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Certain Applications of (ψ, ϕ)–Contractions in C^*-Algebra-Valued S_b-Metric Spaces

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Abstract. We present in this paper unique common coupled fixed-point results for two pairs of ω-compatible mappings that satisfy (ψ, ϕ)-generalized weakly contractive conditions in C^*-algebra-valued S_b-metric spaces. Additionally, we provide an illustration to substantiate our findings. Additionally, the paper offers an application that demonstrates the existence and uniqueness of a solution for a non-linear integral equation, as well as homotopy.

1. Introduction

Stefan Banach first proposed the concept of contraction in 1922 and established the well-known Banach contraction theorem. The Banach Principle of Contraction [1] on metric spaces is of essential relevance in the fields of fixed points and nonlinear analysis, mathematical physics, and applied sciences. Literature has produced new results relating to proving the generalisation of metric space and obtaining a refinement of the contractive condition.

The concept of coupled fixed points was initially introduced by Guo and Lakshmikantham [2] in 1987. Subsequently, utilizing a weak contractivity assumption, Bhaskar and Lakshmikantham [3] formulated an innovative fixed point theorem for a mixed monotone mapping within a metric space governed by partial ordering. In 1998, Jungck and Rhoades [4] proposed the concept of weak compatibility, establishing that compatible mappings are weakly compatible, although the

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converse does not hold. Refer to the study results in ([5]- [8]) and associated references for further findings on coupled fixed point outcomes.

The concept of C^*-algebra-valued metric spaces was first introduced by Ma et al. in the year 2014 [9]. In the year 2015, they presented the concept of C^*-algebra-valued b-metric spaces and analyzed a few findings [10]. Additionally, Razavi and Masiha conducted research on C^*-algebra-valued b-metric spaces [11] in order to comprehend the prevailing concepts.

The authors Sedghi et al. [12] constructed S_b-metric spaces by integrating the notions of S and b-metric spaces. They also demonstrated that these spaces exhibit common fixed point results. Many writers, in an effort to make improvements, have developed a large number of results on S_b-metric spaces (for example, [13]- [19]).

Drawing upon the contributions of Souayah and Mlaiki as referenced in [13], Razavi and colleagues introduced the concept of C^*-algebra-valued S_b-metric space in 2023 [20], and subsequently established several common fixed point results within this framework [21].

In C^*-algebra-valued S_b-metric spaces, this work seeks to offer linked fixed point findings for two sets of ω-compatible mappings that meet (ψ, ϕ)-generalized weakly contractive requirements. Additionally, we are able to give appropriate and pertinent instances pertaining to homotopy and integral equations. First we recall some basic results.

2. Preliminaries

This section offers a concise overview of certain aspects related to the theory of C^*-algebras [22]. Let us consider the scenario where $\mathfrak{A}$ represents a unital C^* algebra, characterized by the presence of the unit element $1_{\mathfrak{A}}$. Define the set $\mathfrak{A}_h$ as follows: $\mathfrak{A}_h = \{s \in \mathfrak{A} : s = s^*\}$. An element $s \in \mathfrak{A}$ is classified as positive, denoted by $s \succeq 0_{\mathfrak{A}}$, if and only if it satisfies two conditions: first, s must equal its adjoint, s^*, and second, the spectrum of s, denoted $\sigma(s)$, must be contained within the interval $[0, \infty)$. Here, $0_{\mathfrak{A}}$ represents the zero element in the algebra $\mathfrak{A}$. On $\mathfrak{A}_h$, a natural partial ordering can be identified, where $\ell \preceq \wp$ holds if and only if $\wp - \ell \succeq 0_{\mathfrak{A}}$. We define $\mathfrak{A}_+ = \{s \in \mathfrak{A} : s \succeq 0_{\mathfrak{A}}\}$ and $\mathfrak{A}' = \{s \in \mathfrak{A} : st = ts\ \forall t \in \mathfrak{A}\}$.

Definition 2.1. *([20])*

Let $\mathcal{G}$ be a non-empty set and $\kappa \in \mathfrak{A}'$ such that $\|\kappa\| \geq 1$. Let $S_b : \mathcal{G}^3 \to \mathfrak{A}$ be a function that satisfies the following properties:

(S_{b_1}) $S_b(æ, œ, ß) \succeq 0_{\mathfrak{A}}$ *for all* $æ, œ, ß \in \mathcal{G}$,

(S_{b_2}) $S_b(æ, œ, ß) = 0_{\mathfrak{A}} \Leftrightarrow æ = œ = ß$,

(S_{b_3}) $S_b(æ, œ, ß) \preceq \kappa(S_b(æ, æ, \theta) + S_b(œ, œ, \theta) + S_b(ß, ß, \theta))$ *for all* $æ, œ, ß, \theta \in \mathcal{G}$.

The combination $(\mathcal{G}, \mathfrak{A}, S_b)$ is then referred to as a C^*-algebra-valued S_b-metric space (C^*-AV-S_bMS) with a coefficient κ, and the function S_b is referred to as a C^*-algebra-valued S_b-metric on $\mathcal{G}$.

Definition 2.2. *([20]) A C^*-AV-S_bMS is symmetric if $S_b(æ, æ, œ) = S_b(œ, œ, æ)$*
$\forall\ æ, œ \in \mathcal{G}$

Definition 2.3. *([20]) Let $(\mathcal{G}, \mathfrak{A}, S_b)$ represent a C*-AV-S_bMS, and let $\{\chi_p\}$ denote a sequence in $\mathcal{G}$.*

(1) *If for all $n \in \mathbb{N}$, $\|S_b(\chi_{p+n}, \chi_{p+n}, \chi_n)\| \to 0$, where $p \to \infty$, then $\{\chi_p\}$ is a Cauchy sequence in $\mathcal{G}$.*
(2) *If $\|S_b(\chi_n, \chi_p, \chi)\| \to 0$, where $p \to \infty$, then $\{\chi_p\}$ converges to χ, and we present it with $\lim_{p\to\infty} \chi_p = \chi$.*
(3) *If every Cauchy sequence converges in $\mathcal{G}$, then the structure $(\mathcal{G}, \mathfrak{A}, S_b)$ qualifies as a complete C*-AV-S_bMS.*

Definition 2.4. *([20])*

Let $(\mathcal{G}_1, \mathfrak{A}_1, S_{b1})$ and $(\mathcal{G}_2, \mathfrak{A}_2, S_{b2})$ represent two C-AV-S_bMS structures. Define a function Γ that maps $(\mathcal{G}_1, \mathfrak{A}_1, S_{b1})$ to $(\mathcal{G}_2, \mathfrak{A}_2, S_{b2})$. Then, Γ is continuous at a point $\chi \in \mathcal{G}_1$ if, for every sequence, $\{\chi_n\}$ in $\mathcal{G}_1$, it holds that $S_b(\chi_n, \chi_n, \chi) \to 0_{\mathfrak{A}}$ as $n \to \infty$ implies $S_b(\Gamma(\chi_n), \Gamma(\chi_n), \gamma(\chi)) \to 0_{\mathfrak{A}'}$ as n approaches infinity. A function Γ is continuous at $\mathcal{G}_1$ if and only if it is continuous at every $\chi \in \mathcal{G}_1$.*

Lemma 2.5. *([23]) Suppose that $\mathfrak{A}$ is a unital C*-algebra with a unit $1_{\mathfrak{A}}$:*

(1) *If $\{\xi_p\}_{p=1}^{\infty} \subseteq \mathfrak{A}$ and $\lim_{p\to\infty} \xi_p = 0_{\mathfrak{A}}$, then for any $\xi \in \mathfrak{A}$, $\lim_{p\to\infty} \xi^* \xi_n \xi = 0_{\mathfrak{A}}$*
(2) *If $\Xi, \xi \in \mathfrak{A}_h$ and $s \in \mathfrak{A}'_+$ then $\xi \preceq \Xi$ yields $s\xi \preceq s\Xi$ in which $\mathfrak{A}'_+ = \mathfrak{A}_+ \cap \mathfrak{A}'$.*
(3) *If $\xi \in \mathfrak{A}_+$ with $\|\xi\| < \frac{1}{2}$ then $1_{\mathfrak{A}} - \xi$ is invertible,and $\|\xi(1_{\mathfrak{A}} - \xi)^{-1}\| < 1$.*
(4) *If $\xi, \Xi \in \mathfrak{A}_+$ such that $\xi\Xi = \Xi\xi$, then $\xi\Xi \succeq 0_{\mathfrak{A}}$.*

3. Main Results

This section begins with the introduction of (ψ, ϕ)-generalized weakly contraction, followed by the proof of our main result.

Definition 3.1. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a C*-AV-S_bMS with coefficient $\|\kappa\| > 1$. Let $\Omega : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ be a mapping, an element $(æ, œ) \in \mathcal{G}^2$ is called coupled fixed point of Ω if $\Omega(æ, œ) = æ$ and $\Omega(œ, æ) = œ$*

Definition 3.2. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a C*-AV-S_bMS with coefficient $\|\kappa\| > 1$ and suppose $\Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda : \mathcal{G} \to \mathcal{G}$ be two mappings:*

(a) *An element $(æ, œ)$ is said to be a coupled coincident point of Ω and Λ if $\Omega(æ, œ) = \Lambdaæ$, $\Omega(œ, æ) = \Lambdaœ$,*
(b) *An element $(æ, œ)$ is said to be a common coupled fixed point of Ω and Λ if $\Omega(æ, œ) = \Lambdaæ = æ$, $\Omega(œ, æ) = \Lambdaœ = œ$,*
(c) *A pair (Ω, Λ) is called weakly compatible if $\Lambda(\Omega(æ, œ)) = \Omega(\Lambdaæ, \Lambdaœ)$ and $\Lambda(\Omega(œ, æ)) = \Omega(\Lambdaœ, \Lambdaæ)$ whenever for all $æ, œ \in \mathcal{G}$ such that*
$\Omega(æ, œ) = \Lambdaæ$, $\Omega(œ, æ) = \Lambdaœ$.

In this manuscript we indicate:

$$(i) \qquad \Psi = \left\{ \begin{array}{c} \psi : \mathfrak{A}_+ \to \mathfrak{A}_+ / \psi \text{ is monotonically non-decreasing, continuous and} \\ \psi(a) = 0_{\mathfrak{A}} \iff a = 0_{\mathfrak{A}} \end{array} \right\}$$

$$(ii) \qquad \Phi = \{\phi : \mathfrak{A}_+ \to \mathfrak{A}_+ / \phi \text{ is lower semi-continuous and } \phi(a) = 0_{\mathfrak{A}} \iff a = 0_{\mathfrak{A}}\}$$

Definition 3.3. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a C^*-AV-S_bMS with coefficient $\|\kappa\| > 1$, $\Gamma, \Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda, \Theta : \mathcal{G} \to \mathcal{G}$ be four mappings. Then we say that $\Gamma, \Omega, \Lambda, \Theta$ be a (ψ, ϕ)-generalized weakly contractive mappings if there exists $\xi \geq 0_{\mathfrak{A}}$ and $\psi \in \Psi$, $\phi \in \Phi$ and $a \in \mathfrak{A}$ in which $\|\sqrt{2}a\| < 1$ such that for all $\ell, \varkappa, æ, œ \in \mathcal{G}$, we have*

$$\psi\left(2\kappa S_b\left(\Gamma(\ell,\varkappa), \Gamma(\ell,\varkappa), \Omega(æ,œ)\right)\right) \leq \psi\left(a^*\mathbb{M}(\ell,\varkappa,æ,œ)a\right) - \phi\left(a^*\mathbb{M}(\ell,\varkappa,æ,œ)a\right) + \xi\mathbb{N}\left(\ell,\varkappa,æ,œ\right) \tag{3.1}$$

where $\mathbb{M}\left(\ell,\varkappa,æ,œ\right) = \max\left\{\begin{array}{c} S_b\left(\Lambda\ell,\Lambda\ell,\Theta æ\right), S_b\left(\Lambda\varkappa,\Lambda\varkappa,\Theta œ\right), \\ S_b\left(\Lambda\ell,\Lambda\ell,\Gamma(\ell,\varkappa)\right), S_b\left(\Theta æ,\Theta æ,\Omega(æ,œ)\right), \\ S_b\left(\Lambda\varkappa,\Lambda\varkappa,\Gamma(\varkappa,\ell)\right), S_b\left(\Theta œ,\Theta œ,\Omega(œ,æ)\right), \\ \frac{S_b(\Lambda\ell,\Lambda\ell,\Omega(æ,œ))+S_b(\Theta æ,\Theta æ,\Gamma(\ell,\varkappa))}{2\kappa^4}, \\ \frac{S_b(\Lambda\varkappa,\Lambda\varkappa,\Omega(œ,æ))+S_b(\Theta œ,\Theta œ,\Gamma(\varkappa,\ell))}{2\kappa^4} \end{array}\right\}$

and $\mathbb{N}\left(\ell,\varkappa,æ,œ\right) = \min\left\{\begin{array}{c} S_b\left(\Lambda\ell,\Lambda\ell,\Gamma(\ell,\varkappa)\right), S_b\left(\Theta æ,\Theta æ,\Omega(æ,œ)\right), S_b\left(\Lambda\ell,\Lambda\ell,\Omega(æ,œ)\right), \\ S_b\left(\Lambda\varkappa,\Lambda\varkappa,\Omega(œ,æ)\right), S_b\left(\Theta æ,\Theta æ,\Gamma(\ell,\varkappa)\right), S_b\left(\Theta œ,\Theta œ,\Gamma(\varkappa,\ell)\right) \end{array}\right\}$

We start our work by proving the following one crucial Lemma.

Lemma 3.4. *If $(\mathcal{G}, \mathfrak{A}, S_b)$ be a C^*-AV-S_bMS with $\|\kappa\| \geq 1$ and suppose that $\{æ_p\}$ is a C^*-AV-S_b-convergent to ℓ, then we have*
$\frac{1}{2\kappa}S_b(\wp,\wp,\ell) \leq \lim_{p\to\infty}\inf\ S_b(\wp,\wp,æ_p) \leq \lim_{n\to\infty}\sup S_b(\wp,\wp,æ_p) \leq 2\kappa S_b(\wp,\wp,\ell)$
for all $\wp \in \mathcal{G}$. In particular, if $\ell = \wp$, then we have $\lim_{p\to\infty} S_b(\ell_p,\ell_p,\wp) = 0_{\mathfrak{A}}$.

Proof. using condition (S_{b_3}) of Definition-2.1, we have
$S_b(\wp,\wp,æ_p) \leq 2\kappa S_b(\wp,\wp,\ell) + \kappa S_b(æ_p,æ_p,\ell)$ and
$S_b(\wp,\wp,\ell) \leq 2\kappa S_b(\wp,\wp,æ_p) + \kappa S_b(\ell,\ell,æ_p)$.

Taking the upper limit as $p \to \infty$ in the first inequality and the lower limit as $p \to \infty$ in the second inequality we obtain the desired result. □

Theorem 3.5. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a complete C^*-AV-S_bMS, suppose $\Gamma, \Omega : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda, \Theta : \mathcal{G} \to \mathcal{G}$ be four mappings with the following assumptions:*

- (i) *$\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ and $\Omega(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$;*
- (ii) *$\Gamma, \Omega, \Lambda, \Theta$ are (ψ, ϕ)- generalized weakly contractive mappings;*
- (iii) *$\{\Gamma, \Lambda\}$ and $\{\Omega, \Theta\}$ are ω-compatible pairs;*
- (iv) *one of $\Lambda(\mathcal{G})$ or $\Omega(\mathcal{G})$ is complete subspaces of $(\mathcal{G}, \mathfrak{A}, S_b)$.*

Then Γ, Ω, Λ and Θ have a unique common coupled fixed point in $\mathcal{G}$.

Proof. Let $æ_0, œ_0 \in \mathcal{G}$ be arbitrary, and from (i), we construct the sequences $\left\{æ_p\right\}, \left\{œ_p\right\}$, in $\mathcal{G}$ as

$$\Gamma\left(æ_{2p}, œ_{2p}\right) = \Theta æ_{2p+1} = \ell_{2p}, \quad \Gamma\left(œ_{2p}, æ_{2p}\right) = \Theta œ_{2p+1} = \wp_{2p},$$
$$\Omega\left(æ_{2p+1}, œ_{2p+1}\right) = \Lambda æ_{2p+2} = \ell_{2p+1}, \quad \Omega\left(œ_{2p+1}, æ_{2p+1}\right) = \Lambda œ_{2p+2} = \wp_{2p+1}, \text{ where } p = 0,1,2,\ldots.$$

Notices that in C^*-algebra, if $a, b \in \mathfrak{A}$ and $a \preceq b$, then for any $\jmath \in \mathfrak{A}_+$ both $\jmath^* a \jmath$ and $\jmath^* b \jmath$ are positive elements and $\jmath^* a \jmath \preceq \jmath^* b \jmath$.

Now we show that Γ, Ω, Λ and Θ have common coupled fixed point in $\mathcal{G}$.

By using (3.1), for each $p \in \mathbb{N}$, we have

$$\begin{aligned}
\psi\left(2\kappa S_b(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p+2})\right) &= \psi\left(2\kappa S_b(\Gamma(æ_{2p+1}, œ_{2p+1}), \Gamma(æ_{2p+1}, œ_{2p+1}), \Omega(æ_{2p+2}, œ_{2p+2}))\right) \\
&\preceq \psi\left(a^* \mathbb{M}(æ_{2p+1}, œ_{2p+1}, æ_{2p+2}, œ_{2p+2}) a\right) \\
&\quad -\phi\left(a^* \mathbb{M}(æ_{2p+1}, œ_{2p+1}, æ_{2p+2}, œ_{2p+2}) a\right) \\
&\quad +\xi \mathbb{N}\left(æ_{2p+1}, œ_{2p+1}, æ_{2p+2}, œ_{2p+2}\right)
\end{aligned} \tag{3.2}$$

Now, by simple computations, we have

$$\begin{aligned}
&\mathbb{M}\left(æ_{2p+1}, œ_{2p+1}, æ_{2p+2}, œ_{2p+2}\right) \\
= &\max\left\{\begin{array}{c}
S_b\left(\Lambda æ_{2p+1}, \Lambda æ_{2p+1}, \Theta æ_{2p+2}\right), \\
S_b\left(\Lambda œ_{2p+1}, \Lambda œ_{2p+1}, \Theta œ_{2p+2}\right), \\
S_b\left(\Lambda æ_{2p+1}, \Lambda æ_{2p+1}, \Gamma(æ_{2p+1}, œ_{2p+1})\right), \\
S_b\left(\Theta æ_{2p+2}, \Theta æ_{2p+2}, \Omega(æ_{2p+2}, œ_{2p+2})\right), \\
S_b\left(\Lambda œ_{2p+1}, \Lambda œ_{2p+1}, \Gamma(œ_{2p+1}, æ_{2p+1})\right), \\
S_b\left(\Theta œ_{2p+2}, \Theta œ_{2p+2}, \Omega(œ_{2p+2}, æ_{2p+2})\right), \\
\frac{S_b\left(\Lambda æ_{2p+1}, \Lambda æ_{2p+1}, \Omega(æ_{2p+2}, œ_{2p+2})\right) + S_b\left(\Theta æ_{2p+2}, \Theta æ_{2p+2}, \Gamma(æ_{2p+1}, œ_{2p+1})\right)}{2\kappa^4}, \\
\frac{S_b\left(\Lambda œ_{2p+1}, \Lambda œ_{2p+1}, \Omega(œ_{2p+2}, æ_{2p+2})\right) + S_b\left(\Theta œ_{2p+2}, \Theta œ_{2p+2}, \Gamma(œ_{2p+1}, æ_{2p+1})\right)}{2\kappa^4}
\end{array}\right\} \\
= &\max\left\{\begin{array}{c}
S_b\left(\ell_{2p}, \ell_{2p}, \ell_{2p+1}\right), S_b\left(\wp_{2p}, \wp_{2p}, \wp_{2p+1}\right), \\
S_b\left(\ell_{2p}, \ell_{2p}, \ell_{2p+1}\right), S_b\left(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p+2}\right), \\
S_b\left(\wp_{2p}, \wp_{2p}, \wp_{2p+1}\right), S_b\left(\wp_{2p+1}, \wp_{2p+1}, \wp_{2p+2}\right), \\
\frac{S_b\left(\ell_{2p}, \ell_{2p}, \ell_{2p+2}\right) + S_b\left(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p+1}\right)}{2\kappa^4}, \\
\frac{S_b\left(\wp_{2p}, \wp_{2p}, \wp_{2p+2}\right) + S_b\left(\wp_{2p+1}, \wp_{2p+1}, \wp_{2p+1}\right)}{2\kappa^4}
\end{array}\right\} \\
= &\max\left\{\begin{array}{c}
S_b\left(\ell_{2p}, \ell_{2p}, \ell_{2p+1}\right), S_b\left(\wp_{2p}, \wp_{2p}, \wp_{2p+1}\right), \\
S_b\left(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p+2}\right), S_b\left(\wp_{2p+1}, \wp_{2p+1}, \wp_{2p+2}\right), \\
\frac{S_b\left(\ell_{2p}, \ell_{2p}, \ell_{2p+2}\right) + S_b\left(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p+1}\right)}{2\kappa^4}, \\
\frac{S_b\left(\wp_{2p}, \wp_{2p}, \wp_{2p+2}\right) + S_b\left(\wp_{2p+1}, \wp_{2p+1}, \wp_{2p+1}\right)}{2\kappa^4}
\end{array}\right\}
\end{aligned}$$

and

$$\mathbb{N}\left(æ_{2p+1}, œ_{2p+1}, æ_{2p+2}, œ_{2p+2}\right) = \min\left\{\begin{array}{c}
S_b\left(\Lambda æ_{2p+1}, \Lambda æ_{2p+1}, \Gamma(æ_{2p+1}, œ_{2p+1})\right), \\
S_b\left(\Theta æ_{2p+2}, \Theta æ_{2p+2}, \Omega(æ_{2p+2}, œ_{2p+2})\right), \\
S_b\left(\Lambda æ_{2p+1}, \Lambda æ_{2p+1}, \Omega(æ_{2p+2}, œ_{2p+2})\right), \\
S_b\left(\Lambda œ_{2p+1}, \Lambda œ_{2p+1}, \Omega(œ_{2p+2}, æ_{2p+2})\right), \\
S_b\left(\Theta æ_{2p+2}, \Theta æ_{2p+2}, \Gamma(æ_{2p+1}, œ2p+1)\right), \\
S_b\left(\Theta œ_{2p+2}, \Theta œ_{2p+2}, \Gamma(œ_{2p+1}, æ_{2p+1})\right)
\end{array}\right\}$$

$$
\begin{aligned}
&= \min\left\{\begin{array}{c} S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right), S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right), \\ S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+2}\right), S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+2}\right), \\ S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+1}\right), S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+1}\right) \end{array}\right\} \\
&= 0_{\mathfrak{A}}.
\end{aligned}
$$

From (3.2), we have

$$
\begin{aligned}
\psi\left(2\kappa S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2})\right) &\preceq \psi\left(a^*\max\left\{\begin{array}{c} S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right), \\ S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right), \\ S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right), \\ S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right), \\ \frac{S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+2}\right)+S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+1}\right)}{2\kappa^4}, \\ \frac{S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+2}\right)+S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+1}\right)}{2\kappa^4} \end{array}\right\}a\right) \\
&\quad -\phi\left(a^*\max\left\{\begin{array}{c} S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right), \\ S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right), \\ S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right), \\ S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right), \\ \frac{S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+2}\right)+S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+1}\right)}{2\kappa^4}, \\ \frac{S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+2}\right)+S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+1}\right)}{2\kappa^4} \end{array}\right\}a\right)
\end{aligned}
$$

Notice that

$$
\begin{aligned}
\frac{S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+2}\right)+S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+1}\right)}{2\kappa^4} &\preceq \frac{2\kappa S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right)+\kappa S_b\left(\ell_{2p+2},\ell_{2p+2},\ell_{2p+1}\right)}{2\kappa^4} \\
&\preceq \max\left\{\begin{array}{c} S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right), \\ S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right) \end{array}\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+2}\right)+S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+1}\right)}{2\kappa^4} &\preceq \frac{2\kappa S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right)+\kappa S_b\left(\wp_{2p+2},\wp_{2p+2},\wp_{2p+1}\right)}{2\kappa^4} \\
&\preceq \max\left\{\begin{array}{c} S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right), \\ S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right) \end{array}\right\}
\end{aligned}
$$

Therefore, we have

$$
\psi\left(2\kappa S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2})\right) \preceq \psi\left(a^*\max\left\{\begin{array}{c} S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right), \\ S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right), \\ S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right), \\ S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right) \end{array}\right\}a\right)
$$

By the definition of ψ, we have that

$$S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right) \preceq a^* \max\left\{\begin{array}{c}\frac{1}{2\kappa}S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right),\frac{1}{2\kappa}S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right),\\ \frac{1}{2\kappa}S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right),\frac{1}{2\kappa}S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)\end{array}\right\}a$$

If for some $p\in N$, $\frac{1}{2\kappa^4}S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right)\prec\frac{1}{2\kappa}S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right)$ and $\frac{1}{2\kappa}S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right)\prec\frac{1}{2\kappa^4}S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)$, then we have

$$S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2})\preceq a^*\max\left\{\begin{array}{c}\frac{1}{2\kappa}S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right),\\ \frac{1}{2\kappa}S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)\end{array}\right\}a. \tag{3.3}$$

Similarly, we can prove that

$$S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right) \preceq a^*\max\left\{\begin{array}{c}\frac{1}{2\kappa}S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right),\\ \frac{1}{2\kappa}S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)\end{array}\right\}a \tag{3.4}$$

Combining (3.3) and (3.4), we can get

$$\begin{aligned}\max\left\{\begin{array}{c}\|S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2})\|,\\ \|S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2})\|\end{array}\right\} &\leq \frac{1}{2\kappa}\|a\|^2\max\left\{\begin{array}{c}\|S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right)\|,\\ \|S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)\|\end{array}\right\}\\ &< \frac{1}{2\kappa}\max\left\{\begin{array}{c}\|S_b\left(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}\right)\|,\\ \|S_b\left(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2}\right)\|\end{array}\right\}.\end{aligned}$$

we get a contradiction. Hence for all $p\in\mathbb{N}$, we have

$$\max\left\{\begin{array}{c}S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}),\\ S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2})\end{array}\right\} \preceq \frac{1}{2\kappa}a^*\max\left\{\begin{array}{c}S_b\left(\ell_{2p},\ell_{2p},\ell_{2p+1}\right),\\ S_b\left(\wp_{2p},\wp_{2p},\wp_{2p+1}\right)\end{array}\right\}a.$$

Let $\Im_{2p+1}=\max\left\{\begin{array}{c}S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2}),\\ S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2})\end{array}\right\}$. Now by above inequality, we have

$$\begin{aligned}\Im_{2p+1} &\preceq \frac{1}{2\kappa}a^*\Im_{2p}a\\ &\preceq \frac{1}{(2\kappa)^2}(a^*)^2\Im_{2p-1}(a)^2\\ &\vdots\\ &\preceq \frac{1}{(2\kappa)^{2p+1}}(a^*)^{2p+1}\Im_0(a)^{2p+1}.\end{aligned}$$

Thus,

$$S_b(\ell_{2p+1},\ell_{2p+1},\ell_{2p+2})\preceq\frac{1}{(2\kappa)^{2p+1}}(a^*)^{2p+1}\Im_0(a)^{2p+1},$$

and

$$S_b(\wp_{2p+1},\wp_{2p+1},\wp_{2p+2})\preceq\frac{1}{(2\kappa)^{2p+1}}(a^*)^{2p+1}\Im_0(a)^{2p+1}$$

Now, we can obtain for any $p \in \mathbb{N}$

$$\begin{aligned}
\mathfrak{I}_p = \max\left\{ \begin{array}{c} S_b(\ell_{2p}, \ell_{2p}, \ell_{2p+1}), \\ S_b(\wp_{2p}, \wp_{2p}, \wp_{2p+1}) \end{array} \right\} &\preceq \frac{1}{2\kappa} a^* \mathfrak{I}_{p-1} a \\
&\preceq \frac{1}{(2\kappa)^2} (a^*)^2 \mathfrak{I}_{p-2} (a)^2 \\
&\vdots \\
&\preceq \frac{1}{(2\kappa)^p} (a^*)^p \mathfrak{I}_0 (a)^p.
\end{aligned}$$

Thus,

$$S_b(\ell_{2p}, \ell_{2p}, \ell_{2p+1}) \preceq \frac{1}{(2\kappa)^p} (a^*)^p \mathfrak{I}_0 (a)^p,$$

and

$$S_b(\wp_{2p}, \wp_{2p}, \wp_{2p+1}) \preceq \frac{1}{(2\kappa)^p} (a^*)^p \mathfrak{I}_0 (a)^p.$$

If $\mathfrak{I}_0 = 0_{\mathfrak{A}}$, then from (S_{b_2}) of Definition-2.1, we know $(\ell_0, \wp_0)$ is a coupled fixed point of Γ, Ω, Λ and Θ. Now letting $\mathfrak{I}_0 \succ 0_{\mathfrak{A}}$, we get for any $p \in \mathbb{N}$, for any $l \in \mathbb{N}$ and using condition (S_{b_3}) of Definition-2.1,

$$\begin{aligned}
S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p}) &\preceq \kappa \left(\begin{array}{c} S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p+l-1}) + S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p+l-1}) \\ + S_b(\ell_{2p}, \ell_{2p}, \ell_{2p+l-1}) \end{array} \right) \\
&\preceq 2\kappa S_b(\ell_{p+l}, \ell_{p+l}, \ell_{p+l-1}) + \kappa S_b(\ell_p, \ell_p, \ell_{p+l-1}) \\
&= 2\kappa S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p+l-1}) + \kappa S_b(\ell_{2p+l-1}, \ell_{2p+l-1}, \ell_{2p}) \\
&\preceq 2\kappa S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p+l-1}) + 2\kappa^2 S_b(\ell_{2p+l-1}, \ell_{2p+l-1}, \ell_{2p+l-2}) \\
&\quad + \kappa^2 S_b(\ell_{2p+l-2}, \ell_{2p+l-2}, \ell_{2p}) \\
&\vdots \\
&\preceq 2\kappa S_b(\ell_{2p+l}, \ell_{2p+l}, \ell_{2p+l-1}) + 2\kappa^2 S_b(\ell_{2p+l-1}, \ell_{2p+l-1}, \ell_{2p+l-2}) \\
&\quad + 2\kappa^3 S_b(\ell_{2p+l-2}, \ell_{2p+l-2}, \ell_{2p+l-3}) + \ldots + 2\kappa^l S_b(\ell_{2p+1}, \ell_{2p+1}, \ell_{2p}) \\
&\preceq 2 \sum_{m=2p}^{2p+l-1} \frac{\kappa^{l-m+2p}}{(2\kappa)^m} (a^*)^m \mathfrak{I}_0 (a)^m \\
&= 2 \sum_{m=2p}^{2p+l-1} \left((a^*)^m \Big(\frac{\kappa^{l-m+2p}}{(2\kappa)^m} \Big)^{\frac{1}{2}} \mathfrak{I}_0^{\frac{1}{2}} \right) \left(\mathfrak{I}_0^{\frac{1}{2}} \Big(\frac{\kappa^{l-m+2p}}{(2\kappa)^m} \Big)^{\frac{1}{2}} (a)^m \right) \\
&= 2 \sum_{m=2p}^{2p+l-1} \left(a^m \Big(\frac{\kappa^{l-m+2p}}{(2\kappa)^m} \Big)^{\frac{1}{2}} \mathfrak{I}_0^{\frac{1}{2}} \right)^* \left(\mathfrak{I}_0^{\frac{1}{2}} \Big(\frac{\kappa^{l-m+2p}}{(2\kappa)^m} \Big)^{\frac{1}{2}} (a)^m \right) \\
&\preceq 2 \sum_{m=2p}^{2p+l-1} \| \mathfrak{I}_0^{\frac{1}{2}} \Big(\frac{\kappa^{l-m+2p}}{(2\kappa)^m} \Big)^{\frac{1}{2}} (a)^m \|^2 1_{\mathfrak{A}}
\end{aligned}$$

$$\begin{aligned} &\leq 2\|\mathfrak{I}_0\| \sum_{m=2p}^{2p+l-1} \|a\|^m \| \frac{\kappa^{l-m+2p}}{(2\kappa)^m} \| 1_{\mathfrak{A}} \\ &\leq 2\|\mathfrak{I}_0\| \frac{\|\kappa\|^{l+1}\|a\|^{2p}}{\|2\kappa\|^{p-1}(\|2\kappa^2\| - \|a\|)} 1_{\mathfrak{A}} \to 0 \text{ as } p \to \infty. \end{aligned}$$

in which $1_{\mathfrak{A}}$ is the unit element in $\mathfrak{A}$. As $\{\ell_{2p}\}$ is a Cauchy sequence in $\mathcal{G}$ with respect to $\mathfrak{A}$. By similar arguments, we obtain $\{\wp_{2p}\}$ is an C^*-AV-S_b-Cauchy sequence in $\mathcal{G}$. Suppose $\Lambda(\mathcal{G})$ is complete subspace of $(\mathcal{G}, \mathfrak{A}, S_b)$, then the sequences $\{\ell_p\}$ and $\{\wp_p\}$ are converge to æ and œ respectively $\Lambda(\mathcal{G})$. Thus there exist $\ell, \wp \in \Lambda(\mathcal{G})$ such that

$$\lim_{p\to\infty} \ell_{2p} = \lim_{p\to\infty} \Lambda æ_{2p+1} = æ = \Lambda\ell \quad \text{ and } \lim_{p\to\infty} \wp_{2p} = \lim_{p\to\infty} \Lambda œ_{2p+1} = œ = \Lambda\wp \tag{3.5}$$

Now we show that $\Gamma(\ell, \wp) = æ$ and $\Gamma(\wp, \ell) = œ$.

Suppose $\Gamma(\ell, \wp) \neq æ$ and $\Gamma(\wp, \ell) \neq œ$ by Lemma (3.4), we have

$$\tfrac{1}{2\kappa} S_b\left(\Gamma(\ell, \wp), \Gamma(\ell, \wp), æ\right) \leq \lim_{p\to\infty} \inf S_b\left(\Gamma(\ell, \wp), \Gamma(\ell, \wp), \ell_{2p}\right).$$

Now from (3.1) and applying ψ on both sides, we have

$$\begin{aligned} \psi\left(S_b\left(\Gamma(\ell, \wp), \Gamma(\ell, \wp), æ\right)\right) &\leq \lim_{p\to\infty} \inf \psi\left(2\kappa S_b\left(\Gamma(\ell, \wp), \Gamma(\ell, \wp), \ell_{2p}\right)\right) \\ &= \lim_{p\to\infty} \inf \psi\left(2\kappa S_b\left(\Gamma(\ell, \wp), \Gamma(\ell, \wp), \Gamma(æ_{2p}, œ_{2p})\right)\right) \\ &\leq \lim_{p\to\infty} \sup \psi\left(a^* \mathbb{M}(\ell, \wp, æ_{2p}, œ_{2p}) a\right) \\ &\quad - \lim_{p\to\infty} \sup \phi\left(a^* \mathbb{M}(\ell, \wp, æ_{2p}, œ_{2p}) a\right) \\ &\quad + \lim_{p\to\infty} \sup \xi \mathbb{N}\left(\ell, \wp, æ_{2p}, œ_{2p}\right) \end{aligned} \tag{3.6}$$

Now, by simple computations, we have

$$\begin{aligned} \lim_{p\to\infty} \sup \mathbb{M}\left(\ell, \wp, æ_{2p}, œ_{2p}\right) &= \lim_{p\to\infty} \sup \max \left\{ \begin{array}{c} S_b\left(\Lambda\ell, \Lambda\ell, \Theta æ_{2p}\right), \\ S_b\left(\Lambda\wp, \Lambda\wp, \Theta œ_{2p}\right), \\ S_b\left(\Lambda\ell, \Lambda\ell, \Gamma(\ell, \wp)\right), \\ S_b\left(\Theta æ_{2p}, \Theta æ_{2p}, \Omega(æ_{2p}, œ_{2p})\right), \\ S_b\left(\Lambda\wp, \Lambda\wp, \Gamma(\wp, \ell)\right), \\ S_b\left(\Theta œ_{2p}, \Theta œ_{2p}, \Omega(œ_{2p}, æ_{2p})\right), \\ \frac{S_b\left(\Lambda\ell, \Lambda\ell, \Omega(æ_{2p}, œ_{2p})\right) + S_b\left(\Theta æ_{2p}, \Theta æ_{2p}, \Gamma(\ell, \wp)\right)}{2\kappa^4}, \\ \frac{S_b\left(\Lambda\wp, \Lambda\wp, \Omega(œ_{2p}, æ_{2p})\right) + S_b\left(\Theta œ_{2p}, \Theta œ_{2p}, \Gamma(\wp, \ell)\right)}{2\kappa^4} \end{array} \right\} \\ &= \lim_{p\to\infty} \sup \max \left\{ \begin{array}{c} S_b\left(\Lambda\ell, \Lambda\ell, \ell_{2p-1}\right), S_b\left(\Lambda\wp, \Lambda\wp, \wp_{2p-1}\right), \\ S_b\left(\Lambda\ell, \Lambda\ell, \Gamma(\ell, \wp)\right), S_b\left(\ell_{2p-1}, \ell_{2p-1}, \ell_{2p}\right), \\ S_b\left(\Lambda\wp, \Lambda\wp, \Gamma(\wp, \ell)\right), S_b\left(\wp_{2p-1}, \wp_{2p-1}, \wp_{2p}\right), \\ \frac{S_b\left(\Lambda\ell, \Lambda\ell, \ell_{2p}\right) + S_b\left(\ell_{2p-1}, \ell_{2p-1}, \Gamma(\ell, \wp)\right)}{2\kappa^4}, \\ \frac{S_b\left(\Lambda\wp, \Lambda\wp, \wp_{2p}\right) + S_b\left(\wp_{2p-1}, \wp_{2p-1}, \Gamma(\wp, \ell)\right)}{2\kappa^4} \end{array} \right\} \end{aligned}$$

$$
\begin{aligned}
&= \max\left\{ S_b\left(æ,æ,\Gamma(\ell,\wp)\right), S_b\left(œ,œ,\Gamma(\wp,\ell)\right) \right\} \\
&= \max\left\{ S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \right\}
\end{aligned}
$$

and

$$
\limsup_{p\to\infty} \mathbb{N}\left(\ell,\wp,æ_{2p},œ_{2p}\right) = \limsup_{p\to\infty} \min\left\{ \begin{array}{c} S_b\left(\Lambda\ell,\Lambda\ell,\Gamma(\ell,\wp)\right), \\ S_b\left(\ell_{2p-1},\ell_{2p-1},\ell_{2p}\right), \\ S_b\left(\Lambda\ell,\Lambda\ell,\ell_{2p}\right), \\ S_b\left(\Lambda\wp,\Lambda\wp,\wp_{2p}\right), \\ S_b\left(\ell_{2p-1},\ell_{2p-1},\Gamma(\ell,\wp)\right), \\ S_b\left(\wp_{2p-1},\wp_{2p-1},\Gamma(\wp,\ell)\right) \end{array} \right\} = 0_{\mathfrak{A}}.
$$

Hence from (3.6), we have

$$
\begin{aligned}
\psi\left(S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right)\right) &\le \psi\left(a^* \max\left\{ \begin{array}{c} S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), \\ S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \end{array} \right\} a\right) \\
&\quad -\phi\left(a^* \max\left\{ \begin{array}{c} S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), \\ S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \end{array} \right\} a\right) + \xi 0_{\mathfrak{A}} \\
&\le \psi\left(a^* \max\left\{ \begin{array}{c} S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), \\ S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \end{array} \right\} a\right)
\end{aligned}
$$

By the definition of ψ, we have that

$$
S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right) \le a^* \max\left\{ \begin{array}{c} S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), \\ S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \end{array} \right\} a.
$$

Similarly, we prove

$$
S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \le a^* \max\left\{ \begin{array}{c} S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right), \\ S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right) \end{array} \right\} a.
$$

Therefore,

$$
\max\left\{ \begin{array}{c} \|S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right)\|, \\ \|S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right)\| \end{array} \right\} \le \|a\|^2 \max\left\{ \begin{array}{c} \|S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right)\|, \\ \|S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right)\| \end{array} \right\}.
$$

Since $\|a\| < 1$, which implies that $\|S_b\left(\Gamma(\ell,\wp),\Gamma(\ell,\wp),æ\right)\| = 0$, and $\|S_b\left(\Gamma(\wp,\ell),\Gamma(\wp,\ell),œ\right)\| = 0$ and hence $\Gamma(\ell,\wp) = æ$, $\Gamma(\wp,\ell) = œ$. It follows that $\Gamma(\ell,\wp) = æ = \Lambda\ell$ and $\Gamma(\wp,\ell) = œ = \Lambda\wp$. Since $\{\Gamma,\Lambda\}$ is weakly compatible pair, we have $\Gamma(æ,œ) = \Lambda æ$ and $\Gamma(œ,æ) = \Lambda œ$, then we prove that $\Lambda æ = æ$ and $\Lambda œ = œ$. From Lemma (3.4), we have

$$
\tfrac{1}{2\kappa} S_b\left(\Lambda æ,\Lambda æ,æ\right) \le \liminf_{p\to\infty} S_b\left(\Lambda æ,\Lambda æ,\ell_{2p}\right).
$$

Now from (3.1) and applying ψ on both sides, we have

$$\begin{aligned}\psi\left(S_b\left(\Lambda æ, \Lambda æ, æ\right)\right) &\leq \lim_{p\to\infty}\inf \psi\left(2\kappa S_b\left(\Lambda æ, \Lambda æ, \ell_{2p}\right)\right) \\ &= \lim_{p\to\infty}\inf \psi\left(2\kappa S_b\left(\Gamma(æ, œ), \Gamma(æ, œ), \Gamma(æ_{2p}, œ_{2p})\right)\right) \\ &\leq \lim_{p\to\infty}\sup \psi\left(a^* \mathbb{M}(æ, œ, æ_{2p}, œ_{2p})a\right) \\ &\quad - \lim_{p\to\infty}\sup \phi\left(a^* \mathbb{M}(æ, œ, æ_{2p}, œ_{2p})a\right) \\ &\quad + \lim_{p\to\infty}\sup \xi \mathbb{N}\left(æ, œ, æ_{2p}, œ_{2p}\right) \end{aligned} \tag{3.7}$$

Now, by simple computations, we have

$$\begin{aligned}\lim_{p\to\infty}\sup \mathbb{M}\left(æ, œ, æ_{2p}, œ_{2p}\right) &= \lim_{p\to\infty}\sup \max\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, \Theta æ_{2p}\right), \\ S_b\left(\Lambda œ, \Lambda œ, \Theta œ_{2p}\right), \\ S_b\left(\Lambda æ, \Lambda æ, \Gamma(æ, œ)\right), \\ S_b\left(\Theta æ_{2p}, \Theta æ_{2p}, \Omega(æ_{2p}, œ_{2p})\right), \\ S_b\left(\Lambda œ, \Lambda œ, \Gamma(œ, æ)\right), \\ S_b\left(\Theta œ_{2p}, \Theta œ_{2p}, \Omega(œ_{2p}, æ_{2p})\right), \\ \frac{S_b\left(\Lambda æ, \Lambda æ, \Omega(æ_{2p}, œ_{2p})\right) + S_b\left(\Theta æ_{2p}, \Theta æ_{2p}, \Gamma(æ, œ)\right)}{2\kappa^4}, \\ \frac{S_b\left(\Lambda œ, \Lambda œ, \Omega(œ_{2p}, æ_{2p})\right) + S_b\left(\Theta œ_{2p}, \Theta œ_{2p}, \Gamma(œ, æ)\right)}{2\kappa^4} \end{array}\right\} \\ &= \lim_{p\to\infty}\sup \max\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, \ell_{2p-1}\right), S_b\left(\Lambda œ, \Lambda œ, \wp_{2p-1}\right), \\ S_b\left(\Lambda æ, \Lambda æ, \Gamma(æ, œ)\right), S_b\left(\ell_{2p-1}, \ell_{2p-1}, \ell_{2p}\right), \\ S_b\left(\Lambda œ, \Lambda œ, \Gamma(œ, æ)\right), S_b\left(\wp_{2p-1}, \wp_{2p-1}, \wp_{2p}\right), \\ \frac{S_b\left(\Lambda æ, \Lambda æ, \ell_{2p}\right) + S_b\left(\ell_{2p-1}, \ell_{2p-1}, \Gamma(æ, œ)\right)}{2\kappa^4}, \\ \frac{S_b\left(\Lambda œ, \Lambda œ, \wp_{2p}\right) + S_b\left(\wp_{2p-1}, \wp_{2p-1}, \Gamma(œ, æ)\right)}{2\kappa^4} \end{array}\right\} \\ &= \max\left\{ S_b\left(\Lambda æ, \Lambda æ, æ\right), S_b\left(\Lambda œ, \Lambda œ, œ\right) \right\}\end{aligned}$$

and

$$\lim_{p\to\infty}\sup \mathbb{N}\left(æ, œ, æ_{2p}, œ_{2p}\right) = \lim_{p\to\infty}\sup \min\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, \Gamma(æ, œ)\right), \\ S_b\left(\ell_{2p-1}, \ell_{2p-1}, \ell_{2p}\right), \\ S_b\left(\Lambda æ, \Lambda æ, \ell_{2p}\right), \\ S_b\left(\Lambda œ, \Lambda œ, \wp_{2p}\right), \\ S_b\left(\ell_{2p-1}, \ell_{2p-1}, \Gamma(æ, œ)\right), \\ S_b\left(\wp_{2p-1}, \wp_{2p-1}, \Gamma(œ, æ)\right) \end{array}\right\} = 0_{\mathfrak{A}}.$$

Hence from (3.7), we have

$$\psi\left(S_b\left(\Lambda æ, \Lambda æ, æ\right)\right) \leq \psi\left(a^* \max\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, æ\right), \\ S_b\left(\Lambda œ, \Lambda œ, œ\right) \end{array}\right\} a\right)$$

By the definition of ψ, we have that

$$S_b\left(\Lambda æ, \Lambda æ, æ\right) \quad \preceq \quad a^* \max\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, æ\right), \\ S_b\left(\Lambda œ, \Lambda œ, œ\right) \end{array}\right\} a.$$

Similarly, we prove

$$S_b\left(\Lambda œ, \Lambda œ, œ\right) \quad \preceq \quad a^* \max\left\{\begin{array}{c} S_b\left(\Lambda æ, \Lambda æ, æ\right), \\ S_b\left(\Lambda œ, \Lambda œ, œ\right) \end{array}\right\} a.$$

Therefore,

$$\max\left\{\begin{array}{c} \|S_b\left(\Lambda æ, \Lambda æ, æ\right)\|, \\ \|S_b\left(\Lambda œ, \Lambda œ, œ\right)\| \end{array}\right\} \quad \leq \quad \|a\|^2 \max\left\{\begin{array}{c} \|S_b\left(\Lambda æ, \Lambda æ, æ\right)\|, \\ \|S_b\left(\Lambda œ, \Lambda œ, œ\right)\| \end{array}\right\}.$$

Since $\|a\| < 1$, which implies that $\|S_b\left(\Lambda æ, \Lambda æ, æ\right)\| = 0$, $\|S_b\left(\Lambda œ, \Lambda œ, œ\right)\| = 0$ and hence $\Lambda æ = æ$, $\Lambda œ = œ$. It follows that $\Gamma(æ, œ) = \Lambda æ = æ$ and
$\Gamma(œ, æ) = \Lambda œ = œ$. Thus $(æ, œ)$ is common coupled fixed point of Γ and Λ. Since $\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ so there exist $\varkappa, ß \in \mathcal{G}$ such that $\Gamma(æ, œ) = æ = \Theta\varkappa$ and $\Gamma(œ, æ) = œ = \Theta ß$. Now we show that $\Omega(\varkappa, ß) = æ$ and $\Omega(ß, \varkappa) = œ$ Now from (3.1), we have

$$\begin{aligned} 0_{\mathfrak{A}} \quad &\preceq \quad \psi\left(S_b\left(æ, æ, \Omega(\varkappa, ß)\right)\right) \preceq \psi\left(2\kappa S_b\left(\Gamma(æ, œ), \Gamma(æ, œ), \Omega(\varkappa, ß)\right)\right) \\ &\preceq \quad \psi\left(a^*\mathbb{M}(æ, œ, \varkappa, ß)a\right) - \phi\left(a^*\mathbb{M}(æ, œ, \varkappa, ß)a\right) + \xi\mathbb{N}\left(æ, œ, \varkappa, ß\right) \\ &\preceq \quad \psi\Big(a^* \max\left\{ S_b\left(æ, æ, \Omega(\varkappa, ß)\right), S_b\left(œ, œ, \Omega(ß, \varkappa)\right) \right\} a\Big) \end{aligned}$$

By the definition of ψ, we have that

$$S_b\left(æ, æ, \Omega(\varkappa, ß)\right) \quad \preceq \quad a^* \max\left\{ S_b\left(æ, æ, \Omega(\varkappa, ß)\right), S_b\left(œ, œ, \Omega(ß, \varkappa)\right) \right\} a.$$

Therefore,

$$\max\left\{ S_b\left(æ, æ, \Omega(\varkappa, ß)\right), S_b\left(œ, œ, \Omega(ß, \varkappa)\right) \right\} \quad \preceq \quad a^* \max\left\{\begin{array}{c} S_b\left(æ, æ, \Omega(\varkappa, ß)\right), \\ S_b\left(œ, œ, \Omega(ß, \varkappa)\right) \end{array}\right\} a.$$

Since $\|a\| < 1$, this implies that $\Omega(\varkappa, ß) = æ$ and $\Omega(ß, \varkappa) = œ$. Since $\{\Omega, \Theta\}$ is weakly compatible pair, we have $\Omega(æ, œ) = \Theta æ$ and $\Omega(œ, æ) = \Theta œ$, then we prove that $\Theta æ = æ$ and $\Theta œ = œ$. Now from (3.1), we have

$$\begin{aligned} 0_{\mathfrak{A}} \quad &\preceq \quad \psi\left(S_b\left(æ, æ, \Theta æ\right)\right) \preceq \psi\left(2\kappa S_b\left(\Gamma(æ, œ), \Gamma(æ, œ), \Omega(æ, œ)\right)\right) \\ &\preceq \quad \psi\left(a^*\mathbb{M}(æ, œ, æ, œ)a\right) - \phi\left(a^*\mathbb{M}(æ, œ, æ, œ)a\right) + \xi\mathbb{N}\left(æ, œ, æ, œ\right) \\ &\preceq \quad \psi\Big(a^* \max\left\{ S_b\left(æ, æ, \Theta æ\right), S_b\left(œ, œ, \Theta œ\right) \right\} a\Big) \end{aligned}$$

By the definition of ψ, we have that

$$S_b\left(æ, æ, \Theta æ\right) \quad \preceq \quad a^* \max\left\{ S_b\left(æ, æ, \Theta æ\right), S_b\left(œ, œ, \Theta œ\right) \right\} a.$$

Therefore,

$$0 \leq \max\left\{ \|S_b(æ, æ, \Theta æ)\|, \|S_b(œ, œ, \Theta œ)\| \right\} \leq \|a\|^2 \max\left\{ \begin{array}{l} \|S_b(æ, æ, \Theta æ)\|, \\ \|S_b(œ, œ, \Theta œ)\| \end{array} \right\}.$$

Since $\|a\| < 1$, we get that $\Theta æ = æ$ and $\Theta œ = œ$. Hence $\Omega(æ, œ) = \Theta æ = æ$ and $\Omega(œ, æ) = \Theta œ = œ$. Thus,$(æ, œ)$ is common coupled fixed point of Γ, Ω, Λ and Θ. in the following we will show the uniqueness of common coupled fixed point in $\mathcal{G}$. Let us take $(æ, œ) \neq (æ', œ')$ be an another fixed point of Γ, Ω, Λ and Θ then,

$$\begin{aligned} 0_{\mathfrak{A}} &\leq \psi\left(S_b(æ, æ, æ')\right) \leq \psi\left(2\kappa S_b\left(\Gamma(æ, œ), \Gamma(æ, œ), \Omega(æ', œ')\right)\right) \\ &\leq \psi\left(a^* \mathbb{M}(æ, œ, æ', œ')a\right) - \phi\left(a^* \mathbb{M}(æ, œ, æ', œ')a\right) + \xi \mathbb{N}\left(æ, œ, æ', œ'\right) \\ &\leq \psi\left(a^* \max\left\{ S_b(æ, æ, æ'), S_b(œ, œ, œ') \right\} a\right) \end{aligned}$$

By the definition of ψ, we have that

$$S_b(æ, æ, æ') \leq a^* \max\left\{ S_b(æ, æ, æ'), S_b(œ, œ, œ') \right\} a.$$

Therefore, we have

$$\max\left\{ \|S_b(æ, æ, æ')\|, \|S_b(œ, œ, œ')\| \right\} \leq \|a\|^2 \max\left\{ \|S_b(æ, æ, æ')\|, \|S_b(œ, œ, œ')\| \right\}.$$

Since $\|a\| < 1$, it is incongruous. Consequently, $æ = æ'$ and $œ = œ'$. Therefore, the UCCFP of Γ, Ω, Λ and Θ is $(æ, œ)$. In order to prove that Γ, Ω, Λ and Θ have a unique fixed point, we only have to prove $æ = œ$, we have

$$\begin{aligned} 0_{\mathfrak{A}} &\leq \psi\left(S_b(æ, æ, œ)\right) \leq \psi\left(2\kappa S_b\left(\Gamma(æ, œ), \Gamma(æ, œ), \Omega(œ, æ)\right)\right) \\ &\leq \psi\left(a^* \mathbb{M}(æ, œ, œ, æ)a\right) - \phi\left(a^* \mathbb{M}(æ, œ, œ, æ)a\right) + \xi \mathbb{N}\left(æ, œ, œ, æ\right) \\ &\leq \psi\left(a^* S_b(æ, æ, œ)\, a\right) \end{aligned}$$

Therefore,

$$0 \leq \|S_b(æ, æ, œ)\| \leq \|a\|^2 \|S_b(æ, æ, œ)\|$$

This is incongruous. Consequently, $æ = œ$, which means that Γ, Ω, Λ and Θ have a unique fixed point of the form $(æ, æ)$ in $\mathcal{G}$. □

Theorem 3.6. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a complete C^*-AV-S_bMS, suppose $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ and $\Lambda : \mathcal{G} \to \mathcal{G}$ be two mappings with the following assumptions:*

(i) *$\Gamma(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$ and $\Lambda(\mathcal{G})$ is closed sub spaces of $\mathcal{G}$;*
(ii) *Γ, Λ are (ψ, ϕ)- generalized weakly contractive mappings;*
(iii) *$\{\Gamma, \Lambda\}$ is ω-compatible pairs.*

Then Γ and Λ have a unique common coupled fixed point in $\mathcal{G}$.

Proof. The proof follows from Theorem (3.5) by taking $\Gamma = \Omega$ and $\Lambda = \Theta$. □

Corollary 3.7. *Let $(\mathcal{G}, \mathfrak{A}, S_b)$ be a complete C^*-AV-S_bMS, $\Gamma : \mathcal{G}^2 \to \mathcal{G}$ is (ψ, ϕ)- generalized weakly contractive mapping then Γ has a unique coupled fixed point in $\mathcal{G}$.*

Proof. The proof follows from Theorem (3.5) by taking $\Gamma = \Omega$ and $\Lambda = \Theta = I_{\mathcal{G}}$. □

Example 3.8. *Let $\mathcal{G} = \{0, 1, 2\}$ and $\mathfrak{A} = M_2(\mathbb{R})$ be all 2×2 matrices whose norm is $\|\mathfrak{A}\| = \max_{1\le j\le 2} \sum_{i=1}^{2} |a_{ij}|$ and define the mapping $d : \mathcal{G}^2 \to [0, \infty)$ as*
$d(0,0) = d(1,1) = d(2,2) = 0, d(0,1) = d(1,0) = 2, d(1,2) = d(2,1) = 3$ *and* $d(0,2) = d(2,0) = 20$. *Then clearly, $(\mathcal{G}, d)$ is b-metric space with $\kappa = 2^{2(b-1)}$ where $b > 1$. Let $S_b : \mathcal{G}^3 \to M_2(\mathbb{R})$ be as $S_b(p,q,r) = (d(p,q) + d(q,r) + d(r,p) \quad 0)$. Then, clearly $(\mathcal{G}, \mathfrak{A}, S_b)$ is a complete C^*-AV-S_bMS with $\|\kappa\| = 4 \ge 1$.*
Let $\psi, \phi : \mathfrak{A}_+ \to \mathfrak{A}_+$ defined by $\psi(\aleph) = \aleph$, $\phi(\aleph) = \frac{\aleph}{10}$ and $a \in \mathfrak{A}$ with $\|a\| = \frac{3}{\sqrt{10}} < 1$. We define mappings $\Gamma, \Omega : \mathcal{G}^2 \to \mathcal{G}$, $\Lambda, \Theta : \mathcal{G} \to \mathcal{G}$ as follows

$$\Gamma(\ae, \oe) = \begin{cases} 1 & \text{if } \ae = \oe = 2 \\ 0 & \text{if } \ae, \oe \in \{0,1\} \end{cases}, \quad \Omega(\ae, \oe) = \begin{cases} 1 & \text{if } \ae = 1, \oe = 1 \\ 0 & \text{if } \ae, \oe \in \{0,2\} \end{cases}$$

$$\Lambda(\ae) = \begin{cases} 2 & \text{if } \ae = 2 \\ 0 & \text{if } \ae \in \{0,1\} \end{cases}, \quad \Theta(\ae) = \begin{cases} 2 & \text{if } \ae = 1 \\ 0 & \text{if } \ae \in \{0,2\} \end{cases}$$

Also put $\xi \succeq 0_{\mathfrak{A}}$ with $\|\xi\| = 2$. Then clearly, $\Gamma(\mathcal{G}^2) \subseteq \Theta(\mathcal{G})$ and $\Omega(\mathcal{G}^2) \subseteq \Lambda(\mathcal{G})$. One can show that $(\ae, \oe)$ is a coupled coincidence point of Γ, Ω, Λ and Θ if and only if $\ae = \oe = 0$. Since $\Gamma(\Lambda 0, \Lambda 0) = \Lambda(\Gamma(0,0))$ and $\Omega(\Theta 0, \Theta 0) = \Theta(\Omega(0,0))$, we get that $\{\Gamma, \Lambda\}$ and $\{\Omega, \Theta\}$ are ω-compatible pairs. Now from inequality (3.1), we have

$$\psi\left(2\kappa S_b\left(\Gamma(\ell, \varkappa), \Gamma(\ell, \varkappa), \Omega(\ae, \oe)\right)\right) \preceq \psi\left(a^* \mathbb{M}(\ell, \varkappa, \ae, \oe) a\right) - \phi\left(a^* \mathbb{M}(\ell, \varkappa, \ae, \oe) a\right) + \xi \mathbb{N}\left(\ell, \varkappa, \ae, \oe\right)$$

implies that

$$\|2\kappa S_b\left(\Gamma(\ell, \varkappa), \Gamma(\ell, \varkappa), \Omega(\ae, \oe)\right)\| \le \frac{9}{10} \|a\|^2 \|\mathbb{M}(\ell, \varkappa, \ae, \oe)\| + \|\xi\| \|\mathbb{N}\left(\ell, \varkappa, \ae, \oe\right)\|$$

Now, we consider the following cases:

(i) $(\ell, \varkappa) = (0,0)$ *and* $(\ae, \oe) = (1,1)$, *then*

$$\|8S_b\left(\Gamma(0,0), \Gamma(0,0), \Omega(1,1)\right)\| = 32 \le \frac{81}{100} \|\mathbb{M}(0,0,1,1)\| + 2\|\mathbb{N}\left(0,0,1,1\right)\| = \frac{324}{10}$$

(ii) $(\ell, \varkappa) = (1,1)$ *and* $(\ae, \oe) = (0,0)$ *then*

$$\|8S_b\left(\Gamma(1,1), \Gamma(1,1), \Omega(0,0)\right)\| = 0 \le \frac{81}{100} \|\mathbb{M}(1,1,0,0)\| + 2\|\mathbb{N}\left(1,1,0,0\right)\| = 0$$

(iii) $(\ell, \varkappa) = (0,0)$ *and* $(\ae, \oe) = (2,2)$ *then*

$$\|8S_b\left(\Gamma(0,0), \Gamma(0,0), \Omega(2,2)\right)\| = 0 \le \frac{81}{100} \|\mathbb{M}(0,0,2,2)\| + 2\|\mathbb{N}\left(0,0,2,2\right)\| = 0$$

(iv) $(\ell, \varkappa) = (2,2)$ *and* $(\ae, \oe) = (0,0)$ *then*

$$\|8S_b\left(\Gamma(2,2), \Gamma(2,2), \Omega(0,0)\right)\| = 32 \le \frac{81}{100} \|\mathbb{M}(2,2,0,0)\| + 2\|\mathbb{N}\left(2,2,0,0\right)\| = \frac{324}{10}$$

(v) $(\ell,\varkappa)=(1,1)$ *and* $(æ,œ)=(2,2)$ *then*

$$\|8S_b\left(\Gamma(1,1),\Gamma(1,1),\Omega(2,2)\right)\|=0\leq\frac{81}{100}\|\mathbb{M}(1,1,2,2)\|+2\|\mathbb{N}\left(1,1,2,2\right)\|=0$$

(vi) $(\ell,\varkappa)=(2,2)$ *and* $(æ,œ)=(1,1)$ *then*

$$\|8S_b\left(\Gamma(2,2),\Gamma(2,2),\Omega(1,1)\right)\|=0\leq\frac{81}{100}\|\mathbb{M}(2,2,1,1)\|+2\|\mathbb{N}\left(2,2,0,0\right)\|=\frac{843}{50}$$

Thus, all conditions of Theorem 3.5 are satisfied and therefore Γ Ω, Λ *and* Θ *have a unique common fixed point (namely,* $(æ,æ)=(0,0)$*) in* $\mathcal{G}$ *.*

4. Application to Integral Equations

We take into account the subsequent integral equation as an application:

Consider the integral equation

$$\ell(t)=\int\limits_{\mathcal{E}}\left(\mathcal{K}_1(t,s)+\mathcal{K}_2(t,s)\right)\left(\mathfrak{f}(s,\ell(s))+\mathfrak{g}(s,\ell(s))\right)ds+\mathcal{A}(t)\ \forall\ t\in\mathcal{E}. \tag{4.1}$$

where $\mathcal{E}$ is a Lebesgue measurable set and $m(\mathcal{E})<\infty$. In what follows, we always let $\mathcal{G}=L^\infty(\mathcal{E})$ denote the class of essentially bounded measurable functions on $\mathcal{E}$. Now, we consider the functions $\mathcal{K}_1,\mathcal{K}_2,\mathfrak{f},\mathfrak{g}$ fulfill the following assumptions:

(i_0) $\mathcal{K}_1:\mathcal{E}^2\to[0,\infty),\mathcal{K}_2:\mathcal{E}^2\to(-\infty,0]$ and $\mathfrak{f},\mathfrak{g}:\mathcal{E}\times\mathbb{R}\to\mathbb{R}$ are integrable, and $\mathcal{A}\in L^\infty(\mathcal{E})$

(i_1) $\exists\ \theta\in(0,1)$ such that for all $x,y\in\mathbb{R}$ and $t\in\mathcal{E}$,
$0\leq\mathfrak{f}(t,x)-\mathfrak{f}(t,y)\leq\frac{\theta}{4\sqrt{6}}(x-y)$ and $-\frac{\theta}{4\sqrt{6}}(x-y)\leq\mathfrak{g}(t,x)-\mathfrak{g}(t,y)\leq 0$

(i_2) $\sup\limits_{t\in\mathcal{E}}\int\limits_{\mathcal{E}}|\mathcal{K}_1(t,s)-\mathcal{K}_2(t,s)|dt\leq 1.$

Theorem 4.1. *Under the assumption* (i_0)*-*(i_2)*, the equation (4.1) has a unique solution in* $L^\infty(\mathcal{E})$*.*

Proof. Suppose $\mathcal{G}=L^\infty(\mathcal{E})$ and $B(L^2(\mathcal{E}))$ is a set of bounded linear operators on a Hilbert space $L^2(\mathcal{E})$. We equip $\mathcal{G}$ with $S_b:\mathcal{G}\times\mathcal{G}\times\mathcal{G}\to B(L^2(\mathcal{E}))$, which is ascertained by $S_b(æ,ß,œ)=\mathbb{M}_{(|æ-œ|+|ß-œ|)^p}$, where $\mathbb{M}_{(|æ-œ|+|ß-œ|)^p}$ is the multiplication operator on $L^2(\mathcal{E})$ ascertained by $\mathbb{M}_h(\alpha)=h.\alpha$, $\alpha\in L^2(\mathcal{E})$. Therefore, $(\mathcal{G},B(L^2(\mathcal{E})),S_b)$ is a complete C*-AV-S_bMS with $\kappa=2^{2(p-1)}$ where $p=2>1$ and $\|\xi\|=0$. Define the mappings $\psi,\phi:\mathfrak{A}_+\to\mathfrak{A}_+$ by $\psi(a)=a$, $\phi(a)=\frac{2a}{3}$ and $\Gamma:\mathcal{G}^2\to\mathcal{G}$ as for all $t\in\mathcal{E}$

$$\Gamma(\ell,\varkappa)(t)=\int\limits_{\mathcal{E}}\mathcal{K}_1(t,s)\left(\mathfrak{f}(s,\ell(s))+\mathfrak{g}(s,\varkappa(s))\right)ds+\int\limits_{\mathcal{E}}\mathcal{K}_2(t,s)\left(\mathfrak{f}(s,\varkappa(s))+\mathfrak{g}(s,\ell(s))\right)ds+\mathcal{A}(t)$$

we have

$$S_b\left(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(æ,œ)\right)=\mathbb{M}_{(2|\Gamma(\ell,\varkappa)-\Gamma(æ,œ)|)^p}$$

Let us first evaluate the following expression:

$$\begin{aligned}
& 2^p|(\Gamma(\ell,\varkappa)-\Gamma(æ,œ))(t)|^p \\
= \;& 2^p \left| \begin{array}{l} \int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)\left(\mathfrak{f}(s,\ell(s))+\mathfrak{g}(s,\varkappa(s))\right)ds + \int\limits_{\mathcal{E}} \mathcal{K}_2(t,s)\left(\mathfrak{f}(s,\varkappa(s))+\mathfrak{g}(s,\ell(s))\right)ds \\ -\int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)\left(\mathfrak{f}(s,æ(s))+\mathfrak{g}(s,œ(s))\right)ds - \int\limits_{\mathcal{E}} \mathcal{K}_2(t,s)\left(\mathfrak{f}(s,œ(s))+\mathfrak{g}(s,æ(s))\right)ds \end{array} \right|^p \\
= \;& 2^p \left(\begin{array}{l} \left|\int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)\left((\mathfrak{f}(s,\ell(s))-\mathfrak{f}(s,æ(s))+\mathfrak{g}(s,\varkappa(s))-\mathfrak{g}(s,œ(s)))\right)ds\right| \\ +\left|\int\limits_{\mathcal{E}} \mathcal{K}_2(t,s)\left((\mathfrak{f}(s,\varkappa(s))-\mathfrak{f}(s,œ(s)))+\mathfrak{g}(s,\ell(s))-\mathfrak{g}(s,æ(s)))\right)ds\right| \end{array} \right)^p \\
\leq \;& 2^p \left(\begin{array}{l} \int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)\left|\left(\mathfrak{f}(s,\ell(s))-\mathfrak{f}(s,æ(s))+\mathfrak{g}(s,\varkappa(s))-\mathfrak{g}(s,œ(s))\right)\right|ds \\ -\int\limits_{\mathcal{E}} \mathcal{K}_2(t,s)\left|\left(\mathfrak{f}(s,\varkappa(s))-\mathfrak{f}(s,œ(s))\right)+\mathfrak{g}(s,\ell(s))-\mathfrak{g}(s,æ(s))\right)\right|ds \end{array} \right)^p \\
\leq \;& 2^p \sup_{s\in\mathcal{E}} \left[\frac{\theta}{4\sqrt{6}}|\ell(s)-æ(s)|+\frac{\theta}{4\sqrt{6}}|\varkappa(s)-œ(s)|\right]^p \left(\int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)-\mathcal{K}_2(t,s)ds\right)^p \\
\leq \;& \frac{\theta}{(4\sqrt{6})^p}\left[2\|\ell-æ\|_\infty+2\|\varkappa-œ\|\|_\infty\right]^p \left(\sup_{t\in\mathcal{E}}\int\limits_{\mathcal{E}} \mathcal{K}_1(t,s)-\mathcal{K}_2(t,s)ds\right)^p \\
\leq \;& \frac{\theta}{(4\sqrt{6})^p}\left[2\|\ell-æ\|_\infty+2\|\varkappa-œ\|\|_\infty\right]^p
\end{aligned}$$

Therefore,

$$\begin{aligned}
& \|\psi\left(2\kappa S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(æ,œ)\right)\| = \|2\kappa S_b(\Gamma(\ell,\varkappa),\Gamma(\ell,\varkappa),\Gamma(æ,œ)\| \\
= \;& 2\kappa \sup_{\|h\|=1} \left\langle \mathbb{M}_{(2|\Gamma(\ell,\varkappa)-\Gamma(æ,œ)|)^p}h,h\right\rangle \\
= \;& 2\kappa \sup_{\|h\|=1} \left\langle 2^p\mathbb{M}_{|\Gamma(\ell,\varkappa)-\Gamma(æ,œ)|^p}h,h\right\rangle \\
= \;& 2\kappa \sup_{\|h\|=1} \int\limits_{\mathcal{E}} (2^p|\Gamma(\ell,\varkappa)(t)-\Gamma(æ,œ)(t)|^p)h(t)\overline{h(t)}dt \\
\leq \;& 2\kappa\frac{\theta}{(4\sqrt{6})^p}2^{p-1}\left[\|S_b(\ell,\ell,æ)\|+\|S_b(\varkappa,\varkappa,œ)\|\right] \\
\leq \;& \frac{\theta}{3}\max\left\{\ \|S_b(\ell,\ell,æ)\|,\|S_b(\varkappa,\varkappa,œ)\|\ \right\} \\
\leq \;& \|\psi\left(a^*M(\ell,\varkappa,æ,œ)a\right)-\phi\left(a^*M(\ell,\varkappa,æ,œ)a\right)+\xi N(\ell,\varkappa,æ,œ)\|
\end{aligned}$$

By setting $a=\sqrt{\theta}1_{B(L^2(\mathcal{E}))}$, then $a\in B(L^2(\mathcal{E}))$ and $\|a\|=\|\sqrt{\theta}\|<1$. Hence, applying our Corollary 3.7, we get the desired result. □

5. Application to Homotopy

In this section, we investigate whether homotopy could have a unique solution.

Theorem 5.1. *If $(\mathcal{G}, \mathfrak{A}, S_b)$ is a complete C^*-AV-S_bMS, then $\mathfrak{U}$ and $\overline{\mathfrak{U}}$ are open and closed subsets of $\mathcal{G}$, respectively, such that $\mathfrak{U} \subseteq \overline{\mathfrak{U}}$. Let $\mathfrak{H}_b : \overline{\mathfrak{U}}^2 \times [0,1] \to \mathcal{G}$ be an homotopy operator meeting the requirements listed below.*

- (τ_0) *$\ell \neq \mathfrak{H}_b(\ell, \varkappa, s)$, and $\varkappa \neq \mathfrak{H}_b(\varkappa, \ell, s)$ for each $\ell, \varkappa \in \partial\mathfrak{U}$ and $s \in [0,1]$ (here $\partial\mathfrak{U}$ is boundary of $\mathfrak{U}$ in $\mathcal{G}$)*
- (τ_1) *there exist $\ell, \varkappa, æ, œ \in \overline{\mathfrak{U}}$, $s \in [0,1]$ and $a \in \mathfrak{A}$ with $\|a\| < 1$ such that*

$$\begin{aligned}\psi\left(2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(æ,œ,s))\right) &\leq \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a\right) \\ &\quad -\phi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a\right)\end{aligned}$$

- (τ_2) *$\exists\, M_b \succeq 0_{\mathfrak{A}} \ni S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,t)) \preceq \|M_b\| |s-t|$ for every $\ell, \varkappa \in \overline{\mathfrak{U}}$ and $s, t \in [0,1]$;*

Then $\mathfrak{H}_b(.,s)$ has a coupled fixed point for some $s \in [0,1] \iff \mathfrak{H}_b(.,t)$ has a coupled fixed point for some $t \in [0,1]$.

Proof. From (τ_2) it follows that $\mathfrak{H}_b$ is continuous in the second variable. From (τ_1) it follows that $\mathfrak{H}_b$ is continuous in the first Variable. We have that

$$\begin{aligned}\psi\Big(2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell_p,\varkappa_p,s))\Big) &\preceq \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_p), \\ S_b(\varkappa,\varkappa,\varkappa_p) \end{array}\right\} a\right) \\ &\quad -\phi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_p), \\ S_b(\varkappa,\varkappa,\varkappa_p) \end{array}\right\} a\right) \\ &\preceq \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_p), \\ S_b(\varkappa,\varkappa,\varkappa_p) \end{array}\right\} a\right) \to 0_{\mathfrak{A}} \text{ as } p \to \infty.\end{aligned}$$

If $\ell_p \to \ell$ and $\varkappa_p \to \varkappa$, then $\psi\Big(2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell_p,\varkappa_p,s))\Big) \to 0_{\mathfrak{A}}$ as $p \to \infty$. Therefore, $S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s)) = 0_{\mathfrak{A}}$ and also we have $S_b(\mathfrak{H}_b(\varkappa,\ell,s), \mathfrak{H}_b(\varkappa,\ell,s), \mathfrak{H}_b(\varkappa,\ell,s)) = 0_{\mathfrak{A}}$.

Now

$$\begin{aligned}S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(æ,œ,t)) &\preceq 2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(æ,œ,s)) \\ &\quad +\kappa S_b(\mathfrak{H}_b(æ,œ,t), \mathfrak{H}_b(æ,œ,t), \mathfrak{H}_b(æ,œ,s)) \\ &\preceq 2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(\ell,\varkappa,s), \mathfrak{H}_b(æ,œ,s)) + \kappa\|M_b\||s-t| \\ &\preceq 2\kappa a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a + \kappa\|M_b\||s-t| \to 0_{\mathfrak{A}} \\ &\quad \text{as } (\ell,\varkappa,s) \to (æ,œ,t).\end{aligned}$$

Hence $\mathfrak{H}_b$ is a continous function on $\overline{\mathfrak{U}}^2 \times [0,1]$. Also

$$\begin{aligned}
& \psi\left(S_b(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(æ,œ,s))\right) \leq \psi\left(2\kappa S_b(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(æ,œ,s))\right) \\
\leq\ & \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a\right) - \phi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a\right) \\
\prec\ & \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a\right) \text{ if } \ell \neq æ, \varkappa \neq œ \\
\Rightarrow\ & \max\left\{\begin{array}{c} S_b(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(æ,œ,s)) \\ S_b(\mathfrak{H}_b(\varkappa,\ell,s),\mathfrak{H}_b(\varkappa,\ell,s),\mathfrak{H}_b(œ,æ,s)) \end{array}\right\} \prec a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,æ), \\ S_b(\varkappa,\varkappa,œ) \end{array}\right\} a.
\end{aligned}$$

Now consider the set
$\mathfrak{B} = \{s \in [0,1] : \ell = \mathfrak{H}_b(\ell,\varkappa,s), \varkappa = \mathfrak{H}_b(\varkappa,\ell,s)$ for some $\ell,\varkappa \in \mathfrak{U}\}$. Suppose s is a limit point of $\mathfrak{B}$. Then there exists a $\{s_p\}$ in $\mathfrak{B}$ such that $s_p \to s$. Then there exist a sequences $\{\ell_p\},\{\varkappa\} \in \mathcal{G}$ such that $\ell_p = \mathfrak{H}_b(\ell_p,\varkappa_p s_p)$ and $\varkappa_p = \mathfrak{H}_b(\varkappa_p,\ell_p,s_p)$. Now we show that $\{\ell_p\},\{\varkappa\}$ are S_b-Cauchy sequences in $(\mathcal{G},\mathfrak{A},S_b)$. Suppose that $\{\ell_p\},\{\varkappa\}$ are not S_b-Cauchy sequences with respect $\mathfrak{A}$. So there exists $\epsilon \succ 0_{\mathfrak{A}}$ and monotonically increasing sequences of natural numbers $\{q_z\}$ and $\{p_z\}$ such that $p_z > q_z$,

$$S_b\left(\ell_{q_z},\ell_{q_z},\ell_{p_z}\right) \succeq \epsilon \quad S_b\left(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_z}\right) \succeq \epsilon \tag{5.1}$$

and

$$S_b\left(\ell_{q_z},\ell_{q_z},\ell_{p_{z-1}}\right) \prec \epsilon \quad S_b\left(\varkappa_{q_z},\varkappa_{q_z},\varkappa_{p_{z-1}}\right) \prec \epsilon. \tag{5.2}$$

From (5.1) and (5.2), we have

$$\begin{aligned}
\epsilon &\preceq S_b\left(\ell_{q_z},\ell_{q_z},\ell_{p_z}\right) \\
&\preceq 2\kappa S_b\left(\ell_{q_z},\ell_{q_z},\ell_{q_{z+1}}\right) + \kappa S_b\left(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\right)
\end{aligned}$$

By applying ψ on both sides, and letting $z \to \infty$ we have that

$$\psi(2\epsilon) \preceq \lim_{z\to\infty} \psi\left(2\kappa S_b\left(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\right)\right) \tag{5.3}$$

Suppose $|s - s_0| < \epsilon$ and $\ell \in \overline{S_b(\ell_0,\delta)}$, $\ell \neq \ell_0$, $\varkappa \in \overline{S_b(\varkappa_0,\delta)}$, $\varkappa \neq \varkappa_0$, then

$$\begin{aligned}
\psi\left(2\kappa S_b\left(\mathfrak{H}_b(\ell,\varkappa,s_0),\mathfrak{H}_b(\ell,\varkappa,s_0),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0)\right)\right) &\preceq \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_0), \\ S_b(\varkappa,\varkappa,\varkappa_0) \end{array}\right\} a\right) \\
&\quad -\phi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_0), \\ S_b(\varkappa,\varkappa,\varkappa_0) \end{array}\right\} a\right) \\
&\prec \psi\left(a^* \max\left\{\begin{array}{c} S_b(\ell,\ell,\ell_0), \\ S_b(\varkappa,\varkappa,\varkappa_0) \end{array}\right\} a\right)
\end{aligned}$$

Therefore,

$$\max\left\{\begin{array}{l}\|S_b\left(\mathfrak{H}_b(\ell,\varkappa,s_0),\mathfrak{H}_b(\ell,\varkappa,s_0),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0)\right)\|,\\ \|S_b\left(\mathfrak{H}_b(\varkappa,\ell,s_0),\mathfrak{H}_b(\varkappa,\ell,s_0),\mathfrak{H}_b(\varkappa_0,\ell_0,s_0)\right)\|\end{array}\right\} \begin{array}{l} < \frac{1}{\|2\kappa\|}\|a\|^2\max\left\{\begin{array}{l}\|S_b(\ell,\ell,\ell_0)\|,\\ \|S_b(\varkappa,\varkappa,\varkappa_0)\|\end{array}\right\} \\ \leq \delta. \end{array}$$

But

$$\begin{aligned} S_b\left(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0)\right) &\leq 2\kappa S_b\left(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s_0)\right)\\ &\quad +\kappa S_b\left(\mathfrak{H}_b(\ell_0,\varkappa_0,s_0),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0),\mathfrak{H}_b(\ell,\varkappa,s_0)\right)\\ &\leq 2\kappa\|M_b\||s-s_0|\\ &\quad +\kappa S_b\left(\mathfrak{H}_b(\ell_0,\varkappa_0,s_0),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0),\mathfrak{H}_b(\ell,\varkappa,s_0)\right) \end{aligned}$$

Therefore,

$$\begin{aligned} \|S_b\left(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0)\right)\| &\leq 2\|\kappa\|\|M_b\||s-s_0|+\|\kappa\|\frac{1}{\|2\kappa\|}\|a\|^2\delta\\ &< 2\|\kappa\|\|M_b\|\epsilon+\frac{\delta}{2}\\ &< 2\|\kappa\|\|M_b|\frac{\delta}{4\|M_b\|\|\kappa\|}+\frac{\delta}{2}=\delta \end{aligned}$$

Hence

$$\|S_b\left(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\ell_0\right)\leq\|S_b\left(\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell,\varkappa,s),\mathfrak{H}_b(\ell_0,\varkappa_0,s_0)\right)\|\leq\delta$$

Therefore, $\mathfrak{H}_b(\ell,\varkappa,s)\in\overline{S_b(\ell_0,\delta)}$ and similarly, we prove that $\mathfrak{H}_b(\varkappa,\ell,s)\in\overline{S_b(\varkappa_0,\delta)}$. Thus, Thus for any s, with $|s-s_0|<\epsilon$ and $s\in[0,1]$, it follows that
$\Gamma:\overline{S_b(\ell_0,\delta)}\times\overline{S_b(\ell_0,\delta)}\to\overline{S_b(\ell_0,\delta)}$ defined by $\Gamma(\ell,\varkappa)=\mathfrak{H}_b(\ell,\varkappa,s)$ satisfies all the hypothesis of the corollary (3.7). Hence Γ has a coupled fixed point. i.e
$\Gamma(\ell,\varkappa)=\ell$ for some $\ell\in\overline{S_b(\ell_0,\delta)}\subseteq\mathfrak{U}$ and $\Gamma(\varkappa,\ell)=\varkappa$ for some $\varkappa\in\overline{S_b(\varkappa_0,\delta)}\subseteq\mathfrak{U}$ therefore, $\Gamma(\ell,\varkappa)=\mathfrak{H}_b(\ell,\varkappa,s)=\ell$ and $\Gamma(\varkappa,\ell)=\mathfrak{H}_b(\varkappa,\ell,s)=\varkappa$ and hence $s\in\mathfrak{B}$. Thus $|s-s_0|<\epsilon\Rightarrow s\in\mathfrak{B}$.
But

$$\begin{aligned} &\lim_{z\to\infty}\psi\left(2\kappa S_b\left(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}\right)\right)\\ =\;&\lim_{z\to\infty}\psi\left(2\kappa S_b\left(\mathfrak{H}_b(\ell_{q_{z+1}},\varkappa_{q_{z+1}},s_{q_{z+1}}),\mathfrak{H}_b(\ell_{q_{z+1}},\varkappa_{q_{z+1}},s_{q_{z+1}}),\mathfrak{H}_b(\ell_{p_z},\varkappa_{p_z},s_{p_z})\right)\right)\\ \preceq\;&\lim_{z\to\infty}\psi\left(a^*\max\left\{\begin{array}{l}S_b(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}),\\ S_b(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})\end{array}\right\}a\right)-\lim_{z\to\infty}\phi\left(a^*\max\left\{\begin{array}{l}S_b(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}),\\ S_b(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})\end{array}\right\}a\right)\\ \prec\;&\lim_{z\to\infty}\psi\left(a^*\max\left\{\begin{array}{l}S_b(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z}),\\ S_b(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})\end{array}\right\}a\right) \end{aligned}$$

It follows that

$$\lim_{z\to\infty}(2\|\kappa\|-\|a\|^2)\max\left\{\begin{array}{l}\|S_b(\ell_{q_{z+1}},\ell_{q_{z+1}},\ell_{p_z})\|,\\ \|S_b(\varkappa_{q_{z+1}},\varkappa_{q_{z+1}},\varkappa_{p_z})\|\end{array}\right\}\leq 0.$$

Thus $\lim_{z\to\infty} S_b(\ell_{q_{z+1}}, \ell_{q_{z+1}}, \ell_{p_z}) = 0_{\mathfrak{A}}$ and $\lim_{z\to\infty} S_b(\varkappa_{q_{z+1}}, \varkappa_{q_{z+1}}, \varkappa_{p_z}) = 0_{\mathfrak{A}}$. Hence from (5.3) and by the def of ψ, we have that $\epsilon \preceq 0_{\mathfrak{A}}$ which is a contradiction. Hence $\{\ell_p\}$ and $\{\varkappa_p\}$ are a C^*-AV-S_b-CS in C^*-AV-S_bMS $(\mathcal{G}, \mathfrak{A}, S_b)$ and by the completeness of $(\mathcal{G}, \mathfrak{A}, S_b)$, there exist æ, œ $\in \mathfrak{U}$ with

$$\lim_{p\to\infty} \ell_p = æ \text{ and } \lim_{p\to\infty} \varkappa_p = œ.$$

Suppose $s_p \to s$, then $(\ell_p, \varkappa_p, s_p) \to (æ, œ, s)$. Since $\mathfrak{H}_b$ is continuous so that $\mathfrak{H}_b(\ell_p, \varkappa_p, s_p) \to \mathfrak{H}_b(æ, œ, s)$ and as well as $\mathfrak{H}_b(\varkappa_p, \ell_p, s_p) \to \mathfrak{H}_b(œ, æ, s)$. But $\mathfrak{H}_b(\ell_p, \varkappa_p, s_p) = \ell_p \to æ$ and $\mathfrak{H}_b(\varkappa_p, \ell_p, s_p) = \varkappa_p \to œ$. Therefore, we have $\mathfrak{H}_b(æ, œ, s) = æ$ and $\mathfrak{H}_b(œ, æ, s) = œ$. Hence $\mathfrak{B}$ is closed.

Now we show that $\mathfrak{B}$ is open. Let $\mathfrak{B}$ be s_0. Then, $\ell_0, \varkappa_0$ exists in $\mathfrak{U}$ such that $\ell_0 = \mathfrak{H}_b(\ell_0, \varkappa_0, s_0)$ and $\varkappa_0 = \mathfrak{H}_b(\varkappa_0, \ell_0, s_0)$. Because $\mathfrak{U}$ is open, $\delta > 0$ must exist for $S_b(\ell, \ell, \ell_0) \preceq \delta$ and $S_b(\varkappa, \varkappa, \varkappa_0) \preceq \delta$ implies that $\ell, \varkappa \in \mathfrak{U}$. Choose ϵ such that $0 < \epsilon < \frac{\delta}{4\|M_b\|\|\kappa\|}$, Then s_0 is an interior point of $\mathfrak{B}$. Hence $\mathfrak{B}$ is open. Consequently $\mathfrak{B}$ is both closed and open. Therefore, either $\mathfrak{B} = \emptyset$ or $\mathfrak{B} = [0, 1]$. Now suppose $\mathfrak{H}_b(.; s)$ has a coupled fixed point for some $s \in [0, 1]$,then $\mathfrak{B} \neq \emptyset$ so that $\mathfrak{B} = [0, 1]$. Therefore, $\mathfrak{H}_b(.; t)$ has a coupled fixed point for all $t \in [0, 1]$.

□

Conclusion

In this paper we conclude some applications to homotopy theory and integral equations by using (ψ, ϕ)-generalized weakly contractive type coupled fixed point theorems in the context of complete C^*-algebra valued S_b-metric spaces .

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