implies

$$\|x-x^*\|_1 < \epsilon,$$

where x^* is the unique solution of

$$x^*(t) = -f_1(t,\lambda_1\int_0^{\phi^*(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) + f_2(t,\lambda_2\, x^*(t)) \tag{2.8}$$

which can be given by equation (2.6), where

$$y^*(t) = -\beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi^*(t)} y^*(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t,\lambda_2\,\frac{t^{1-\beta}}{\beta} y^*(t)). \tag{2.9}$$

Theorem 2.4. *Let the assumptions of Theorem 2.2 be satisfied. Then the unique solution of equation (1.1) depends continuously on the delay function ϕ.*

Proof. Let x and x^* be two solutions of (2.3) and (2.6) respectively, where y and y^* be two solutions of (2.2) and (2.9) respectively, then

$$\begin{aligned}|y(t)-y^*(t)| &= |-\beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1}\, f_2(t,\lambda_2\,\frac{t^{1-\beta}}{\beta} y(t))\\ &+ \beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi^*(t)} y^*(s)\, ds) - \beta\, t^{\beta-1}\, f_2(t,\lambda_2\,\frac{t^{1-\beta}}{\beta} y^*(t))\,|\\ &= |-\beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi(t)} y(s)\, ds) + \beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi^*(t)} y(s)\, ds)\\ &- \beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi^*(t)} y(s)\, ds) + \beta\, t^{\beta-1} f_1(t,\lambda_1\int_0^{\phi^*(t)} y^*(s)\, ds)\\ &+ \beta\, t^{\beta-1}\, f_2(t,\lambda_2\,\frac{t^{1-\beta}}{\beta} y(t)) - \beta\, t^{\beta-1}\, f_2(t,\lambda_2\,\frac{t^{1-\beta}}{\beta} y^*(t))\,|\\ &\le \beta\, t^{\beta-1}\,|\, f_1(t,\lambda_1\int_0^{\phi^*(t)} y(s)\, ds) - f_1(t,\lambda_1\int_0^{\phi(t)} y(s)\, ds)\,|\end{aligned}$$

$$\begin{aligned}
&+ \quad \beta\, t^{\beta-1} \mid f_1(t, \lambda_1 \int_0^{\phi^*(t)} y^*(s)\, ds) - f_1(t, \lambda_1 \int_0^{\phi^*(t)} y(s)\, ds) \mid \\
&+ \quad \beta\, t^{\beta-1} \mid f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) - f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y^*(t)) \mid \\
&\leq \quad \beta\, t^{\beta-1}\, b_1\, \lambda_1 \mid \int_{\phi(t)}^{\phi^*(t)} y(s)\, ds \mid + \beta\, t^{\beta-1}\, b_1\, \lambda_1 \int_0^{\phi^*(t)} \mid y^*(s) - y(s) \mid ds \\
&+ \quad \beta\, t^{\beta-1}\, b_2\, \lambda_2\, \frac{t^{1-\beta}}{\beta} \mid y(t) - y^*(t) \mid \\
&\leq \quad \beta\, t^{\beta-1}\, b\, \lambda \mid \int_{\phi(t)}^{\phi^*(t)} y(s)\, ds \mid + \beta\, t^{\beta-1}\, b\, \lambda \int_0^{T} \mid y(s) - y^*(s) \mid ds + b\, \lambda \mid y(t) - y^*(t) \mid \\
&\leq \quad \beta\, t^{\beta-1}\, b\, \lambda \mid \int_{\phi(t)}^{\phi^*(t)} y(s)\, ds \mid + \beta\, t^{\beta-1}\, b\, \lambda\, \|y - y^*\|_1 + b\, \lambda \mid y(t) - y^*(t) \mid,
\end{aligned}$$

since $y \in L_1(I)$ and $\mid \phi(t) - \phi^*(t) \mid < \delta$, then from the absolute continuity of Lebesgue integral [16], we obtain that $\mid \int_{\phi(t)}^{\phi^*(t)} y(s)\, ds \mid < \epsilon^*$ and

$$|y(t) - y^*(t)| \;\leq\; \beta\, t^{\beta-1}\, b\, \lambda\, \epsilon^* + \beta\, t^{\beta-1}\, b\, \lambda\, \|y - y^*\|_1 + b\, \lambda \mid y(t) - y^*(t) \mid,$$

thus

$$\|y - y^*\|_1 \;\leq\; b\, \lambda\, T^{\beta}\, \epsilon^* + b\, \lambda\, T^{\beta}\, \|y - y^*\|_1 + b\, \lambda\, \|y - y^*\|_1$$

and

$$[1 - b\, \lambda\, (1 + T^{\beta})]\, \|y - y^*\|_1 \;\leq\; b\, \lambda\, T^{\beta}\, \epsilon^*,$$

then

$$\|y - y^*\|_1 \;=\; \frac{b\, \lambda\, T^{\beta}\, \epsilon^*}{1 - b\, \lambda\, (1 + T^{\beta})} \;=\; \epsilon_1^*.$$

Now,

$$\begin{aligned}
|x(t) - x^*(t)| \quad &= \quad \mid \frac{t^{1-\beta}}{\beta} y(t) - \frac{t^{1-\beta}}{\beta} y^*(t) \mid \\
&\leq \quad \frac{T^{1-\beta}}{\beta} \mid y(t) - y^*(t) \mid,
\end{aligned}$$

then

$$\|x - x^*\|_1 \quad \leq \quad \frac{T^{1-\beta}}{\beta}\, \|y - y^*\|_1 \;\leq\; \frac{T^{1-\beta}}{\beta}\, \epsilon_1^* \;=\; \epsilon.$$

2.4. **Hyers-Ulam Stability.**

Definition 2.3. *([6], [7], [19], [15]) Let the solution $x \in L_1(I)$ of equation (1.1) be exists, then the integral equation (1.1) is Hyers-Ulam stable, if $\forall \epsilon > 0, \exists \delta > 0$ such that for any δ-approximate solution x_s of equation (1.1) satisfying*

$$\mid x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, \theta^{\beta-1}\, x_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t)) \mid < \delta,$$

implies

$$\|x - x_s\|_1 < \epsilon.$$

Theorem 2.5. *Let the assumptions of Theorem 2.2 be satisfied. Then the integral equation (1.1) is Hyers-Ulam stable.*

Proof. We have $|\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, \theta^{\beta-1}\, x_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t))\,| < \delta$, then

$$-\delta < \; x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, \theta^{\beta-1}\, x_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t)) \; < \; \delta$$

multiplying by $\beta\, t^{\beta-1}$, we get

$$-\delta\beta t^{\beta-1} < \; \beta t^{\beta-1}\, x_s(t) + \beta t^{\beta-1}\, f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\theta^{\beta-1}\, x_s(\theta)\, d\theta) - \beta t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, \beta t^{\beta-1}\, x_s(t)) \; < \; \delta\beta t^{\beta-1}$$

and let $y_s(t) = \; \beta\, t^{\beta-1}\, x_s(t)$, then

$$-\delta\, \beta\, t^{\beta-1} \; < \; y_s(t) + \beta\, t^{\beta-1}\, f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, y_s(t)) \; < \; \delta\, \beta\, t^{\beta-1},$$

thus we get

$$|\, y_s(t) + \beta\, t^{\beta-1}\, f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, y_s(t))\,| \; < \; \delta\, \beta\, t^{\beta-1}.$$

Now,

$$\begin{aligned}
|y(t) - y_s(t)| \;&=\; |\, -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \; - \; y_s(t)\,| \\
&=\; |\, -\beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta) + \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) \\
&-\; (\, y_s(t) + \beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_s(t))\,) \\
&+\; \beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_s(t))\,| \\
&\leq\; \beta\, t^{\beta-1}\, |\, f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta)\,| \\
&+\; \beta\, t^{\beta-1}\, |\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y(t)) - f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_s(t))\,| \\
&+\; |\, y_s(t) + \beta\, t^{\beta-1} f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - \beta\, t^{\beta-1}\, f_2(t, \lambda_2\, \frac{t^{1-\beta}}{\beta} y_s(t))\,| \\
&\leq\; \beta\, t^{\beta-1}\, b_1\, \lambda_1 \int_0^{\phi(t)} |\, y_s(\theta) - y(\theta)\,|\, d\theta \; + \; \beta\, t^{\beta-1}\, b_2\, \lambda_2\, \frac{t^{1-\beta}}{\beta}\, |\, y(t) - y_s(t)\,| \; + \; \delta\, \beta\, t^{\beta-1} \\
&\leq\; \beta\, t^{\beta-1}\, b_1\, \lambda_1 \int_0^{t} |\, y(\theta) - y_s(\theta)\,|\, d\theta \; + \; b_2\, \lambda_2\, |\, y(t) - y_s(t)\,| \; + \; \delta\, \beta\, t^{\beta-1} \\
&\leq\; \beta\, t^{\beta-1}\, b\, \lambda\, \|y - y_s\|_1 \; + \; b\, \lambda\, |\, y(t) - y_s(t)\,| \; + \; \delta\, \beta\, t^{\beta-1},
\end{aligned}$$

then

$$\int_0^T |y(t) - y_s(t)|\, dt \ \le\ b\,\lambda\, \|y - y_s\|_1 \int_0^T \beta\, t^{\beta-1}\, dt \ +\ b\,\lambda \int_0^T |\, y(t) - y_s(t)\,|\, dt \ +\ \delta \int_0^T \beta\, t^{\beta-1}\, dt$$

and

$$\|y - y_s\|_1 \ \le\ b\,\lambda\, T^{\beta}\, \|y - y_s\|_1 \ +\ b\,\lambda\, \|y - y_s\|_1 \ +\ \delta\, T^{\beta},$$

$$[1 - b\,\lambda\, (1 + T^{\beta})]\, \|y - y_s\|_1 \ \le\ \delta\, T^{\beta},$$

then

$$\|y - y_s\|_1 \ \le\ \frac{\delta\, T^{\beta}}{1 - b\,\lambda\,(1 + T^{\beta})} \ =\ \epsilon_1. \tag{2.10}$$

Now,

$$|x(t) - x_s(t)| \ = \ |\, -f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, \theta^{\beta-1}\, x(\theta)\, d\theta) + f_2(t, \lambda_2\, x(t)) - x_s(t)\,|,$$

let $y(t) = \ \beta\, t^{\beta-1}\, x(t)$, then

$$\begin{aligned}
|x(t) - x_s(t)| \ &= \ |\, -f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta) + f_2(t, \lambda_2\, x(t)) - x_s(t)\,| \\
&= \ |\, -f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta) + f_2(t, \lambda_2\, x(t)) \\
&+ \ f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t)) \\
&- \ (\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t))\ )\,| \\
&\le \ |\, f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - f_1(t, \lambda_1 \int_0^{\phi(t)} y(\theta)\, d\theta)\,| \\
&+ \ |\, f_2(t, \lambda_2\, x(t)) - f_2(t, \lambda_2\, x_s(t))\,| \\
&+ \ |\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} y_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t))\,| \\
&\le \ b_1\, \lambda_1 \int_0^{\phi(t)} |\, y_s(\theta) - y(\theta)\,|\, d\theta \ +\ b_2\, \lambda_2\, |\, x(t) - x_s(t)\,| \\
&+ \ |\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, \theta^{\beta-1} x_s(\theta)\, d\theta) - f_2(t, \lambda_2\, x_s(t))\,| \\
&\le \ b\,\lambda \int_0^{t} |\, y(\theta) - y_s(\theta)\,|\, d\theta \ +\ b\,\lambda\, |\, x(t) - x_s(t)\,| \ +\ \delta \\
&\le \ b\,\lambda\, \|y - y_s\|_1 \ +\ b\,\lambda\, |\, x(t) - x_s(t)\,| \ +\ \delta
\end{aligned}$$

and

$$\int_0^T |x(t) - x_s(t)|\, dt \ \ \le\ \ b\,\lambda\, T\, \|y - y_s\|_1 \ +\ b\,\lambda \int_0^T |\, x(t) - x_s(t)\,|\, dt \ +\ \delta\, T$$

by using (2.10), we get

$$\|x - x_s\|_1 \ \le \ b\, \lambda\, T\, \epsilon_1 \ + \ b\, \lambda\, \|x - x_s\|_1 \ + \ \delta\, T,$$

then

$$(1 - b\, \lambda)\ \|x - x_s\|_1 \ \le \ b\, \lambda\, T\, \epsilon_1 + \delta\, T$$

and

$$\|x - x_s\|_1 \ \le \ \frac{b\, \lambda\, T\, \epsilon_1 + \delta\, T}{1 - b\, \lambda} \ = \ \epsilon.$$

3. Continuous solution

3.1. **Existence of solution.** Consider equation (1.1) under the following assumptions
$(i)^{**}$ $f_i : I \times R \to R,\ i = 1,2$ satisfies the Lipschitz condition

$$|f_i(t_2, x_2) - f_i(t_1, x_1)| \le b_i\ (\ |t_2 - t_1| + |x_2 - x_1|\ ),\ b_i > 0.$$

Theorem 3.1. *Let the condition $(i)^{**}$, (iii) and (iv) be satisfied, then the fractal integral equation (1.1) has a unique solution $x \in C(I)$.*

Proof. Define the operator F by

$$Fx(t) = -f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds) + f_2(t, \lambda_2\, x(t)).$$

Let $x \in C(I)$, and $t_1, t_2 \in I$ where $t_1 < t_2$ be such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Fx(t_2) - Fx(t_1)| \ &= \ |-f_1(t_2, \lambda_1 \int_0^{\phi(t_2)} \beta\, s^{\beta-1}\, x(s)\, ds) + f_2(t_2, \lambda_2\, x(t_2)) \\
&+ \ f_1(t_1, \lambda_1 \int_0^{\phi(t_1)} \beta\, s^{\beta-1}\, x(s)\, ds) - f_2(t_1, \lambda_2\, x(t_1))\ | \\
&\le \ |\ f_1(t_1, \lambda_1 \int_0^{\phi(t_1)} \beta\, s^{\beta-1}\, x(s)\, ds) - f_1(t_2, \lambda_1 \int_0^{\phi(t_2)} \beta\, s^{\beta-1}\, x(s)\, ds)\ | \\
&+ \ |\ f_2(t_2, \lambda_2\, x(t_2)) - f_2(t_1, \lambda_2\, x(t_1))\ | \\
&\le \ b_1\, |t_1 - t_2| + b_1\, \lambda_1\, |\int_0^{\phi(t_1)} \beta\, s^{\beta-1}\, x(s)\, ds - \int_0^{\phi(t_2)} \beta\, s^{\beta-1}\, x(s)\, ds\ | \\
&+ \ b_2\, |t_2 - t_1| + b_2\, \lambda_2\, |x(t_2) - x(t_1)| \\
&\le \ b\, |t_1 - t_2| + b\, \lambda \int_{\phi(t_2)}^{\phi(t_1)} \beta\, s^{\beta-1}\, |x(s)|\, ds \ + \ b\, |t_2 - t_1| + b\, \lambda\, |x(t_2) - x(t_1)|,
\end{aligned}$$

which means that the operator F maps from $C(I)$ into $C(I)$.

Now,

$$\begin{aligned}
|Fx(t)-Fy(t)| &= |-f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)+f_2(t,\lambda_2\, x(t))\\
&+ f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, y(s)\, ds)-f_2(t,\lambda_2\, y(t))\,|\\
&\leq |\, f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, y(s)\, ds)-f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)\,|\\
&+ |\, f_2(t,\lambda_2\, x(t))-f_2(t,\lambda_2\, y(t))\,|\\
&\leq b_1\,\lambda_1\,|\int_0^{\phi(t)}\beta\, s^{\beta-1}\, y(s)\, ds-\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds\,|\;+\;b_2\,\lambda_2\,|x(t)-y(t)|\\
&\leq b\,\lambda\int_0^{\phi(t)}\beta\, s^{\beta-1}\,|y(s)-x(s)|\, ds\;+\;b\,\lambda\,|x(t)-y(t)|\\
&\leq b\,\lambda\int_0^{\phi(t)}\beta\, s^{\beta-1}\sup_s|x(s)-y(s)|\, ds\;+\;b\,\lambda\,\sup_t|x(t)-y(t)|\\
&\leq b\,\lambda\,\|x-y\|_c\int_0^t\beta\, s^{\beta-1}\, ds\;+\;b\,\lambda\,\|x-y\|_c\\
&= b\,\lambda\, t^{\beta}\,\|x-y\|_c\;+\;b\,\lambda\,\|x-y\|_c\\
&\leq b\,\lambda\,(1+T^{\beta})\,\|x-y\|_c,
\end{aligned}$$

then

$$\begin{aligned}
\sup_t|Fx(t)-Fy(t)| &\leq b\,\lambda\,(1+T^{\beta})\,\|x-y\|_c\\
\|Fx-Fy\|_c &\leq b\,\lambda\,(1+T^{\beta})\,\|x-y\|_c,
\end{aligned}$$

which means that the operator F is contraction, as $b\,\lambda\,(1+T^{\beta})<1$.
Consequently, from Banach fixed point theorem [10], equation (1.1) has a unique solution.

3.2. **Continuous Dependence.**

Definition 3.1. *The solution $x\in C(I)$ of equation (1.1) depends continuously on the parameters λ_i and on the functions f_i [9], if $\forall\epsilon>0,\ \exists\delta>0$ such that*

$$\max\{\,|\lambda_i-\lambda_i^*|,\ |f_i-f_i^*|\,\}<\delta,\ i=1,2$$

implies

$$\|x-x^*\|_c<\epsilon,$$

where x^* is the unique solution of equation (2.5)

Theorem 3.2. *Let the assumptions of Theorem 3.1 be satisfied. Then the unique solution of equation (1.1) depends continuously on the parameters λ_i and on the functions f_i.*

Proof. Let x, x^* be two solutions of (1.1) and (2.5) respectively, then

$$\begin{aligned}
|x(t)-x^*(t)| &= |-f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)+f_2(t,\lambda_2\, x(t))\\
&+ f_1^*(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)-f_2^*(t,\lambda_2^*\, x^*(t))\,|\\
&= |-f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)+f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)\\
&- f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)+f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)\\
&- f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)+f_1^*(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)\\
&+ f_2(t,\lambda_2\, x(t))-f_2(t,\lambda_2^*\, x(t))\ +\ f_2(t,\lambda_2^*\, x(t))-f_2(t,\lambda_2^*\, x^*(t))\\
&+ f_2(t,\lambda_2^*\, x^*(t))-f_2^*(t,\lambda_2^*\, x^*(t))\,|\\
&\le |\,f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)-f_1(t,\lambda_1\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)\,|\\
&+ |\,f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)-f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x(s)\, ds)\,|\\
&+ |\,f_1^*(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)-f_1(t,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\, x^*(s)\, ds)\,|\\
&+ |\,f_2(t,\lambda_2\, x(t))-f_2(t,\lambda_2^*\, x(t))\,|\ +\ |\,f_2(t,\lambda_2^*\, x(t))-f_2(t,\lambda_2^*\, x^*(t))\,|\\
&+ |\,f_2(t,\lambda_2^*\, x^*(t))-f_2^*(t,\lambda_2^*\, x^*(t))\,|\\
&\le b_1\,|\lambda_1^*-\lambda_1|\int_0^{\phi(t)}\beta\, s^{\beta-1}\,|x(s)|\, ds\ +\ b_1\,\lambda_1^*\int_0^{\phi(t)}\beta\, s^{\beta-1}\,|x^*(s)-x(s)|\, ds\\
&+ \delta\ +\ b_2\,|\lambda_2-\lambda_2^*|\,|x(t)|\ +\ b_2\,\lambda_2^*\,|x(t)-x^*(t)|\ +\ \delta\\
&\le b\,\delta\,\|x\|_c\int_0^{t}\beta\, s^{\beta-1}\, ds\ +\ b\,\lambda\,\|x-x^*\|_c\int_0^{t}\beta\, s^{\beta-1}\, ds\ +\ 2\,\delta\ +\ b\,\delta\,|x(t)|\\
&+ b\,\lambda\,\|x-x^*\|_c\\
&\le b\,\delta\, T^\beta\,\|x\|_c\ +\ b\,\lambda\, T^\beta\,\|x-x^*\|_c\ +\ 2\,\delta\ +\ b\,\delta\,\|x\|_c\ +\ b\,\lambda\,\|x-x^*\|_c,
\end{aligned}$$

thus

$$\|x-x^*\|_c\ \le\ b\,\delta\, T^\beta\,\|x\|_c\ +\ b\,\lambda\, T^\beta\,\|x-x^*\|_c\ +\ 2\,\delta\ +\ b\,\delta\,\|x\|_c\ +\ b\,\lambda\,\|x-x^*\|_c$$

and

$$[1-b\,\lambda\,(1+T^\beta)]\,\|x-x^*\|_c\ \le\ 2\,\delta\ +\ b\,\delta\,(1+T^\beta)\,\|x\|_c,$$

then

$$\|x-x^*\|_c\ \le\ \frac{2\,\delta\ +\ b\,\delta\,(1+T^\beta)\,\|x\|_c}{1-b\,\lambda\,(1+T^\beta)}\ =\ \epsilon,\ \ \text{where } b\,\lambda\,(1+T^\beta)<1.$$

Definition 3.2. *The solution $x \in C(I)$ of equation (1.1) depends continuously on the delay function ϕ, if $\forall\epsilon > 0,\ \exists\delta > 0$ such that*

$$|\phi - \phi^*| \ < \ \delta,$$

implies

$$\|x - x^*\|_c < \epsilon,$$

where x^* is the unique solution of equation (2.8)

Theorem 3.3. *Let the assumptions of Theorem 3.1 be satisfied. Then the unique solution of equation (1.1) depends continuously on the delay function ϕ.*

Proof. Let x, x^* be two solutions of (1.1) and (2.8) respectively, then

$$\begin{aligned}
|x(t) - x^*(t)| &= |-f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds) + f_2(t, \lambda_2\, x(t)) \\
&+ f_1(t, \lambda_1 \int_0^{\phi^*(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) - f_2(t, \lambda_2\, x^*(t))\, | \\
&= |-f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) \\
&- f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) + f_1(t, \lambda_1 \int_0^{\phi^*(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) \\
&+ f_2(t, \lambda_2\, x(t)) - f_2(t, \lambda_2\, x^*(t))\, | \\
&\leq |\, f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) - f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x(s)\, ds)\, | \\
&+ |\, f_1(t, \lambda_1 \int_0^{\phi^*(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds) - f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds)\, | \\
&+ |\, f_2(t, \lambda_2\, x(t)) - f_2(t, \lambda_2\, x^*(t))\, | \\
&\leq b_1\, \lambda_1 \int_0^{\phi(t)} \beta\, s^{\beta-1}\, |x^*(s) - x(s)|\, ds \ + \ b_1\, \lambda_1\, |\int_{\phi(t)}^{\phi^*(t)} \beta\, s^{\beta-1}\, x^*(s)\, ds\, | \\
&+ b_2\, \lambda_2\, |x(t) - x^*(t)| \\
&\leq b\, \lambda\, \|x - x^*\|_c \int_0^t \beta\, s^{\beta-1}\, ds \ + \ b\, \lambda\, \|x^*\|_c\, |\int_{\phi(t)}^{\phi^*(t)} \beta\, s^{\beta-1}\, ds\, | \ + \ b\, \lambda\, \|x - x^*\|_c,
\end{aligned}$$

since $\beta\, s^{\beta-1} \in L_1(I)$ and $|\, \phi(t) - \phi^*(t)\, | < \delta$, then from the absolute continuity of Lebesgue integral [16], we obtain that $|\int_{\phi(t)}^{\phi^*(t)} \beta\, s^{\beta-1}\, ds\, | < \epsilon^*$ and

$$|x(t) - x^*(t)| \ \leq \ b\, \lambda\, T^\beta\, \|x - x^*\|_c \ + \ b\, \lambda\, \|x^*\|_c\, \epsilon^* \ + \ b\, \lambda\, \|x - x^*\|_c,$$

thus

$$\|x - x^*\|_c \ \leq \ b\, \lambda\, T^\beta\, \|x - x^*\|_c \ + \ b\, \lambda\, \epsilon^*\, \|x^*\|_c \ + \ b\, \lambda\, \|x - x^*\|_c$$

and

$$[1 - b\, \lambda\, (1 + T^\beta)]\, \|x - x^*\|_c \ \leq \ b\, \lambda\, \epsilon^*\, \|x^*\|_c,$$

then

$$\|x - x^*\|_c \;\leq\; \frac{b\,\lambda\,\epsilon^*\,\|x^*\|_c}{1 - b\,\lambda\,(1 + T^\beta)} \;=\; \epsilon.$$

3.3. **Hyers-Ulam Stability.**

Definition 3.3. *([6], [7], [19], [15]) Let the solution $x \in C(I)$ of equation (1.1) be exists, then the integral equation (1.1) is Hyers-Ulam stable, if $\forall \epsilon > 0, \exists \delta > 0$ such that for any δ-approximate solution x_s of equation (1.1) satisfying*

$$|\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_2(t, \lambda_2\,x_s(t))\,| < \delta,$$

implies

$$\|x - x_s\|_c < \epsilon.$$

Theorem 3.4. *Let the assumptions of Theorem 3.1 be satisfied. Then the integral equation (1.1) is Hyers-Ulam stable.*

Proof. Let $|\, x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_2(t, \lambda_2\,x_s(t))\,| < \delta$, then

$$\begin{aligned}
|x(t) - x_s(t)| &= |- f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x(\theta)\,d\theta) + f_2(t, \lambda_2\,x(t)) - x_s(t)\,| \\
&= |- f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x(\theta)\,d\theta) + f_2(t, \lambda_2\,x(t)) \\
&+ \; f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_2(t, \lambda_2\,x_s(t)) \\
&- \; f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) + f_2(t, \lambda_2\,x_s(t)) - x_s(t)\,| \\
&= |\,(\,f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x(\theta)\,d\theta)\,) \\
&+ \; (\,f_2(t, \lambda_2\,x(t)) - f_2(t, \lambda_2\,x_s(t))\,) \\
&- \; (\,x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_2(t, \lambda_2\,x_s(t))\,)\,| \\
&\leq |\,f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x(\theta)\,d\theta)\,| \\
&+ \; |\,f_2(t, \lambda_2\,x(t)) - f_2(t, \lambda_2\,x_s(t))\,| \\
&+ \; |\,x_s(t) + f_1(t, \lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,x_s(\theta)\,d\theta) - f_2(t, \lambda_2\,x_s(t))\,| \\
&\leq b_1\,\lambda_1 \int_0^{\phi(t)} \beta\,\theta^{\beta-1}\,|x_s(\theta) - x(\theta)|\,d\theta \;+\; b_2\,\lambda_2\,|x(t) - x_s(t)| \;+\; \delta \\
&\leq b\,\lambda\,\|x - x_s\|_c \int_0^t \beta\,\theta^{\beta-1}\,d\theta \;+\; b\,\lambda\,\|x - x_s\|_c \;+\; \delta \\
&\leq b\,\lambda\,T^\beta\,\|x - x_s\|_c \;+\; b\,\lambda\,\|x - x_s\|_c \;+\; \delta,
\end{aligned}$$

thus

$$\|x - x_s\|_c \leq b\,\lambda\,(1 + T^{\beta})\,\|x - x_s\|_c + \delta$$

and

$$[\,1 - b\,\lambda\,(1 + T^{\beta})\,]\,\|x - x_s\|_c \leq \delta,$$

then

$$\|x - x_s\|_c \leq \frac{\delta}{1 - b\,\lambda\,(1 + T^{\beta})} = \epsilon.$$

4. Conclusions

This research paper focuses on investigate the existence of solutions for the delayed Abel fractal integral equation (1.1) and properties associated with these solutions. Firstly, we studied the existence of at least one solution $x \in L_1[0, T]$ of (1.1) by applying Schauder's fixed point Theorem. Furthermore, we established sufficient conditions to ensure the uniqueness of the solution and its dependence on the parameters λ_1 and λ_2 and on the functions f_1, f_2 and the delay function ϕ. We also investigated the Hyers-Ulam stability of the equation (1.1). Secondly, we studied the uniqueness of the solution $x \in C[0, T]$ by applying Banach's fixed point Theorem. Also, we examined the continuous dependence and the Hyers-Ulam stability of this equation.

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