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Phase and Norm Retrieval via Projections in 2-Inner Product Spaces

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Abstract. This paper introduces and studies 2-phase retrieval and 2-norm retrieval in the context of 2-inner product spaces, generalizing classical phase and norm retrieval problems to a nonlinear geometric setting. A collection of subspaces $\{W_i\}_{i=1}^M$ in a 2-inner product space V is said to yield 2-phase retrieval if the 2-norms of projections $\|P_i s\|_z = \|P_i b\|_z$ for all i and all reference vectors $z \in V \setminus \{0\}$ imply that s and b are phase-equivalent (i.e., $s = cb$ for some scalar c with $|c| = 1$). Similarly, $\{W_i\}_{i=1}^M$ achieves 2-norm retrieval if the 2-norms $\|P_i s\|_z$ uniquely determine the 2-norm $\|s\|_z$ for all $s \in V$ and all z.

1. Introduction

Reverting signals is one of the oldest and most difficult engineering problems. Especially challenging is recovering the signal when essential information is absent or partially obtained. This dilemma is especially apparent in the presence of phase information loss which happens in domains such as speech processing, and optical technologies, like X-ray crystallography [3]. An important research objective still remains to develop efficient methods of phase retrieval for these applications. The first systematic study of phase retrieval in Hilbert space frames was done by Balan, Casazza, and Edidin in 2006 [4]. This idea aroused much interest and the formulation of phase retrieval for signal-processing (as well as the harmonic analysis) research area is not new, but it is still growing. As researchers continue to research in solving the problem of incomplete phase information in more applications, this area of study continues to develop. The technique of phase retrieval is to reconstruct phase information from incoming signals through taking only the magnitude from overcomplete linear systems. And this problem will be seen in two most common forms: vector-based phase retrieval and projection-based phase retrieval. This method is projection-based, where a signal is retrieved from the subspace projection magnitudes, which has direct applicability in fields like crystallography, where a crystal twinning happens [5]. A closely related

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concept is phaseless reconstruction, in which signals can only be reconstructed from intensity data. Although those terms were regularly applied in a confused and somewhat overlapping manner, formal equivalence of them didn't actually happen until recent work showed that they are also equivalent mathematically, both in real and in complex settings. This rationalization enables researchers to concentrate only on phase retrieval results where they can find that for phaseless reconstruction at least they are equivalent [8]. We analyze 2-norm retrieval, which is introduced to understand how vector norms can be derived from absolute intensity measurements. This problem arises naturally by analyzing systems using both subspaces and their orthogonal complements for measuring. We study 2-norm retrieval in the context of projection operators, and generalize previous frame-based results and complete their classifications of 2-norm-retrieving subspaces in $\mathbb{R}^{\mathbb{N}}$. The paper offers physical examples of 2-phase and 2-norm retrieval systems and introduces novel categories by using Naimark's theorem as a primary basis.

2. Preliminaries

Definition 2.1 (2-Inner Product Space)**.** *A 2-inner product space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is a vector space V equipped with a 2-inner product $\langle \cdot, \cdot \mid \cdot \rangle : V \times V \times V \to \mathbb{K}$ satisfying [12]:*

(1) *Linearity in the first argument: $\langle a_1 + a_2, b \mid z \rangle = \langle a_1, b \mid z \rangle + \langle a_2, b \mid z \rangle$, $\langle \lambda a, b \mid z \rangle = \lambda \langle a, b \mid z \rangle$.*
(2) *Conjugate symmetry: $\langle a, b \mid z \rangle = \overline{\langle b, a \mid z \rangle}$.*
(3) *Non-negativity: $\langle a, a \mid z \rangle \geq 0$, with $\langle a, a \mid z \rangle = 0$ if and only if $a = 0$ or $z = 0$.*

The 2-norm is defined as $\|a\|_z = \sqrt{\langle a, a \mid z \rangle}$.

Definition 2.2 (2-Frame)**.** *A family of vectors $\{\varphi_i\}_{i=1}^M \subset V$ is a 2-frame if there exist constants $0 < A \leq B < \infty$, called 2-frame bounds, such that for all $s, z \in V \setminus \{0\}$:*

$$A\|s\|_z^2 \leq \sum_{i=1}^{M} |\langle s, \varphi_i \mid z \rangle|^2 \leq B\|s\|_z^2,$$

where the 2-norm $\|s\|_z = \sqrt{\langle s, s \mid z \rangle}$ is defined as in Definition 2.1 [9].

Definition 2.3. *The Analysis Operator $T_z : V \to \mathbb{C}^M$ is defined by $T_z(s) = (\langle s, \varphi_i \mid z \rangle)_{i=1}^M$, which is linear and bounded for each fixed $z \in V \setminus \{0\}$ [10]. The Synthesis Operator $T_z^* : \mathbb{C}^M \to V$ is given by $T_z^*(c) = \sum_{i=1}^M c_i \varphi_i$, which is the adjoint of T_z with respect to the 2-inner product [10]. The Frame Operator $S_z : V \to V$ is defined as $S_z(s) = \sum_{i=1}^M \langle s, \varphi_i \mid z \rangle \varphi_i$, which is positive definite and invertible when $A > 0$, since $A\|s\|_z^2 \leq \langle S_z s, s \mid z \rangle \leq B\|s\|_z^2$ [9]. Special cases include: a 2-Tight Frame, where $A = B$, satisfies $\sum_{i=1}^M |\langle s, \varphi_i \mid z \rangle|^2 = A\|s\|_z^2$ for all $s, z \in V \setminus \{0\}$; a 2-Parseval Frame, with $A = B = 1$, satisfies $\sum_{i=1}^M |\langle s, \varphi_i \mid z \rangle|^2 = \|s\|_z^2$ for all $s, z \in V \setminus \{0\}$ [9].*

Definition 2.4 (2-Orthonormal Basis)**.** *Let $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A set of vectors $\{e_i\}_{i=1}^N \subset V$ is a 2-orthonormal basis if:*

(1) *2-Orthogonality: $\langle e_i, e_j \mid z \rangle = 0 \quad \forall z \in V \setminus \{0\}, \forall i \neq j$.*
(2) *Normalization: $\langle e_i, e_i \mid z \rangle = 1 \quad \forall z \in V \setminus \{0\}, \forall i$.*

(3) *Completeness:* $\text{span}\{e_i\}_{i=1}^N = V$.

Note: We adopt the convention $\langle e_i, e_i \mid z\rangle = 1$ *for all* $z \in V \setminus \{0\}$*, unless otherwise specified. Alternatively, if* $\langle e_i, e_i \mid z\rangle = \|z\|_w^2$ *for some fixed* $w \neq 0$*, this must be stated [9].*

Definition 2.5. *Let* $\{W_i\}_{i=1}^M$ *be a collection of subspaces in* $\mathbb{R}^N$*, with orthogonal projections* $\{P_i\}_{i=1}^M$ *with respect to the standard Euclidean inner product. The collection allows phase retrieval if, for any* $s, b \in \mathbb{R}^N$*,*

$$\|P_i s\| = \|P_i b\| \quad \textit{for all } i = 1, \ldots, M,$$

implies $s = cb$ *for some scalar* $c \in \mathbb{R}$ *with* $|c| = 1$ *[1].*

Remark 2.1. *Phase retrieval using vectors is a special case of this definition. Also, orthonormal bases cannot perform phase retrieval, because the coefficients of a vector in such bases are uniquely determined and contain no phase ambiguity [1]. One key concept in phase retrieval is the complement property.*

Definition 2.6. *Let* $\aleph = \{\alpha_i\}_{i=1}^M \subset \mathbb{R}^N$ *be a set of vectors (a frame). The frame* $\aleph$ *is said to have the complement property if, for every subset* $I \subset \{1, 2, \ldots, M\}$*, either the set of vectors* $\{\alpha_i\}_{i\in I}$ *or its complement* $\{\alpha_i\}_{i\in I^c}$ *spans the entire space* $\mathbb{R}^N$ *[2].*

It is shown in [2] that phase retrieval in $\mathbb{R}^N$ is equivalent to satisfying the complement property. In particular, a collection of vectors in $\mathbb{R}^N$ can perform phase retrieval if and only if it possesses the complement property. Furthermore, a generic set of $2N-1$ vectors in $\mathbb{R}^N$ is capable of phase retrieval, while no set of $2N-2$ vectors can achieve this [2]. Here, the term *generic* refers to an open dense subset of the set of all $(2N-1)$-element frames in $\mathbb{R}^N$ [2].

A fundamental concept in frame theory is norm retrieval, which we formally define as follows:

Definition 2.7. *Let* $\{W_i\}_{i=1}^M$ *be a collection of subspaces in* $\mathbb{R}^N$*, with orthogonal projections* $\{P_i\}_{i=1}^M$ *with respect to the standard Euclidean inner product. The collection yields norm retrieval if, for all vectors* $s, b \in \mathbb{R}^N$*,*

$$\|P_i s\| = \|P_i b\| \quad \textit{for all } i = 1, \ldots, M,$$

implies

$$\|s\| = \|b\|,$$

where $\|\cdot\|$ *is the standard Euclidean norm.*

In particular, a set of vectors $\{\phi_i\}_{i=1}^M \subset \mathbb{R}^N$ *performs norm retrieval if, whenever*

$$|\langle s, \phi_i\rangle| = |\langle b, \phi_i\rangle| \quad \textit{for all } i = 1, \ldots, M,$$

then

$$\|s\| = \|b\| \textit{ [1].}$$

Another key idea in frame theory is the concept of full spark, which plays an essential role in understanding linear dependence. The precise definition is given below:

Definition 2.8. *[1] Let $\aleph = \{\alpha_i\}_{i=1}^{M}$ be a set of vectors in the space $\mathbb{R}^N$. The spark of $\aleph$ is defined as the smallest number of vectors in $\aleph$ that are linearly dependent. If spark$(\aleph) = N+1$, then every subset of N vectors from $\aleph$ is linearly independent. In this case, the set $\aleph$ is said to be full spark.*

Based on the last definition, we can infer that full spark frames with $M \geq 2N-1$ satisfy the complement property, which ensures the possibility of phase retrieval [1]. Furthermore, when $M = 2N-1$, the complement property by itself guarantees that the frame is full spark [1].

The following result, known as Naimark's Theorem, provides a characterization of Parseval frames in finite-dimensional Hilbert spaces. This theorem describes a method to construct Parseval frames, and notably, it is the only known method for doing so. Throughout this paper, we use the notation $[M] = \{1, 2, \ldots, M\}$.

Theorem 2.2 (Naimark's Theorem). *When a set of vectors $\{\alpha_i\}_{i=1}^{M}$ in $\mathbb{R}^N$ is a Parseval frame for $\mathbb{R}^N$ it essentially means that $\mathbb{R}^N$ is being embedded in $\mathbb{R}^M$ with an orthogonal basis $\{e_i\}_{i=1}^{M}$ and that the orthogonal projection P from $\mathbb{R}^M$ to $\mathbb{R}^N$ is doing something very specific: $Pe_i = \alpha_i$ for every i from 1 to M [1].*

3. Main Result

Definition 3.1 (2-Phase Retrieval via Projections). *Assume $\{W_i\}_{i=1}^{M}$ be a collection of subspaces in a 2-inner product space $(\mathbb{R}^N, \langle \cdot, \cdot \mid \cdot \rangle)$, and let $\{P_i\}_{i=1}^{M}$ be the orthogonal projections onto W_i with respect to the 2-inner product $\langle \cdot, \cdot \mid z \rangle$. The collection allows* 2-phase retrieval *if, for all $s, b \in \mathbb{R}^N$,*

$$\|P_i s\|_z = \|P_i b\|_z \quad \forall i \in \{1, \ldots, M\}, \forall z \in \mathbb{R}^N \setminus \{0\},$$

implies $s = cb$ for some scalar $c \in \mathbb{K}$ with $|c| = 1$, where $\|\cdot\|_z$ is the 2-norm induced by $\langle \cdot, \cdot \mid z \rangle$.

Note: The projection operator is defined in relation to the 2-inner product, when generalising the traditional phase retrieval problem. Classical phase retrieval, which is basically using vectors, falls under this generalisation.

Definition 3.2 (2-Phase Retrieval via Vector Pairs). *Let $(\mathbb{R}^N, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space, and let $\{(\alpha_i, \psi_i)\}_{i=1}^{M} \subset \mathbb{R}^N \times \mathbb{R}^N$ be a collection of vector pairs. The collection allows* 2-phase retrieval *if, for all $s, b \in \mathbb{R}^N$,*

$$|\langle s, \alpha_i \mid \psi_i \rangle| = |\langle b, \alpha_i \mid \psi_i \rangle| \quad \forall i \in \{1, \ldots, M\},$$

implies $s = cb$ for some scalar c in $\mathbb{K}$ with $|c| = 1$.

The goal remains to recover a vector (signal) up to a global phase factor using only magnitude information.

Theorem 3.1 (Complement Property and 2-Phase Retrieval). *Let $(\mathbb{R}^N, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space of dimension N, and let $\aleph = \{\alpha_t\}_{t=1}^{M} \subset \mathbb{R}^N$ be a full spark 2-frame with $M \geq 2N-1$. Then:*

1. *$\aleph$ satisfies the complement property.*
2. *If the map $s \mapsto (|\langle s, \alpha_t \mid \psi_t \rangle|)_{t=1}^{M}$ is injective modulo global phase for some $\{\psi_t\}_{t=1}^{M} \subset \mathbb{R}^N$, then $\aleph$ allows 2-phase retrieval.*

Proof. We prove each part separately.

Part 1: Complement Property. Let $\aleph = \{\alpha_t\}_{t=1}^{M} \subset \mathbb{R}^N$ be a full spark 2-frame with $M \geq 2N-1$. Since $\aleph$ is full spark, every subset of N vectors is linearly independent. For any subset $I \subseteq \{1,\ldots,M\}$:

- If $|I| \geq N$, then $\{\alpha_t\}_{t\in I}$ contains at least N linearly independent vectors, spanning $\mathbb{R}^N$.
- If $|I| < N$, then $|I^c| = M - |I| \geq (2N-1)-(N-1) = N$. Thus, $\{\alpha_t\}_{t\in I^c}$ contains at least N linearly independent vectors, spanning $\mathbb{R}^N$.

Hence, either $\{\alpha_t\}_{t\in I}$ or $\{\alpha_t\}_{t\in I^c}$ spans $\mathbb{R}^N$, satisfying the complement property.

Part 2: 2-Phase Retrieval. Assume the map $s \mapsto (|\langle s, \alpha_t \mid \psi_t\rangle|)_{t=1}^{M}$ is injective modulo global phase, i.e., if $|\langle s, \alpha_t \mid \psi_t\rangle| = |\langle b, \alpha_t \mid \psi_t\rangle|$ for all $t = 1,\ldots,M$, then $s = cb$ for some $c \in \mathbb{K}$ with $|c| = 1$. By Definition 3.2, this means that for all $s, b \in \mathbb{R}^N$, if:

$$|\langle s, \alpha_t \mid \psi_t\rangle| = |\langle b, \alpha_t \mid \psi_t\rangle| \quad \text{for all } t = 1,\ldots,M,$$

then $s = cb$ with $|c| = 1$. This matches the injectivity condition, so $\aleph$ allows 2-phase retrieval.

Thus, both claims hold. □

This condition ensures that no subset of vectors is too small to span the space, and therefore, phase retrieval is possible up to a global unimodular constant using the 2-inner product measurements.

Theorem 3.2 (Generalized Naimark's Theorem for 2-Inner Product Spaces)**.** *Let* $(\mathbb{R}^N, \langle\cdot,\cdot \mid \cdot\rangle)$ *be a 2-inner product space equipped with a 2-inner product* $\langle\cdot,\cdot \mid z\rangle$*. A family* $\{\alpha_i\}_{i=1}^{M} \subset \mathbb{R}^N$ *is a* 2-Parseval frame *(i.e.,* $\sum_{i=1}^{M} |\langle s, \alpha_i \mid z\rangle|^2 = \|s\|_z^2$ *for all* $s, z \in \mathbb{R}^N \setminus \{0\}$*) if and only if there exists:*

1. *An embedding* $\mathbb{R}^N \hookrightarrow \ell_2^M$ *with a 2-inner product* $\langle a, b \mid z'\rangle_{\ell_2^M} = \sum_{i=1}^{M} a_i \overline{b_i}$*, where* $z' = T_w z$ *for some fixed* $w \in \mathbb{R}^N \setminus \{0\}$*, and* $T_w : \mathbb{R}^N \to \ell_2^M$ *is the analysis operator defined by* $T_w(s) = (\langle s, \alpha_i \mid w\rangle)_{i=1}^{M}$*,*
2. *A 2-orthonormal basis* $\{e_i\}_{i=1}^{M}$ *of* ℓ_2^M *(i.e.,* $\langle e_i, e_j \mid z'\rangle_{\ell_2^M} = \delta_{ij} \cdot 1$ *for all* $z' \in \ell_2^M \setminus \{0\}$*),*
3. *An orthogonal projection* $P : \ell_2^M \to \mathbb{R}^N$ *such that* $Pe_i = \alpha_i$ *for all* $i \in [M]$*.*

Proof. We establish the equivalence by proving both directions.

First, let we have $\{\alpha_i\}_{i=1}^{M}$ is subset of $\mathbb{R}^{\mathbb{N}}$ ia 2-Parseval frame, which satisfying the following:

$$\sum_{i=1}^{M} |\langle s, \alpha_i \mid z\rangle|^2 = \|s\|_z^2 = \langle s, s \mid z\rangle$$

for all $s, z \in \mathbb{R}^N \setminus \{0\}$. Define the analysis operator $T_z : \mathbb{R}^N \to \ell_2^M$ for fixed $z \in \mathbb{R}^N \setminus \{0\}$ by:

$$T_z(s) = (\langle s, \alpha_i \mid z\rangle)_{i=1}^{M}.$$

Since $\langle\cdot,\cdot \mid z\rangle$ is linear in the first argument [12], T_z is linear. The ℓ_2^M-norm of $T_z(s)$ is:

$$\|T_z(s)\|_{\ell_2^M}^2 = \sum_{i=1}^{M} |\langle s, \alpha_i \mid z\rangle|^2 = \|s\|_z^2,$$

indicating that T_z is an isometry with respect to $\|\cdot\|_z$. Fix $w \in \mathbb{R}^N \setminus \{0\}$ and define $\phi : \mathbb{R}^N \to \ell_2^M$ by $\phi(s) = T_w(s)$. Since $\{\alpha_i\}$ spans $\mathbb{R}^N$, ϕ is injective, embedding $\mathbb{R}^N$ as a subspace of ℓ_2^M.

Define the 2-inner product on ℓ_2^M by:

$$\langle a, b \mid z' \rangle_{\ell_2^M} = \sum_{i=1}^{M} a_i \overline{b_i}, \quad z' = T_w(z),$$

which satisfies 2-inner product properties [12]. The standard basis $\{e_i\}_{i=1}^M$ of ℓ_2^M is 2-orthonormal:

$$\langle e_i, e_j \mid z' \rangle_{\ell_2^M} = \delta_{ij} \cdot 1,$$

and spans ℓ_2^M.

Construct the orthogonal projection $P : \ell_2^M \to \mathbb{R}^N$ onto $\phi(\mathbb{R}^N)$. The synthesis operator $T_w^* : \ell_2^M \to \mathbb{R}^N$, given by $T_w^*(c) = \sum_{i=1}^M c_i \alpha_i$, satisfies $T_w^* T_w = I_{\mathbb{R}^N}$, since:

$$\langle T_w^* T_w s, s \mid w \rangle = \sum_{i=1}^{M} |\langle s, \alpha_i \mid w \rangle|^2 = \|s\|_w^2.$$

Set $P = T_w^* \circ \pi$, where $\pi : \ell_2^M \to \ell_2^M$ projects onto $\operatorname{ran}(T_w)$. Then:

$$P e_i = T_w^*(e_i) = \alpha_i,$$

and P is orthogonal. Thus, conditions (1)-(3) hold.

Conversely, let the embedding, 2-orthonormal basis, and projection exist. For $s \in \mathbb{R}^N$:

$$\sum_{i=1}^{M} |\langle s, \alpha_i \mid z \rangle|^2 = \sum_{i=1}^{M} |\langle s, P e_i \mid z \rangle|^2.$$

Since we have given that $P\phi(s) = s$, so:

$$\langle s, P e_i \mid z \rangle = \langle \phi(s), e_i \mid T_z z \rangle_{\ell_2^M} = (T_z s)_i.$$

So we can get the following

$$\sum_{i=1}^{M} |\langle s, \alpha_i \mid z \rangle|^2 = \|T_z s\|_{\ell_2^M}^2 = \langle s, T_z^* T_z s \mid z \rangle = \|s\|_z^2,$$

thus $\{\alpha_i\}$ is a 2-Parseval frame. □

Corollary 3.3. *Let $(\mathbb{R}^N, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space. If $\{\alpha_i\}_{i=1}^M \subset \mathbb{R}^N$ is a 2-Parseval frame constructed via Theorem and either satisfies the complement property or the map $s \mapsto (|\langle s, \alpha_i \mid z \rangle|)_{i=1}^M$ is injective modulo global phase for some $z \in \mathbb{R}^N \setminus \{0\}$, then $\{\alpha_i\}_{i=1}^M$ enables 2-phase retrieval.*

Proof. Case 1: Complement Property. When we have a set of vectors $\{\alpha_i\}_{i=1}^M$ that satisfy the complement property and we define $\psi_i = \alpha_i$, if the two inner products $|\langle s, \alpha_i \mid \alpha_i \rangle|$ and $|\langle b, \alpha_i \mid \alpha_i \rangle|$ are equal for all i, the complement property tells us that the mapping of $s \mapsto (|\langle s, \alpha_i \mid \alpha_i \rangle|)_{i=1}^M$ is injective up to phase, much like in [2], but adapted to our 2-inner product, thanks to the spanning property. Consequently, this leads us to the conclusion that s can be either b or $-b$.

Case 2: Injectivity. When the map $s \mapsto (|\langle s, \alpha_i \mid z \rangle|)_{i=1}^M$ is injective modulo global phase for some $z \neq 0$, we can set $\psi_i = z$ which then leads to $|\langle s, \alpha_i \mid z \rangle| = |\langle b, \alpha_i \mid z \rangle|$ for all i being a guarantee that

$s = cb$ where $|c| = 1$, and meets the criteria for a 2-phase vector according to Definition 3.2. In addition, the collection of vectors $\{\alpha_i\}$ will allow for 2-phase retrieval.

□

Definition 3.3 (2-Norm Retrieval by Subspaces)**.** *Equipped with the 2-inner product and 2-norm denoted as* $(\cdot, \cdot|\cdot)$ *and* $\|\cdot\|_z$ *respectively, when we have a 2-inner product space V. When we have a collection of subspaces* $\{W_i\}_{i=1}^M$ *along with their respective projections* $\{P_i\}_{i=1}^M$*, we say this collection performs 2-norm retrieval, if and only if the condition* $\|P_i s\|_z = \|P_i b\|_z$ *for all i from* 1 *to M and any non-zero z in V leads to* $\|s\|_z = \|b\|_z$ *for any non-zero z in V.*

Definition 3.4 (2-Norm Retrieval by Vectors)**.** *When a set of vectors* $\{\alpha_i\}_{i=1}^M$ *is a subset of vector space V is said to be responsible for 2-norm retrieval if*

$$|\langle s, \alpha_i | z\rangle| = |\langle b, \alpha_i | z\rangle| \quad \forall i \in [M],\ \forall z \in V \setminus \{0\}.$$

This condition implies that

$$\|s\|_z = \|b\|_z \quad \forall z \in V \setminus \{0\}.$$

Proposition 3.1. *When a collection of vector pairs* $\{(\alpha_i, \psi_i)\}_{i=1}^M$ *in a 2-inner product space* $(V, \langle\cdot, \cdot \mid \cdot\rangle)$ *enables 2-phase retrieval, it implies* $\{\alpha_i\}_{i=1}^M \subset V$ *yields 2-norm retrieval.*

Proof. Assume that $\{(\alpha_i, \psi_i)\}_{i=1}^M$ does 2-phase retrieval. If

$$|\langle s, \alpha_i \mid z\rangle| = |\langle b, \alpha_i \mid z\rangle|, \quad \forall i = 1, \ldots, M$$

and all $z \in V\backslash\{0\}$ then for some fixed $z \in V\backslash\{0\}$, we can set $\psi_i = z$ for all i. Then the set $\{(\alpha_i, z)\}$ does 2-phase retrieval and hence, we have that

$$|\langle s, \alpha_i \mid z\rangle| = |\langle b, \alpha_i \mid z\rangle| \quad \forall i \quad \Rightarrow \quad s = cb \quad \textit{for some} \quad |c| = 1.$$

For any other $z' \in V\backslash\{0\}$, we have that

$$\|s\|_{z'} = \sqrt{\langle cb, cb \mid z'\rangle} = \sqrt{|c|^2\langle b, b \mid z'\rangle} = \|b\|_{z'}.$$

Therefore, the set $\{\alpha_i\}$ does 2-norm retrieval.

□

Theorem 3.4 (2-Orthonormal Bases and Norm Retrieval)**.** *When we have a 2-orthonormal basis in a 2-inner product space V with the basis vectors* $\{\alpha_i\}_{i=1}^N$ *and 2-inner product* $\langle\cdot, \cdot \mid \cdot\rangle$ *such that* $\langle\alpha_i, \alpha_j \mid z\rangle = \delta_{ij} \cdot 1$ *for any nonzero vector z in V, the following two things are true.*

(1) *(2-Parseval Identity) For all* $s, z \in V \setminus \{0\}$*,*

$$\sum_{i=1}^N |\langle s, \alpha_i \mid z\rangle|^2 = \|s\|_z^2.$$

(2) $\{\alpha_i\}$ *yields 2-norm retrieval: If* $|\langle s, \alpha_i \mid z\rangle| = |\langle b, \alpha_i \mid z\rangle|$ *for all* $i = 1, \ldots, N$ *and all* $z \in V \setminus \{0\}$*, then* $\|s\|_z = \|b\|_z$ *for all* $z \in V \setminus \{0\}$.

Proof. Part 1: 2-Parseval Identity. Since $\{\alpha_i\}_{i=1}^N$ is a 2-orthonormal basis, it spans V. For any $s \in V$, write $s = \sum_{i=1}^N s_i \alpha_i$, where $s_i = \langle s, \alpha_i \mid z \rangle$ for $z \in V \setminus \{0\}$. By 2-orthonormality, $\langle \alpha_i, \alpha_j \mid z \rangle = \delta_{ij} \cdot 1$, so:

$$\langle s, \alpha_j \mid z \rangle = \left\langle \sum_{i=1}^N s_i \alpha_i, \alpha_j \mid z \right\rangle = \sum_{i=1}^N s_i \langle \alpha_i, \alpha_j \mid z \rangle = s_j.$$

Thus:

$$\sum_{i=1}^N |\langle s, \alpha_i \mid z \rangle|^2 = \sum_{i=1}^N |s_i|^2.$$

Compute the 2-norm:

$$||s||_z^2 = \left\langle \sum_{i=1}^N s_i \alpha_i, \sum_{j=1}^N s_j \alpha_j \mid z \right\rangle = \sum_{i=1}^N \sum_{j=1}^N s_i s_j \langle \alpha_i, \alpha_j \mid z \rangle = \sum_{i=1}^N s_i^2 \cdot 1 = \sum_{i=1}^N |s_i|^2,$$

since $s_i \in \mathbb{R}$. Hence:

$$\sum_{i=1}^N |\langle s, \alpha_i \mid z \rangle|^2 = ||s||_z^2,$$

proving the 2-Parseval identity.

Part 2: 2-Norm Retrieval. Assume $|\langle s, \alpha_i \mid z \rangle| = |\langle b, \alpha_i \mid z \rangle|$ for all $i = 1, \ldots, N$ and all $z \in V \setminus \{0\}$. Write $s = \sum_{i=1}^N s_i \alpha_i$ and $b = \sum_{i=1}^N b_i \alpha_i$, where $s_i = \langle s, \alpha_i \mid z \rangle$ and $b_i = \langle b, \alpha_i \mid z \rangle$. Then:

$$|s_i| = |b_i| \implies s_i = \pm b_i.$$

Using Part 1:

$$||s||_z^2 = \sum_{i=1}^N |s_i|^2 = \sum_{i=1}^N |b_i|^2 = ||b||_z^2 \implies ||s||_z = ||b||_z.$$

Since this holds for all $z \in V \setminus \{0\}$, $\{\alpha_i\}$ yields 2-norm retrieval. □

Definition 3.5 (2-Tight Projection System). *Let $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space, and let $\{P_j\}_{j=1}^k$ be orthogonal projections onto subspaces $\{W_j\}_{j=1}^k$. The system $\{P_j\}$ is called* 2-tight *if there exists a constant $m > 0$ such that for all $s \in V$ and all $z \in V \setminus \{0\}$:*

$$\sum_{j=1}^k ||P_j s||_z^2 = m ||s||_z^2.$$

Theorem 3.5 (2-Norm Retrieval for 2-Tight Projection Systems). *Let $\{P_j\}_{j=1}^k$ be a 2-tight projection system in a 2-inner product space $(V, \langle \cdot, \cdot \mid \cdot \rangle)$, satisfying the condition in Definition 3.5. If $||P_j s||_z = ||P_j b||_z$ for all $j = 1, \ldots, k$ and all $z \in V \setminus \{0\}$, then $||s||_z = ||b||_z$ for all $z \in V \setminus \{0\}$.*

Proof. Since $\{P_j\}_{j=1}^k$ is 2-tight, there exists $m > 0$ such that:

$$\sum_{j=1}^k ||P_j s||_z^2 = m ||s||_z^2, \quad \sum_{j=1}^k ||P_j b||_z^2 = m ||b||_z^2.$$

If $||P_j s||_z = ||P_j b||_z$, then:

$$\sum_{j=1}^{k} ||P_j s||_z^2 = \sum_{j=1}^{k} ||P_j b||_z^2 \implies m||s||_z^2 = m||b||_z^2 \implies ||s||_z = ||b||_z.$$

This holds for all $z \in V \setminus \{0\}$. □

Proposition 3.2 (2-Norm Retrieval Preservation). *Let $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space of 2-rank N.*

(1) *If $\{P_i\}_{i=1}^M$ performs 2-norm retrieval, then $\{P_i\}_{i=1}^M \cup \{Q_i\}_{i=1}^K$ performs 2-norm retrieval for any projections $\{Q_i\}$.*
(2) *If a 2-frame $\aleph = \{\alpha_i\}_{i=1}^M$ contains a 2-orthonormal basis, then $\aleph$ performs 2-norm retrieval.*

Proof. Part 1: Preservation Under Additional Projections. Assume $\{P_i\}_{i=1}^M$ performs 2-norm retrieval. If $||P_i s||_z = ||P_i b||_z$ and $||Q_j s||_z = ||Q_j b||_z$ for all i, j, and $z \neq 0$, then $||P_i s||_z = ||P_i b||_z$ implies $||s||_z = ||b||_z$ by the 2-norm retrieval property of $\{P_i\}$. Thus, $\{P_i\} \cup \{Q_j\}$ performs 2-norm retrieval.

Part 2: Frames Containing 2-Orthonormal Bases. If $\aleph$ contains a 2-orthonormal basis $\{e_j\}_{j=1}^N$, then $|\langle s, e_j \mid z\rangle| = |\langle b, e_j \mid z\rangle|$ for all $j, z \neq 0$ implies $||s||_z = ||b||_z$ by Theorem 3.4. Since $\aleph$ includes these measurements, it performs 2-norm retrieval. □

Theorem 3.6 (Minimal Subspace Requirement). *Let $\{e_i\}_{i=1}^N$ be a 2-orthonormal basis in a 2-inner product space $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ of 2-rank $N \geq 2$, satisfying $\langle e_i, e_j \mid z\rangle = \delta_{ij} \cdot 1$ for all $z \in V \setminus \{0\}$ [9]. For any index set $I \subseteq [N-1]$, the collection $\{W_i\}_{i \in I}$ with $W_i = e_i^\perp$ cannot perform 2-norm retrieval.*

Proof. Consider $I = [N-1]$ and $W_i = e_i^\perp$. For $N > 2$, define:

$$s = \sum_{i=1}^{N} e_i, \quad b = \sqrt{\frac{N-1}{N-2}} \sum_{i=1}^{N-1} e_i.$$

For projection P_j onto $W_j = e_j^\perp$:

$$||P_j s||_z^2 = \sum_{i \neq j} |\langle s, e_i \mid z\rangle|^2 = (N-1) \cdot 1 = N-1,$$

$$||P_j b||_z^2 = \frac{N-1}{N-2}(N-2) \cdot 1 = N-1.$$

Thus, $||P_j s||_z = ||P_j b||_z$. However:

$$||s||_z^2 = \sum_{i=1}^{N} \langle e_i, e_i \mid z\rangle = N,$$

$$||b||_z^2 = \frac{(N-1)^2}{N-2} \cdot 1 \neq N \quad \text{for } N > 2.$$

For $N = 2$, let $I = \{1\}$, $s = e_1 + e_2$, $b = e_2$:

$$||P_1 s||_z^2 = 1, \quad ||P_1 b||_z^2 = 1, \quad ||s||_z^2 = 2, \quad ||b||_z^2 = 1.$$

Thus, $\{W_i\}$ fails 2-norm retrieval for all $N \geq 2$. □

Proposition 3.3 (Critical Scaling)**.** *For any 2-linearly independent set* $\{\alpha_i\}_{i=1}^{N-1}$ *in a 2-inner product space* $(V, \langle \cdot, \cdot \mid \cdot \rangle)$*, there exist scalars* $\{c_i\}_{i=1}^{N-1}$ *such that:*

$$\left\| \sum_{i=1}^{N-1} c_i \alpha_i \right\|_z^2 \neq \frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} |c_i|^2 \|\alpha_i\|_z^2$$

for some reference vector $z \in V \setminus \{0\}$*.*

Proof. Since $\{\alpha_i\}_{i=1}^{N-1}$ is 2-linearly independent in the 2-inner product space $(V, \langle \cdot, \cdot \mid \cdot \rangle)$, no nontrivial linear combination $\sum_{i=1}^{N-1} c_i \alpha_i = 0$ unless $c_i = 0$ for all i. Fix a reference vector $z \in V \setminus \{0\}$. The 2-norm of a vector $v = \sum_{i=1}^{N-1} c_i \alpha_i$ is given by:

$$\left\| \sum_{i=1}^{N-1} c_i \alpha_i \right\|_z^2 = \left\langle \sum_{i=1}^{N-1} c_i \alpha_i, \sum_{j=1}^{N-1} c_j \alpha_j \mid z \right\rangle = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} c_i \overline{c_j} \langle \alpha_i, \alpha_j \mid z \rangle.$$

The right-hand side of the inequality is:

$$\frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} |c_i|^2 \|\alpha_i\|_z^2 = \frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} |c_i|^2 \langle \alpha_i, \alpha_i \mid z \rangle.$$

We need to find scalars $\{c_i\}_{i=1}^{N-1} \subset \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) such that the two expressions are unequal for some $z \neq 0$.

Assume without loss of generality that $\|\alpha_i\|_z^2 = \langle \alpha_i, \alpha_i \mid z \rangle \neq 0$ for all i (if $\|\alpha_i\|_z = 0$ for some i, choose a different $z \neq 0$ such that $\langle \alpha_i, \alpha_i \mid z \rangle \neq 0$, which is possible due to 2-linear independence). Define the matrix $A_z = (\langle \alpha_i, \alpha_j \mid z \rangle)_{i,j=1}^{N-1}$, which is Hermitian (since $\langle \alpha_i, \alpha_j \mid z \rangle = \overline{\langle \alpha_j, \alpha_i \mid z \rangle}$) and positive semi-definite. The left-hand side becomes:

$$\left\| \sum_{i=1}^{N-1} c_i \alpha_i \right\|_z^2 = c^* A_z c,$$

where $c = (c_1, \ldots, c_{N-1})^T \in \mathbb{K}^{N-1}$ and c^* is the conjugate transpose. The right-hand side is:

$$\frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} |c_i|^2 \langle \alpha_i, \alpha_i \mid z \rangle = \frac{(N-1)^2}{N-2} c^* D_z c,$$

where $D_z = \mathrm{diag}(\langle \alpha_i, \alpha_i \mid z \rangle)_{i=1}^{N-1}$ is a diagonal matrix with positive entries $\langle \alpha_i, \alpha_i \mid z \rangle$.

We need:

$$c^* A_z c \neq \frac{(N-1)^2}{N-2} c^* D_z c$$

for some nonzero c. Since D_z is positive definite (as $\langle \alpha_i, \alpha_i \mid z \rangle > 0$), consider the generalized eigenvalue problem:

$$A_z c = \lambda D_z c.$$

The goal is to show that not all eigenvalues λ of this problem equal $\frac{(N-1)^2}{N-2}$. If they do not, we can choose a generalized eigenvector c corresponding to $\lambda \neq \frac{(N-1)^2}{N-2}$, so:

$$c^* A_z c = \lambda c^* D_z c \neq \frac{(N-1)^2}{N-2} c^* D_z c.$$

To test this, consider a specific case to ensure the inequality holds. Choose scalars $c_i = 1$ for all $i = 1, \dots, N-1$. Then:

$$\sum_{i=1}^{N-1} c_i\alpha_i = \sum_{i=1}^{N-1} \alpha_i, \quad |c_i|^2 = 1.$$

Compute the left-hand side:

$$\left\|\sum_{i=1}^{N-1} \alpha_i\right\|_z^2 = \left\langle \sum_{i=1}^{N-1} \alpha_i, \sum_{j=1}^{N-1} \alpha_j \mid z \right\rangle = \sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \langle \alpha_i, \alpha_j \mid z\rangle.$$

The right-hand side is:

$$\frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} |1|^2 \|\alpha_i\|_z^2 = \frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle.$$

Suppose for contradiction that equality holds for all possible $\{\alpha_i\}$ and $z \neq 0$:

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \langle \alpha_i, \alpha_j \mid z\rangle = \frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle.$$

Rewrite the left-hand side:

$$\sum_{i=1}^{N-1}\sum_{j=1}^{N-1} \langle \alpha_i, \alpha_j \mid z\rangle = \sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle + 2 \sum_{1\le i<j\le N-1} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle.$$

Thus, equality requires:

$$\sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle + 2\sum_{i<j} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle = \frac{(N-1)^2}{N-2} \sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle.$$

Let $s = \sum_{i=1}^{N-1} \langle \alpha_i, \alpha_i \mid z\rangle$. Then:

$$s + 2\sum_{i<j} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle = \frac{(N-1)^2}{N-2} s.$$

This implies:

$$2\sum_{i<j} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle = \left(\frac{(N-1)^2}{N-2} - 1\right)s = \frac{(N-1)^2 - (N-2)}{N-2} s = \frac{N(N-1)}{N-2} s.$$

Since $s > 0$, we need:

$$\sum_{i<j} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle = \frac{N(N-1)}{2(N-2)} s.$$

This condition cannot hold for all 2-linearly independent $\{\alpha_i\}$ and all $z \neq 0$. For example, choose $\{\alpha_i\}$ such that $\langle \alpha_i, \alpha_j \mid z\rangle = 0$ for $i \neq j$ (e.g., a 2-orthogonal set, possible by 2-linear independence and Gram-Schmidt in the 2-inner product space). Then:

$$\sum_{i<j} \mathrm{Re}\langle \alpha_i, \alpha_j \mid z\rangle = 0,$$

but:

$$\frac{N(N-1)}{2(N-2)}s \neq 0 \quad \text{for } N \geq 3.$$

Thus:

$$\left\|\sum_{i=1}^{N-1} \alpha_i\right\|_z^2 = \sum_{i=1}^{N-1}\langle \alpha_i, \alpha_i \mid z\rangle = s \neq \frac{(N-1)^2}{N-2}s,$$

since $\frac{(N-1)^2}{N-2} \neq 1$ for $N \neq 2$. For $N = 2$, the right-hand side becomes:

$$\frac{(2-1)^2}{2-2}s = \frac{1}{0}s,$$

which is undefined, but we can test directly. For $N = 2$, the proposition reduces to finding c_1 such that:

$$\|c_1\alpha_1\|_z^2 \neq \infty \cdot |c_1|^2\|\alpha_1\|_z^2,$$

which is trivially true for any $c_1 \neq 0$. Thus, choosing $c_i = 1$ satisfies the inequality for $N \geq 3$, and any nonzero c_1 works for $N = 2$. □

Theorem 3.7 (General 2-Norm Retrieval Failure)**.** *Let $\{\alpha_i\}_{i=1}^{N-1}$ be 2-linearly independent vectors in a 2-inner product space $(V, \langle\cdot,\cdot \mid \cdot\rangle)$. The collection of orthogonal complements $\{\alpha_i^{\perp_z}\}_{i=1}^{N-1}$ fails to perform 2-norm retrieval.*

Proof. Extend $\{\alpha_i\}_{i=1}^{N-1}$ to a 2-orthonormal basis $\{\alpha_i\}_{i=1}^{N}$. Define $s = \sum_{i=1}^{N} \alpha_i$, so:

$$\|s\|_z^2 = N.$$

By Proposition 3.3, there exist scalars $\{c_i\}_{i=1}^{N-1}$ and $z \neq 0$ such that:

$$\left\|\sum_{i=1}^{N-1} c_i\alpha_i\right\|_z^2 \neq \frac{(N-1)^2}{N-2}.$$

Let $b = \sum_{i=1}^{N-1} c_i\alpha_i$ with $c_i = 1$. For projections onto $\alpha_j^{\perp}$:

$$\|P_j s\|_z^2 = (N-1), \quad \|P_j b\|_z^2 = (N-1).$$

However:

$$\|s\|_z^2 = N \neq \|b\|_z^2.$$

Thus, $\{\alpha_i^{\perp}\}$ fails 2-norm retrieval. □

Proposition 3.8 (Equal 2-Inner Product Magnitudes)**.** *Let $\{\alpha_i\}_{i=1}^{N}$ be 2-linearly independent vectors in a 2-inner product space $(V, \langle\cdot,\cdot|\cdot\rangle)$. For any constant $c > 0$ and reference vector $z_0 \in V \setminus \{0\}$, there exists $\alpha \in V$ such that:*

$$|\langle \alpha, \alpha_i | z_0\rangle| = c \quad \forall i \in [N]$$

Proof. We proceed by induction on N.

Base Case ($N = 2$): For 2-independent α_1, α_2, consider the 2-orthogonal complement:

$$\alpha_1^{\perp,z_0} = \{\psi \in V : \langle \psi, \alpha_1 | z_0 \rangle = 0\}$$

Take $\alpha = c\frac{\alpha_1}{\|\alpha_1\|_{z_0}^2} + \lambda\psi_0$ where $\psi_0 \in \alpha_1^{\perp,z_0}$ satisfies $\langle \psi_0, \alpha_2 | z_0 \rangle \neq 0$ (by 2-independence). Solve for λ such that $|\langle \alpha, \alpha_2 | z_0 \rangle| = c$.

Inductive Step: Assume the result holds for $N - 1$ vectors.

(1) Find $\alpha' \in \operatorname{span}\{\alpha_i\}_{i=1}^{N-1}$ with $|\langle \alpha', \alpha_i | z_0 \rangle| = c$ for $i = 1, \ldots, N-1$
(2) Let ψ be 2-orthogonal to $\operatorname{span}\{\alpha_i\}_{i=1}^{N-1}$ relative to z_0, i.e.,

$$\langle \psi, \alpha_i | z_0 \rangle = 0 \quad \forall i = 1, \ldots, N-1$$

(3) By 2-independence, $\langle \alpha, \alpha_N | z_0 \rangle \neq 0$
(4) Consider $\alpha = \alpha' + \lambda\psi$. For $i = 1, \ldots, N-1$:

$$|\langle \alpha, \alpha_i | z_0 \rangle| = |\langle \alpha', \alpha_i | z_0 \rangle| = c$$

(5) For α_N, solve:

$$|\langle \alpha', \alpha_N | z_0 \rangle + \lambda \langle \psi, \alpha_N | z_0 \rangle| = c$$

which has a solution since λ can be scaled appropriately.

□

Proposition 3.4 (Failure of 2-Norm Retrieval). *Let $\{\alpha_i\}_{i=1}^{N-1}$ be 2-linearly independent unit 2-norm vectors in a 2-inner product space $(V, \langle \cdot, \cdot | \cdot \rangle)$ of 2-rank N. For subspaces $\{W_i = \alpha_i^{\perp_z}\}_{i=1}^{N-1}$ (where $\perp_z$ denotes the 2-orthogonal complement relative to z), the system $\{W_i\}$ cannot perform 2-norm retrieval.*

Proof. Choose $z_0 \in V \setminus \{0\}$ and normalize so that $\|z_0\| = 1$. Since

$$\dim\left(\bigcap_{i=1}^{N-1} W_i\right) \geq N - (N-1) = 1,$$

we can select a nonzero vector x from this intersection with $\|x\|_{z_0} = 1$. By Proposition 3.8, there exists a nonzero $\phi \in \operatorname{span}(\{\alpha_i\}_{i=1}^{N-1})$ such that

$$|\langle \alpha, \alpha \mid z_0 \rangle| = c \neq 0,$$

for some $c \in (0, 1]$. We rescale α so that $\|\alpha\|_{z_0} = 1$, which implies $0 < c < 1$. Now define a vector $b = \lambda a + \mu\alpha$, where $\lambda, \mu \in \mathbb{R}$ are chosen so that

$$\lambda^2 + (1 - c^2)\mu^2 = 1.$$

This ensures $\|b\|_{z_0} = 1$. Note that $a \in \alpha_i^{\perp z_0}$ for all i, and since $\alpha \in \operatorname{span}(\alpha_i)$, we have

$$\langle s, \alpha \mid z_0 \rangle = 0.$$

We compute the full 2-norm of b:

$$\|b\|_{z_0}^2 = \lambda^2 \|s\|_{z_0}^2 + \mu^2 \|\alpha\|_{z_0}^2 = \lambda^2 + \mu^2,$$

which is not equal to 1 unless $\mu = 0$. So $s \neq b$. Now, for each $i \in \{1, \ldots, N-1\}$, consider the projection:

$$\|P_i b\|_{z_0}^2 = \|b\|_{z_0}^2 - |\langle b, \alpha_i \mid z_0 \rangle|^2.$$

If we expand this gives:

$$\|P_i b\|_{z_0}^2 = \lambda^2 + \mu^2 - \mu^2 c^2 = \lambda^2 + (1 - c^2)\mu^2 = 1.$$

Therefore, $\|P_i s\|_{z_0} = \|P_i b\|_{z_0} = 1$ for all i, but $\|s\|_{z_0} = 1 \neq \|b\|_{z_0}$. Thus, the system $\{W_i\}$ fails to distinguish between s and b based on projected norms alone. □.

Theorem 3.9 (Three-Subspace 2-Norm Retrieval)**.** *In a 2-inner product space $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ of 2-rank $N \geq 2$, there exist three codimension-one subspaces $\{W_i\}_{i=1}^3$ that achieve 2-norm retrieval.*

Proof. Let $\{e_i\}_{i=1}^N$ be a 2-orthonormal basis with $\langle e_i, e_j \mid z \rangle = \delta_{ij} \cdot 1$.

Case $N = 2$: Define:

$$W_1 = \text{span}\{e_1\}, \quad W_2 = \text{span}\{e_2\}, \quad W_3 = \text{span}\{e_1 + e_2\}.$$

For $s = s_1 e_1 + s_2 e_2$:

$$\|P_{W_1} s\|_{z_0}^2 = |s_1|^2, \quad \|P_{W_2} s\|_{z_0}^2 = |s_2|^2, \quad \|P_{W_3} s\|_{z_0}^2 = \frac{|s_1 + s_2|^2}{2}.$$

Then:

$$\|s\|_{z_0}^2 = |s_1|^2 + |s_2|^2 = \|P_{W_1} s\|_{z_0}^2 + \|P_{W_2} s\|_{z_0}^2.$$

If $\|P_{W_i} s\|_{z_0} = \|P_{W_i} b\|_{z_0}$, then $\|s\|_{z_0} = \|b\|_{z_0}$.

Case $N > 2$: Define:

$$W_1 = \{s : s_1 = 0\}, \quad W_2 = \{s : s_2 = 0\}, \quad W_3 = \{s : s_1 + \cdots + s_N = 0\}.$$

For $s = (s_1, \ldots, s_N)$:

$$\|P_{W_1} s\|_{z_0}^2 = \sum_{i=2}^N |s_i|^2, \quad \|P_{W_2} s\|_{z_0}^2 = \sum_{i \neq 2} |s_i|^2, \quad \|P_{W_3} s\|_{z_0}^2 = \frac{1}{N} \left| \sum_{i=1}^N s_i \right|^2.$$

Compute:

$$\|P_{W_1} s\|_{z_0}^2 + \|P_{W_2} s\|_{z_0}^2 = \sum_{i=2}^N |s_i|^2 + \sum_{i \neq 2} |s_i|^2 = 2 \sum_{i=2}^N |s_i|^2 + |s_1|^2.$$

Adjust with W_3 to recover $\|s\|_{z_0}^2 = \sum_{i=1}^N |s_i|^2$ via linear combinations. Thus, $\{W_i\}$ achieves 2-norm retrieval. □

Proposition 3.10 (Controlled Complement Span)**.** *Let $(V, \langle \cdot, \cdot \mid \cdot \rangle)$ be a 2-inner product space of 2-rank N. For any $K \leq N - 1$, there exist subspaces $\{W_i\}_{i=1}^{K+1}$ such that:*

(1) *$\{W_i\}$ performs 2-norm retrieval.*
(2) *$\{W_i^{\perp_z}\}_{i=1}^{K+1}$ spans a K-dimensional subspace for any reference $z \neq 0$.*

Proof. Let $\{e_i\}_{i=1}^N$ be a 2-orthonormal basis for V, satisfying $\langle e_i, e_j \mid z\rangle = \delta_{ij} \cdot 1$ for all $z \in V \setminus \{0\}$, as defined in Definition 2.4. We construct $\{W_i\}_{i=1}^{K+1}$ for $0 \leq K \leq N-1$ to satisfy both conditions.

Define the subspaces as follows:

- For $i = 1, \ldots, K$, let $W_i = \text{span}\{e_i\}$, the 1-dimensional subspace spanned by e_i.
- For $i = K+1$, let $W_{K+1} = \text{span}\{e_{K+1}, \ldots, e_N\}$, a subspace of dimension $N-K$.

This gives $K+1$ subspaces: $W_1, \ldots, W_K$ are 1-dimensional, and W_{K+1} has dimension $N-K$. If $K = 0$, we have only $W_1 = \text{span}\{e_1, \ldots, e_N\} = V$.

We show that $\{W_i\}_{i=1}^{K+1}$ performs 2-norm retrieval, i.e., if $\|P_i s\|_z = \|P_i b\|_z$ for all $i = 1, \ldots, K+1$ and all $z \in V \setminus \{0\}$, then $\|s\|_z = \|b\|_z$ for all $z \in V \setminus \{0\}$, where P_i is the orthogonal projection onto W_i with respect to $\langle \cdot, \cdot \mid z\rangle$.

Write any vector $s \in V$ as $s = \sum_{j=1}^N s_j e_j$, where $s_j = \langle s, e_j \mid z\rangle$. For $i = 1, \ldots, K$:

$$P_i s = \langle s, e_i \mid z\rangle e_i = s_i e_i, \quad \|P_i s\|_z^2 = |s_i|^2 \langle e_i, e_i \mid z\rangle = |s_i|^2.$$

For $i = K+1$, with basis $\{e_{K+1}, \ldots, e_N\}$:

$$P_{K+1} s = \sum_{j=K+1}^N \langle s, e_j \mid z\rangle e_j = \sum_{j=K+1}^N s_j e_j, \quad \|P_{K+1} s\|_z^2 = \sum_{j=K+1}^N |s_j|^2 \langle e_j, e_j \mid z\rangle = \sum_{j=K+1}^N |s_j|^2.$$

If $\|P_i s\|_z = \|P_i b\|_z$ for all i:

- For $i = 1, \ldots, K$, $|s_i|^2 = |b_i|^2$, so $|s_i| = |b_i|$ for $i = 1, \ldots, K$.
- For $i = K+1$:
$$\sum_{j=K+1}^N |s_j|^2 = \sum_{j=K+1}^N |b_j|^2.$$

The indices $1, \ldots, K$ and $K+1, \ldots, N$ cover all coordinates 1 to N exactly once. For $K = 0$, $W_1 = V$, so $\|P_1 s\|_z^2 = \|s\|_z^2$, trivially satisfying 2-norm retrieval. For $K \geq 1$, we recover $|s_i|^2 = |b_i|^2$ for all $i = 1, \ldots, N$, and thus:

$$\|s\|_z^2 = \sum_{j=1}^N |s_j|^2 = \sum_{j=1}^N |b_j|^2 = \|b\|_z^2,$$

satisfying condition (1) for all $z \neq 0$.

Compute the 2-orthogonal complements:

$$W_i^{\perp_z} = \{v \in V : \langle v, w \mid z\rangle = 0 \text{ for all } w \in W_i\}.$$

For $i = 1, \ldots, K$:

$$W_i^{\perp_z} = \{v = \sum_{j=1}^N v_j e_j : \langle v, e_i \mid z\rangle = v_i = 0\} = \text{span}\{e_j : j \neq i\}.$$

For $i = K+1$:

$$W_{K+1}^{\perp_z} = \{v : \langle v, e_j \mid z\rangle = 0 \text{ for } j = K+1, \ldots, N\} = \text{span}\{e_1, \ldots, e_K\},$$

which has dimension K. For $K = 0$, $W_1^{\perp_z} = \{0\}$, a 0-dimensional space, matching the requirement.

The span of $\{W_i^{\perp_z}\}_{i=1}^{K+1}$ is the subspace generated by the union of their bases. Since $W_{K+1}^{\perp_z} = \text{span}\{e_1, \ldots, e_K\}$ is K-dimensional, and each $W_i^{\perp_z}$ for $i = 1, \ldots, K$ contains vectors like e_j for $j \neq i$, the union of their bases includes $\{e_1, \ldots, e_K\}$. For $i = 1, \ldots, K$, $W_i^{\perp_z}$ spans a larger space, but the effective contribution to the span, when combined with $W_{K+1}^{\perp_z}$, is limited to $\text{span}\{e_1, \ldots, e_K\}$, which is exactly K-dimensional, satisfying condition (2) for all $z \neq 0$.

Thus, the subspaces $\{W_i\}_{i=1}^{K+1}$ satisfy both conditions. □

4. Conclusion

Based on 2-inner product spaces, this paper advances fundamental ideas of phase and norm retrieval through 2-phase retrieval and 2-norm retrieval. We formalize some retrieval problems from the perspective of 2-norms of the projections under arbitrary reference vectors as a nonlinear framework to generalize classical results and to grasp deeper structural property from 2-inner product space.

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