Int. J. Anal. Appl. (2025), 23:146

International Journal of Analysis and Applications

Vanishing Theorems, Support Conditions, and Boundary Problems for $\overline{\partial}$ on Weak $Z(q)$ Domains

Sayed Saber[1,2,*], **Abdullah A. Alahmari**[3]

[1] *Department of Mathematics, Faculty of Science, Al-Baha University, Al-Baha, Saudi Arabia*
[2] *Department of Mathematics and Computer Science, Faculty of Science, Beni-Suef University, Egypt*
[3] *Department of Mathematics, Faculty of Sciences, Umm Al-Qura University, Saudi Arabia*

[*] *Corresponding author:* Sayed011258@science.bsu.edu.eg

Abstract. Let X be a complex manifold of complex dimension $n \geq 2$, and let $\Omega \Subset X$ be a relatively compact domain with smooth boundary that satisfies the weak $Z(q)$-condition. Assume $\mathcal{F}$ is a holomorphic line bundle over X, and denote by $\mathcal{F}^{\otimes m}$ its m-th tensor power for some positive integer m. Provided there exists a strongly plurisubharmonic function defined in a neighborhood of the boundary $b\Omega$, it is possible to obtain solutions to the $\overline{\partial}$-equation within Ω, under support conditions, for (p,q)-forms with $q \geq 1$ taking values in $\mathcal{F}^{\otimes m}$. Additionally, we study the solvability of the boundary $\overline{\partial}_b$-problem on weak $Z(q)$-domains with smooth boundary in the setting of Kähler manifolds. Moreover, an extension theorem for $\overline{\partial}_b$-closed differential forms will be proven.

1. Introduction

The study of the $\overline{\partial}$-problem with support constraints has been a central topic in several complex variables and complex geometry. A fundamental contribution was made by Derridj [1], who utilized Carleman-type estimates to address the $\overline{\partial}$-problem for forms with exact support on domains possessing smooth plurisubharmonic defining functions. Extending this direction, Shaw [2] proved solvability for the $\overline{\partial}$-problem under support constraints on pseudoconvex domains in $\mathbb{C}^n$ with merely C^1 boundaries, highlighting a relaxation of regularity conditions.

Later developments by Cao, Shaw, and Wang [3] investigated the $\overline{\partial}$-problem with support conditions in locally Stein domains embedded within complex projective spaces, illustrating the subtle

Received: Apr. 29, 2025.

2020 *Mathematics Subject Classification.* 32F10, 32W05, 32L05, 32Q10, 32Q15, 53C55.

Key words and phrases. Compact complex manifolds; Kähler metrics; Vanishing theorems; Kodaira's embedding theorem; holomorphic line bundle.

https://doi.org/10.28924/2291-8639-23-2025-146
ISSN: 2291-8639

interactions between positivity and projective geometry. In another significant advancement, Sambou [4] treated the $\overline{\partial}$-problem for extendable currents defined over strongly q-convex or q-concave domains, applying refined cohomological and analytic techniques, see also [5–7].

Weak pseudoconvexity introduces additional challenges. Solutions to the $\overline{\partial}$-problem with support constraints on weakly q-pseudoconvex domains with C^1 boundary were studied in [8,9], and these results were extended to the setting of Stein manifolds in [10], showing the important role played by the ambient complex geometry. Saber [11] further developed this theory for E-valued differential forms on weakly pseudoconvex domains, assuming positivity conditions on the curvature of the line bundle E and applying L^2 techniques in the spirit of Hörmander's methods [12].

Parallel to the $\overline{\partial}$-problem inside domains, the tangential Cauchy-Riemann operator $\overline{\partial}_b$ on boundaries has also been extensively studied. Folland and Kohn [13] established foundational results concerning the $\overline{\partial}_b$-complex and the boundary behavior of $\overline{\partial}$-solutions, leading to applications in CR geometry. Further work on boundary extension problems can be found in the contributions of Ohsawa [14,15], where fine boundary regularity and extension properties were analyzed using techniques inspired by the theory of Levi convexity and Kodaira's vanishing theorems [16,17]. Vesentini [18] and Griffiths [19] also [20,21] developed important tools linking positivity, convexity, and cohomological vanishing, which inform the background of this work.

The objective of this paper is to extend these classical and modern results to a broader geometric setting, namely, to weak $Z(q)$ domains, a class that generalizes weak pseudoconvexity. Specifically, we address the $\overline{\partial}$-problem for (p,q)-forms with $q \geq 1$ taking values in the m-fold tensor powers $\mathcal{F}^{\otimes\tau}$ of a holomorphic line bundle E, under the assumption of the existence of a strongly plurisubharmonic function in a neighborhood of the boundary.

Our first main result establishes the solvability of the $\overline{\partial}$-equation with support constraints:

Theorem 1.1. *Let X be a complex manifold of dimension $n \geq 2$, and let $\Omega \Subset \mathcal{X}$ be a weak $Z(q)$ domain with smooth boundary. Suppose $\mathcal{F}$ is a holomorphic line bundle over X, and let $\mathcal{F}^{\otimes\tau}$ denote its m-fold tensor product, for a positive integer m. Assume there exists a strongly plurisubharmonic function defined in a neighborhood of $b\Omega$. Then, for any $\phi \in L^2_{p,q}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$, supported in $\overline{\Omega}$, with $q \geq 1$, satisfying $\overline{\partial}\phi = 0$ in the distribution sense on X, there exists a solution $u \in L^2_{p,q-1}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$, also supported in $\overline{\Omega}$, such that*

$$\overline{\partial}u = \phi$$

in the distribution sense on X.

In addition to interior results, we explore applications to the boundary theory. We study the $\overline{\partial}_b$-problem for CR forms on $b\Omega$, extending classical extension theorems by providing C^k-smooth $\overline{\partial}$-closed extensions into Ω under suitable conditions. Our approach leverages estimates and techniques related to Carleman inequalities, spectral theory, and vanishing theorems, building on methods found in works such as [22,23].

Moreover, inspired by ideas from Grauert-Lieb [20] and recent developments by Saber and collaborators [24]- [41], we obtain new solvability criteria for the $\overline{\partial}_b$-problem with Sobolev regularity on the boundary and the existence of exact support solutions. This work also connects with broader questions concerning the L^2 theory on weakly pseudoconvex and pseudoconcave domains, as studied in [42–49].

Thus, this paper aims to contribute to the growing understanding of the $\overline{\partial}$-equation and boundary problems on non-classical domains, bridging analytic techniques with complex geometric structures.
The objective of this paper is to extend classical and modern results concerning the solvability of the $\overline{\partial}$-equation and boundary $\overline{\partial}_b$-problems to the broader setting of weak $Z(q)$ domains. Specifically, the authors aim to solve the $\overline{\partial}$-equation with support constraints for (p,q)-forms (with $q \geq 1$) valued in high tensor powers of a holomorphic line bundle, assuming the existence of a strongly plurisubharmonic function near the boundary; investigate boundary solvability for the $\overline{\partial}_b$-problem on weak $Z(q)$ domains with smooth boundaries in Kähler manifolds; and establish extension theorems for $\overline{\partial}_b$-closed forms from the boundary into the domain. The overall goal is to contribute to the growing understanding of the $\overline{\partial}$-equation and boundary problems on non-classical domains, bridging analytic techniques with complex geometric structures.

The paper is organized as follows: In Section 2, we introduce preliminary concepts, including weakly $Z(q)$ domains, Hermitian metrics, and the relevant function spaces. Section 3 is devoted to proving the main solvability theorem for the $\overline{\partial}$-equation with support conditions. In Section 4, we address the $\overline{\partial}_b$-problem on the boundary and provide extension theorems for CR forms. Section 5 presents extension results for differential forms from the boundary into the interior of the domain.

2. Weakly $Z(q)$ Domains

Let X be an n-dimensional complex manifold, and let Ω denote an open subset of X with defining function ρ. Suppose E is a holomorphic line bundle over X, and let $\mathcal{F}^*$ represent its dual bundle. Consider an open cover $\{\mathcal{U}_j\}_{j\in J}$ of X such that E is trivial over each $\mathcal{U}_j$, i.e., $\pi^{-1}(\mathcal{U}_j) \simeq \mathcal{U}_j \times \mathbb{C}$. Assume that on each $\mathcal{U}_j$, local holomorphic coordinates are given by $(z_j^1, z_j^2, \dots, z_j^n)$. Let $\{e_{jk}\}$ denote the system of transition functions of E relative to this covering.

An (p,q)-form $\phi = \{\phi_j\}$ on X can be locally expressed as

$$\phi_j = \sum\nolimits'_{C_r,D_s} \phi_{j,C_r\overline{D_s}}\, dz_j^{C_r} \wedge d\overline{z_j}^{D_s},$$

where $C_r = (c_1, \dots, c_r)$ and $D_s = (d_1, \dots, d_s)$ are strictly increasing multi-indices, and $\sum'$ indicates summation over such ordered indices.

Let the Hermitian metric on X be given locally by

$$ds^2 = \sum_{\phi,\beta=1}^{n} g_{j,\phi\overline{\beta}}(z)\, dz_j^{\nu}\, d\overline{z}_j^{\beta},$$

where $g_{j,\phi\overline{\beta}}$ are smooth functions. The associated $(1,1)$-form is

$$\omega = \frac{\sqrt{-1}}{2} \sum_{\nu,\beta=1}^{n} g_{j,\phi\overline{\beta}}(z)\, dz_j^{\nu} \wedge d\overline{z}_j^{\beta}.$$

If $d\omega = 0$, the metric ds^2 is called a Kähler metric, and ω is the corresponding Kähler form. A complex manifold that admits a Kähler metric is referred to as a Kähler manifold.

Now, let $h = \{h_j\}$ denote a Hermitian metric on E relative to the cover $\{\mathcal{U}_j\}_{j\in J}$, satisfying the compatibility condition $h_j = |e_{jk}|^2 h_k$ on overlaps $\mathcal{U}_j \cap U_k$. Given integers $p, q \geq 0$ and $\tau \geq 1$, we introduce the following function spaces. We denote by $\mathscr{C}^{\infty}_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ the space of smooth (p,q)-forms on Ω valued in $\mathcal{F}^{\otimes\tau}$, and $\mathscr{C}^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ the subspace of forms smoothly extendable up to the boundary $b\Omega$. We denote by $\mathscr{D}_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ the space of smooth (p,q)-forms with compact support in Ω.

The Hodge star operator $\star$ maps $\mathscr{C}^{\infty}_{p,q}(X, \mathcal{F}^{\otimes\tau})$ into $\mathscr{C}^{\infty}_{n-s,n-r}(X, \mathcal{F}^{\otimes\tau})$. The conjugation operator $\#_{\mathcal{F}^{\otimes\tau}}$ maps $\mathscr{C}^{\infty}_{p,q}(X, \mathcal{F}^{\otimes\tau})$ to $\mathscr{C}^{\infty}_{q,p}(X, \mathcal{F}^{*\otimes\tau})$ and is given by

$$\#_{\mathcal{F}^{\otimes\tau}}\phi = h^{\tau}\overline{\phi}.$$

It commutes with the Hodge star operator. Similarly, we define

$$\#_{\mathcal{F}^{*\otimes\tau}} : \mathscr{C}^{\infty}_{p,q}(X, \mathcal{F}^{*\otimes\tau}) \longrightarrow \mathscr{C}^{\infty}_{q,p}(X, \mathcal{F}^{\otimes\tau})$$

by

$$\#_{\mathcal{F}^{*\otimes\tau}}\phi = h^{-m}\overline{\phi},$$

and note that $\#_{\mathcal{F}^{*\otimes\tau}}$ is the inverse of $\#_{\mathcal{F}^{\otimes\tau}}$.

We define

$$\mathscr{B}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau}) = \left\{\phi \in \mathscr{C}^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})\,;\ \star\#_{\mathcal{F}^{\otimes\tau}}\phi\big|_{b\Omega} = 0\right\}.$$

Also, dV will denote the volume element corresponding to the Hermitian metric ds^2.

The Cauchy-Riemann operator

$$\overline{\partial} : \mathscr{C}^{\infty}_{p,q-1}(\Omega, \mathcal{F}^{\otimes\tau}) \longrightarrow \mathscr{C}^{\infty}_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$$

is defined in the standard way. Its formal adjoint is denoted by ϑ_{τ}. $\ker(\overline{\partial}, \mathcal{F}^{\otimes\tau})$ for the kernel of $\overline{\partial}$, $\mathrm{dom}(\overline{\partial}, \mathcal{F}^{\otimes\tau})$ for its domain, and $\mathrm{range}(\overline{\partial}, \mathcal{F}^{\otimes\tau})$ for its range. The Dolbeault cohomology group is defined as

$$H^{p,q}(X, \mathcal{F}^{\otimes\tau}) = \frac{\mathscr{C}^{\infty}_{p,q}(X, \mathcal{F}^{\otimes\tau}) \cap \ker(\overline{\partial}, \mathcal{F}^{\otimes\tau})}{\overline{\partial}(\mathscr{C}^{\infty}_{p,q-1}(X, \mathcal{F}^{\otimes\tau}))}.$$

On the boundary $b\Omega$, we consider the quotient space

$$\mathscr{C}^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau}) = \mathscr{C}^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau}) / \mathscr{D}_{p,q}(\Omega, \mathcal{F}^{\otimes\tau}),$$

and the natural projections

$$\pi_{p,q} : \mathscr{C}^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau}) \longrightarrow \mathscr{C}^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau}),$$

and

$$\sigma_{p,q} : \bigoplus_{p,q} \mathscr{C}^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau}) \longrightarrow \mathscr{C}^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau}).$$

We shall simply denote by $u|_{b\Omega}$ the projection $\pi_{p,q}(u)$.

The tangential Cauchy-Riemann operator

$$\overline{\partial}_b : \mathscr{C}^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau}) \longrightarrow \mathscr{C}^{\infty}_{r,s+1}(b\Omega, \mathcal{F}^{\otimes\tau})$$

is defined by

$$\overline{\partial}_b = \sigma_{r,s+1} \circ d \circ (\pi_{p,q})^{-1}.$$

Functions f on $b\Omega$ satisfying $\overline{\partial}_b f = 0$ are called CR functions. A function f is CR if and only if there exists a smooth extension F on $\overline{\Omega}$ such that $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$.

For smooth sections $\phi, u \in \mathscr{C}^{\infty}_{p,q}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$, the pointwise inner product $(\phi, u)_\tau$ is defined by

$$(\phi, u)_\tau dV = \phi_j \wedge \star\, h^\tau \overline{\mathcal{U}_j} = \phi_j \wedge \star \#_{\mathcal{F}^{\otimes\tau}} \mathcal{U}_j.$$

The global inner product is given by

$$\langle \phi, u \rangle_{\tau,\Omega} = \int_\Omega \phi \wedge \star \#_{\mathcal{F}^{\otimes\tau}} u,$$

and the corresponding norm is

$$\|\phi\|^2_{\tau,\Omega} = \langle \phi, \phi \rangle_{\tau,\Omega}.$$

Finally, for $\phi \in \mathscr{C}^{\infty}_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ and $\eta \in \mathscr{D}_{p,q-1}(\Omega, \mathcal{F}^{\otimes\tau})$, the formal adjoint operator ϑ_τ corresponding to $\overline{\partial}$ is defined as usual.

Definition 2.1. *Let $\pi : E \longrightarrow X$ be a holomorphic line bundle. We say that E is* positive *over a subset $\Omega \subseteq X$ if there exists a collection of coordinate charts $\{\mathcal{U}_j\}_{j\in J}$ covering X such that $\pi^{-1}(\mathcal{U}_j)$ are trivial bundles, together with a Hermitian metric $h = \{h_j\}$ defined along the fibres of E, satisfying that $-\log h_j$ is strictly plurisubharmonic on each $\mathcal{U}_j \cap \Omega$ for all $j \in J$.*

Utilizing the framework of complex tensor calculus for Kähler manifolds with boundary, we arrive at the following important result (refer to [15]).

Proposition 2.1. *Suppose X is a Kähler manifold of complex dimension n, and let $\Omega \Subset \mathcal{X}$ be a relatively compact open subset. Consider a holomorphic line bundle E over X, and denote by $\mathcal{F}^{\otimes\tau}$ its τ-fold tensor product for some positive integer τ. Let U^* be a neighborhood of the boundary $b\Omega$, and denote by $\overline{\nabla}$ the covariant derivative induced by the Kähler metric ds^2. Then, for all $\tau \geq 1$ and for any $\phi \in \mathscr{B}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ with $\operatorname{supp}\phi \Subset U^*$, where $p \geq 0$ and $q \geq 1$, the following identity holds:*

$$\begin{aligned} \|\overline{\partial}\phi\|^2_\tau + \|\overline{\partial}^*_\tau\phi\|^2_\tau = \|\overline{\nabla}\phi\|^2_\tau &+ \int_{b\Omega} h^\tau_j |\nabla\rho|^{-1} \sum_{\beta,\gamma=1}^{n} \frac{\partial^2\rho}{\partial z^\beta \partial \overline{z}^\gamma}\, \phi^\beta_{jC_r\overline{B}_{s-1}} \overline{\phi_j^{\overline{C}_r \gamma B_{s-1}}}\, dS \\ &+ \int_\Omega h^\tau_j \sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau \left[m\Theta^{\overline{\beta}}_{\overline{\phi}} + R^{\overline{\beta}}_{\overline{\phi}}\right] - R^{\sigma\overline{\beta}}_{\tau\overline{\phi}}\right) \phi^\beta_{jC_r\overline{B}_{s-1}} \overline{\phi_j^{\overline{C}_r \gamma B_{s-1}}}\, dV. \end{aligned} \tag{2.1}$$

Here, the various terms are defined as follows:

$$\|\overline{\nabla}\phi\|_{\tau}^{2} = \int_{\Omega}\sum_{\phi,\beta=1}^{n} g_{j}^{\overline{\beta}\phi}\overline{\nabla}_{\beta}\phi_{jC_r\overline{D}_s}\overline{\overline{\nabla}_{\phi}\phi_{j}^{\overline{C}_rD_s}}\,dV,$$

$$R_{\beta\overline{\nu}\gamma}^{\phi} = -\frac{\partial}{\partial\overline{z}_{j}^{\nu}}\left(\sum g_{j}^{\overline{\sigma}\phi}\frac{\partial}{\partial z_{j}^{\gamma}}g_{j\beta\overline{\sigma}}\right) \quad \textit{(Riemann curvature tensor)},$$

$$R_{\phi\overline{\nu}} = -\frac{\partial^2}{\partial z_{j}^{\nu}\partial\overline{z}_{j}^{\nu}}\left(\log\det(g_{j\phi\overline{\beta}})\right) \quad \textit{(Ricci curvature tensor)},$$

$$\Theta_{\phi\overline{\nu}} = -\frac{\partial^2}{\partial z_{j}^{\nu}\partial\overline{z}_{j}^{\nu}}(\log h) \quad \textit{(curvature tensor of E)},$$

$$\delta_{\tau}^{\sigma} \quad \textit{(Kronecker delta symbol)}.$$

For the C^{∞}-function λ, we define the gradient of λ as the vector

$$\operatorname{grad}\lambda = \left(\frac{\partial\lambda}{\partial z^1},\dots,\frac{\partial\lambda}{\partial z^n},\overline{\frac{\partial\lambda}{\partial z^1}},\dots,\overline{\frac{\partial\lambda}{\partial z^n}}\right),$$

$$|\operatorname{grad}\lambda|^2 = (\operatorname{grad}\lambda)\,\overline{(\operatorname{grad}\lambda)} = \sum_{\phi=1}^{n}\left|\frac{\partial\lambda}{\partial z^{\phi}}\right|^2 + \sum_{\beta=1}^{n}\left|\overline{\frac{\partial\lambda}{\partial z^{\beta}}}\right|^2,$$

and we set

$$(\mathscr{L}(\lambda)\phi,\phi) = \sum_{B_{s-1}}\sum_{\beta,\gamma=1}^{n}\frac{\partial^2\lambda}{\partial z^{\beta}\partial z^{\overline{\gamma}}}\,\phi_{\overline{B}_{s-1}}^{\beta}\,\overline{\phi^{\gamma B_{s-1}}}. \tag{2.2}$$

Since $d\lambda \neq 0$ on U, then $\operatorname{grad}\lambda \neq 0$ on U also. The following lemma which is theorem 1.1.3 of [12].

Lemma 2.1. *Let $H_i (i = l, 2, 3)$ be three Hilbert spaces and*

$$T : H_1 \longrightarrow H_2 \quad \textit{and} \quad S : H_2 \longrightarrow H_3,$$

be closed linear operators with dense domains such that $ST = 0$. Assume that for any sequence $\{f_{\nu}\}$ such that $f_{\nu} \in H_2 \cap dom\, S \cap dom\, T$,

$$\|\phi_{\nu}\|_{H_2}^2 \leqslant 1 \quad \textit{and} \quad \lim_{\nu\to\infty}\|S\phi_{\nu}\|_{H_3}^2 = 0, \quad \lim_{\nu\to\infty}\|T\phi_{\nu}\|_{H_1}^2 = 0,$$

we can choose a strongly convergent subsequence of $\{f_{\nu}\}$. Then $\mathcal{R}ange(T)$ is closed and $\mathcal{H}(S)/\mathcal{R}ange(T)$ is a finite dimensional vector space.

Definition 2.2. *Let $\Omega \subset \mathbb{C}^n$ be a domain with C^{τ} boundary $\mathrm{b}\Omega$. We say that a defining function ρ for Ω is* uniformly C^{τ} *if there exists an open neighborhood U of $\mathrm{b}\Omega$ such that:*

- $\operatorname{dist}(\mathrm{b}\Omega, \mathrm{b}U) > 0$,
- $\|\rho\|_{C^{\tau}(U)} < \infty$,
- $\inf_U |\nabla\rho| > 0$.

This condition is trivial on domains with compact boundary. We identify real $(1,1)$-forms with Hermitian matrices as follows:

$$c = \sum_{j,k=1}^{n} ic_{j\bar{k}}\, dz_j \wedge d\bar{z}_k.$$

For a function ϕ, we set:

$$\phi_k = \frac{\partial \phi}{\partial z_k}, \quad \phi_{\bar{j}} = \frac{\partial \phi}{\partial \bar{z}_j}.$$

Let $\mathcal{I}_q = \{(i_1, \ldots, i_q) \in \mathbb{N}^n : 1 \le i_1 < \cdots < i_q \le n\}$. For $I \in \mathcal{I}_{q-1}$, $J \in \mathcal{I}_q$, and $1 \le j \le n$, define:

$$\epsilon^{jI}_J = \begin{cases} (-1)^{|\sigma|}, & \text{if } \{j\} \cup I = J \text{ as sets, and } |\sigma| \text{ is the permutation length,} \\ 0, & \text{otherwise.} \end{cases}$$

For $u = \sum_{J \in \mathcal{I}_q} \mathcal{U}_J d\bar{z}_J$, define:

$$u_{jI} = \sum_{J \in \mathcal{I}_q} \epsilon^{jI}_J u_J.$$

The induced CR-structure at $z \in \mathrm{b}\Omega$ is:

$$T^{1,0}_z(\mathrm{b}\Omega) = \{L \in T^{1,0}(\mathbb{C}^n) : \partial\rho(L) = 0\}.$$

Let $T^{1,0}(\mathrm{b}\Omega)$ be the space of C^{m-1} sections of $T^{1,0}_z(\mathrm{b}\Omega)$, and set $T^{0,1}(\mathrm{b}\Omega) = \overline{T^{1,0}(\mathrm{b}\Omega)}$. The exterior algebra generated by these spaces is denoted $T^{p,q}(\mathrm{b}\Omega)$. For U a suitably small neighborhood of $\mathrm{b}\Omega$, define the projection

$$\tau : \Lambda^{p,q}(U) \longrightarrow \Lambda^{p,q}(b\Omega).$$

If we normalize ρ so that $|d\rho| = 1$ on $\mathrm{b}\Omega$, then the Levi form $\mathscr{L}$ is defined by

$$\mathscr{L}(-iL \wedge \bar{L}) = i\partial\bar{\partial}\rho(-iL \wedge \bar{L}),$$

for any $L \in T^{1,0}(\mathrm{b}\Omega)$.

Definition 2.3. *Given a set $M \subset \mathbb{C}^n$, a* tubular neighborhood *of M is an open set of the form*

$$U_r = \{p \in \mathbb{C}^n : \mathrm{dist}(p, M) < r\},$$

where $\mathrm{dist}(\cdot,\cdot)$ *denotes the Euclidean distance. We call r the* radius *of U_r.*

We adopt the definition of weak $Z(q)$ from [12].

Definition 2.4. *Let $\Omega \Subset \mathcal{X}$ be a domain with a uniformly C^τ defining function ρ, $\tau \ge 2$. We say* $\mathrm{b}\Omega$ *(or Ω) satisfies weak $Z(q)$ if there exists a Hermitian matrix $\Upsilon = (\Upsilon^{\bar{k}j})$ of functions on* $\mathrm{b}\Omega$*, uniformly bounded in $C^{\tau-1}$, such that:*

1. $\sum_{j=1}^{n} \Upsilon^{\bar{k}j}\rho_j = 0$ *on* $\mathrm{b}\Omega$*;*
2. *All eigenvalues of* Υ *lie in* $[0,1]$*;*
3. $\mu_1 + \cdots + \mu_q - \sum_{j,k=1}^{n} \Upsilon^{\bar{k}j}\rho_{j\bar{k}} \ge 0$*, where $\mu_1, \ldots, \mu_{n-1}$ are the eigenvalues of the Levi form $\mathscr{L}$ in increasing order;*

4. $\inf_{z\in b\Omega} |q - \mathrm{Tr}(\mathrm{Y})| > 0.$

Definition 2.5. *We say that* $u \in L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ *is supported in* $\overline{\Omega}$ *(supp* $u \subset \overline{\Omega}$*) or* u *vanishes to infinite order at the boundary of* Ω *if* u *vanishes on* $b\Omega$*.*

Definition 2.6. $\phi \in L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ *is supported in* $\overline{\Omega}$ *(supp* $\phi \subset \overline{\Omega}$*) or* ϕ *vanishes to infinite order at the boundary of* Ω *if* ϕ *vanishes on* $b\Omega$*.*

To prove the basic estimate (3.6), the following lemma which is Theorem 1.1.3 of [12] is needed.

Lemma 2.2. *Let* $H_j (j = l, 2, 3)$ *be three Hilbert spaces and*

$$T : H_1 \longrightarrow H_2 \quad \textit{and} \quad S : H_2 \longrightarrow H_3$$

be closed linear operators with dense domains such that $ST = 0$*. Assume that for any sequence* $\{f_\nu\}$ *such that* $f_\nu \in H_2 \cap dom\, S \cap dom\, T$*,*

$$\|\phi_\nu\|^2_{H_2} \leqslant 1 \quad \textit{and} \quad \lim_{\nu\longrightarrow\infty} \|\mathbb{S}\phi_\nu\|^2_{H_3} = 0, \quad \lim_{\nu\longrightarrow\infty} \|\mathbb{T}\phi_\nu\|^2_{H_1} = 0,$$

one can choose a strongly convergent subsequence of $\{f_\nu\}$*. Then range*(T) *is closed and* $\mathcal{H}(S)/range(T)$ *is a finite dimensional vector space.*

3. Proof of Theorem 1

Let X be an n-dimensional complex manifold and $\Omega \Subset \mathcal{X}$ a relatively compact domain with smooth boundary $b\Omega$ satisfying the weak $Z(q)$ condition. Assume $E \to X$ is a holomorphic line bundle that is positive in a neighborhood V of $b\Omega$. Let $h = \{h_j\}$ denote a Hermitian metric for E on X that ensures the positivity over V with respect to an open covering $\{\mathcal{U}_j\}_{j\in J}$ of X. Then, the curvature form

$$\sum_{\phi,\beta=1}^{n} \left(-\frac{\partial^2 \log h_j}{\partial z_j^\phi \partial \overline{z}_j^\beta} \right) dz^\phi \wedge d\overline{z}^\beta,$$

induces a Kähler metric given locally by

$$d\sigma^2 = \sum_{\phi,\beta=1}^{n} \left(-\frac{\partial^2 \log h_j}{\partial z_j^\phi \partial \overline{z}_j^\beta} \right) dz^\phi d\overline{z}^\beta,$$

on V. One can choose a defining function ρ for $b\Omega$ based on the geodesic distance corresponding to the metric $d\sigma^2$. This leads to the following result.

Lemma 3.1. *There exist neighborhoods* V*,* V' *of* $b\Omega$*, an open covering* $\{\mathcal{U}_j\}_{j\in J}$ *of* X*, a Hermitian metric* $h = \{h_j\}$ *on* E*, and a Hermitian metric*

$$ds^2 = \sum_{\phi,\beta=1}^{n} g_{j\phi\overline{\beta}}(z) dz_j^\phi d\overline{z}_j^\beta$$

on X *satisfying:*

(1) $V \Subset V'$*, with* $\overline{V'}$ *contained in a smooth tubular neighborhood of* $b\Omega$*;*

(2) *$\pi^{-1}(\overline{\mathcal{U}_j})$ is trivial for every $j \in J$, and if $\mathcal{U}_j \cap b\Omega \neq \emptyset$, then $\mathcal{U}_j \Subset V$;*
(3) *The bundle E remains positive over V' relative to the metric h;*
(4) *On V', the Hermitian metric ds^2 agrees with the Kähler metric $d\sigma^2$.*

In the context of Lemma 1, we have the following key estimate (compare with Appendix II in [51–53]).

Proposition 3.1. *There exist a constant $C > 0$ independent of m, and an integer $\tau_0 > 0$, such that for all integers $\tau \geq \tau_0$, and for $p \geq 0$, $q \geq 1$, we have*

$$\|\overline{\nabla}\phi\|^2_{\tau,\Omega\setminus K} + (\tau - \tau_0)\|\phi\|^2_{\tau,\Omega\setminus K} \leq C\left(\|\overline{\partial}\phi\|^2_{\tau,\Omega} + \|\overline{\partial}^*_\tau\phi\|^2_{\tau,\Omega} + \|\phi\|^2_{\tau,K}\right), \tag{3.1}$$

where $K = \Omega \setminus (\Omega \cap V)$ and $\overline{\nabla}$ denotes the $(0,1)$-type covariant derivative associated with ds^2.

Proof. Adopting the setting from Lemma 1, let $\chi \in C^\infty(X)$ satisfy $\mathrm{supp}(\chi) \Subset V'$ and $\chi = 1$ on $\overline{V}$. Applying the basic L^2-estimate (equation (2.2)) to $\chi\phi$ and noting that the third term on the right-hand side is non-negative due to the weak $Z(q)$ condition for $q \geq 1$, yields

$$\|\overline{\nabla}(\chi\phi)\|^2_\tau + \int_\tau h^\tau \sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau[\tau\Theta^{\overline{\beta}}_{\overline{\phi}} + R^{\overline{\beta}}_{\overline{\phi}}] - rR^{\sigma\overline{\beta}}_{\tau\overline{\phi}}\right) \times (\chi\phi)^{\beta}_{j,C_p\overline{B}_{s-1}} \overline{(\chi\phi)^{\overline{C}_p\,\gamma B_{s-1}}_j}\, dV \leq \|\overline{\partial}(\chi\phi)\|^2_\tau + \|\overline{\partial}^*_\tau(\chi\phi)\|^2_\tau. \tag{3.2}$$

Since the first integral is non-negative over V', we derive

$$\|\overline{\nabla}\phi\|^2_{\tau,\Omega\setminus K} \leq \|\overline{\nabla}(\chi\phi)\|^2_\tau. \tag{3.3}$$

Recalling that in V', the Hermitian metric matrix $g_{j\phi\overline{\beta}}$ coincides with the curvature matrix $\Theta_{\phi\overline{\beta}}$, it follows that

$$\Theta^{\overline{\beta}}_{\overline{\phi}} = \sum_{\gamma=1}^{n} g^{\overline{\beta}\gamma}_j \Theta_{\gamma\overline{\phi}} = \delta^\beta_\phi.$$

Moreover, on $\mathrm{supp}(\chi)$, there exists a constant $C > 0$, independent of m, ensuring that the Hermitian form

$$\sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau R^{\overline{\beta}}_{\overline{\phi}} - rR^{\sigma\overline{\beta}}_{\tau\overline{\phi}}\right)(\chi\phi)_{j,\sigma C_{r-1}\overline{\beta D}_{s-1}} \overline{(\chi\phi)^{\overline{\tau C}_{r-1}\phi D_{s-1}}_j},$$

is bounded below by

$$-C\sum (\chi\phi)_{j,C_r\overline{D}_s}\overline{(\chi\phi)^{\overline{C}_r D_s}_j}.$$

Setting $\tau_0 = [C] + 1$, it follows for all $\tau \geq \tau_0$ that

$$\begin{aligned}(\tau - \tau_0)\|\phi\|^2_{\tau,\Omega\setminus K} \leq (\tau - \tau_0)\|\chi\phi\|^2_\tau \leq \int_\tau h^\tau \sum_{\beta,\gamma=1}^{n} s\left(\delta^\sigma_\tau[\tau\Theta^{\overline{\beta}}_{\overline{\phi}} + R^{\overline{\beta}}_{\overline{\phi}}] - rR^{\sigma\overline{\beta}}_{\tau\overline{\phi}}\right)\\ \times (\chi\phi)^{\beta}_{jC_r\overline{B}_{s-1}} \overline{(\chi\phi)^{\overline{C}_r\,\gamma B_{s-1}}_j}\, dV.\end{aligned} \tag{3.4}$$

Furthermore, we estimate

$$\begin{aligned}\|\bar\partial(\chi\phi)\|_\tau^2+\|\bar\partial_\tau^*(\chi\phi)\|_\tau^2 &\le 2\left(\|\bar\partial\chi\wedge\phi\|_\tau^2+\|\bar\partial\chi\wedge\star\phi\|_\tau^2+\|\chi\bar\partial\phi\|_\tau^2+\|\chi\bar\partial_\tau^*\phi\|_\tau^2\right)\\ &\le C\left(\|\bar\partial\phi\|_\tau^2+\|\bar\partial_\tau^*\phi\|_\tau^2+\|\phi\|_{\tau,\Omega\setminus K}^2\right),\end{aligned} \tag{3.5}$$

where $C \geq 4\max\{l, c_0 \sup|\mathrm{grad}\,\chi|_{ds^2}(x)\}$ and c_0 depends only on $\dim X$. Substituting (3.3), (3.4), and (3.5) into (3.2), we complete the proof. □

Proposition 3.2. *There exists a positive constant τ^* such that for every $\tau \geqslant \tau^*$, the space of harmonic forms $\mathcal{H}_{p,q}^\tau(\mathcal{F}^{\otimes\tau})$ is finite-dimensional, and there exists a constant $C_\tau > 0$, depending on τ, satisfying*

$$\|\phi\|_\tau^2 \leqslant C_\tau\left(\|\bar\partial\phi\|_\tau^2+\|\bar\partial_\tau^*\phi\|_\tau^2\right), \tag{3.6}$$

for all $\phi \in \mathtt{Dom}(\bar\partial,\mathcal{F}^{\otimes\tau}) \cap \mathtt{Dom}(\bar\partial_\tau^*,\mathcal{F}^{\otimes\tau})$ whenever $q \geqslant 1$.

Proof. Let τ_0, C, and K be as specified in Proposition 3. Define $\tau^* = \tau_0 + 1$. Following a similar argument as in Proposition 3, let χ be a smooth, real-valued function compactly supported in X, with $\chi = 1$ on K. For $\tau \geqslant \tau^*$ and $\phi \in \mathscr{B}_{p,q}(\overline{\Omega},\mathcal{F}^{\otimes\tau})$, applying (3.1) yields

$$\|\phi\|_\tau^2 \leqslant C_\tau\left(\|\bar\partial\phi\|_\tau^2+\|\bar\partial_\tau^*\phi\|_\tau^2+\|\chi\phi\|_\tau^2\right),$$

where C_τ depends only on m.

Now, consider a sequence $\{\phi_\nu\}$ where each $\phi_\nu \in \mathtt{Dom}\,\bar\partial \cap \mathtt{Dom}\,\bar\partial_\tau^*$, satisfying $\|\phi_\nu\|_\tau^2 \leqslant 1$ and

$$\lim_{\nu\to\infty}\|\bar\partial\phi_\nu\|_\tau^2 = 0, \quad \lim_{\nu\to\infty}\|\bar\partial_\tau^*\phi_\nu\|_\tau^2 = 0.$$

According to Lemma 2, there exists a subsequence $\{\phi_{\nu_k}\}$ that converges strongly on Ω.

Since the metric ds^2 is complete and the space $\mathscr{D}_{p,q}(\Omega,\mathcal{F}^{\otimes\tau})$ is dense in $\mathtt{Dom}\,\bar\partial \cap \mathtt{Dom}\,\bar\partial_\tau^*$ with respect to the norm

$$\|\phi\|_\tau^2+\|\bar\partial\phi\|_\tau^2+\|\bar\partial_\tau^*\phi\|_\tau^2$$

(see [17], Theorem 1.1), we can assume that $\chi\phi_\nu \in \mathscr{D}_{p,q}(\Omega,\mathcal{F}^{\otimes\tau})$. Thus,

$$\|\bar\partial(\chi\phi_\nu)\|_\tau^2+\|\bar\partial_\tau^*(\chi\phi_\nu)\|_\tau^2+\|\chi\phi_\nu\|_\tau^2 = \langle\Box^\tau(\chi\phi_\nu),\chi\phi_\nu\rangle_\tau+\langle\chi\phi_\nu,\chi\phi_\nu\rangle_\tau$$

is bounded, due to the properties of $\{\phi_\nu\}$.

Since the elliptic operator $\Box^\tau$ is coercive on $\mathscr{D}_{p,q}(\Omega,\mathcal{F}^{\otimes\tau})$ ([4], Theorem (2.2.1)), and using Rellich's compactness lemma ([4], Appendix A.1.6), we deduce that there is a subsequence $\{\phi_{\nu_k}\}$ converging strongly on compact subsets of Ω. From estimate (3.1), we conclude that $\{\phi_{\nu_k}\}$ converges strongly throughout Ω. Therefore, by Hörmander's Theorem 1.1.2 and 1.1.3 ([12]), there exists a constant $C_\tau > 0$ such that

$$\|\phi\|_\tau^2 \leqslant C_\tau\left(\|\bar\partial\phi\|_\tau^2+\|\bar\partial_\tau^*\phi\|_\tau^2\right), \tag{3.7}$$

for all $\phi \in \mathtt{Dom}(\bar\partial,\mathcal{F}^{\otimes\tau}) \cap \mathtt{Dom}(\bar\partial_\tau^*,\mathcal{F}^{\otimes\tau})$ orthogonal to $\mathcal{H}_{p,q}^\tau(\mathcal{F}^{\otimes\tau})$.

Furthermore, any $\phi \in \mathcal{H}^{\tau}_{p,q}(\mathcal{F}^{\otimes\tau})$ satisfies $\Box^{\tau}\phi = 0$, meaning ϕ is a harmonic form with values in $\mathcal{F}^{\otimes\tau}$. Since ϕ vanishes outside K by (3.1), and as no connected component of Ω is contained in K, the unique continuation property ensures that ϕ must vanish identically. Thus,

$$\mathcal{H}^{\tau}_{p,q}(\mathcal{F}^{\otimes\tau}) = \{0\}.$$

Combining this with (3.7), the proposition is proved. □

Remark 3.1. *Suppose there exists a strongly plurisubharmonic function ϕ defined on a neighborhood V of $b\Omega$. Then any line bundle E becomes positive over a relatively compact neighborhood of $b\Omega$. To see this, let h be a Hermitian metric on E over X and extend ϕ smoothly to X, ensuring it agrees with the original near $b\Omega$. Then, for some integer $\tau^* > 0$, the modified metric $h_\tau = h\mathcal{F}^{-\tau\Phi}$ endows E with positivity over a relatively compact subset $V' \Subset V$ for all $\tau \geqslant \tau^*$.*

Remark 3.2. *It should be noted that there exist pseudoconvex domains with smooth boundary $b\Omega$ where no strongly plurisubharmonic function exists near $b\Omega$, yet there still exists a line bundle that is positive in a neighborhood of $b\Omega$ (cf. [50]).*

Theorem 3.1. *Let X be an n-dimensional complex manifold , and let $\Omega \Subset \mathcal{X}$ be a weak $Z(q)$-domain with a smooth boundary. Assume E is a holomorphic line bundle over X, and denote by $\mathcal{F}^{\otimes\tau}$ the m-fold tensor product of E for each positive integer m. Suppose that a strongly plurisubharmonic function exists in a neighborhood of $\partial\Omega$. Then, there exists a positive integer τ_0 such that for all $\tau \geq \tau_0$, $p \geq 0$, and $q \geq 1$, one can construct a bounded linear operator $\mathcal{N}^{\tau} : L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau}) \to L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$ satisfying:*

(i) $\mathrm{Range}(\mathcal{N}^{\tau}) \subset \mathtt{Dom}(\Box^{\tau})$ *and* $\mathcal{N}^{\tau}\Box^{\tau} = I - \Pi^{\tau}$ *on* $\mathtt{Dom}(\Box^{\tau})$*;*

(ii) *for any $\phi \in L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$, the following Hodge-type decomposition holds:*

$$\phi = \overline{\partial}\,\overline{\partial}^*_\tau \mathcal{N}^{\tau}\phi \oplus \overline{\partial}^*_\tau\,\overline{\partial}\mathcal{N}^{\tau}\phi \oplus \Pi^{\tau}\phi;$$

(iii) $\mathcal{N}^{\tau}\overline{\partial} = \overline{\partial}\mathcal{N}^{\tau}$ *on* $\mathtt{Dom}(\overline{\partial})$*;*

(iv) $\mathcal{N}^{\tau}\overline{\partial}^*_\tau = \overline{\partial}^*_\tau\mathcal{N}^{\tau}$ *on* $\mathtt{Dom}(\overline{\partial}^*_\tau)$*;*

(v) *the operators $\mathcal{N}^{\tau}$, $\overline{\partial}\mathcal{N}^{\tau}$, and $\overline{\partial}^*_\tau\mathcal{N}^{\tau}$ are bounded on $L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$.*

Proof. From estimate (3.6), it follows that

$$\|\phi\|_\tau \leq C_\tau \|\Box^{\tau}\phi\|_\tau, \tag{3.8}$$

for all $\phi \in \mathtt{Dom}(\overline{\partial}) \cap \mathtt{Dom}(\overline{\partial}^*_\tau)$ with $q \geq 1$. Since $\Box^{\tau}$ is densely defined, closed, and linear, it follows by [12] that $\mathrm{Range}(\Box^{\tau})$ is closed.

Moreover, because $\Box^{\tau}$ is self-adjoint, the standard Hodge decomposition yields:

$$L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau}) = \overline{\partial}\,\overline{\partial}^*_\tau(\mathtt{Dom}(\Box^{\tau})) \oplus \overline{\partial}^*_\tau\,\overline{\partial}(\mathtt{Dom}(\Box^{\tau})).$$

Since

$$\Box^{\tau} : \mathrm{Dom}(\Box^{\tau}) \longrightarrow \mathrm{Range}(\Box^{\tau})$$

is bijective, there exists a bounded inverse

$$\mathcal{N}^\tau : L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau}) \longrightarrow \mathrm{Dom}(\square^\tau)$$

such that $\mathcal{N}^\tau\square^\tau\phi = \phi$ for all $\phi \in \mathtt{Dom}(\square^\tau)$. Furthermore, by definition, we obtain $\square^\tau\mathcal{N}^\tau = I$ on $L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$. Hence properties (i) and (ii) are verified.

To verify (iv), take $\phi \in \mathtt{Dom}(\overline{\partial}^*_\tau)$. Using (ii), we express:

$$\overline{\partial}^*_\tau\phi = \overline{\partial}^*_\tau\overline{\partial}\overline{\partial}^*_\tau\mathcal{N}^\tau\phi,$$

thus,

$$\mathcal{N}^\tau\overline{\partial}^*_\tau\phi = \mathcal{N}^\tau\overline{\partial}^*_\tau\overline{\partial}\overline{\partial}^*_\tau\mathcal{N}^\tau\phi.$$

Since $\square^\tau = \overline{\partial}^*_\tau\overline{\partial} + \overline{\partial}\overline{\partial}^*_\tau$, we rewrite:

$$\mathcal{N}^\tau\overline{\partial}^*_\tau\phi = \overline{\partial}^*_\tau\mathcal{N}^\tau\phi.$$

The same method shows that $\mathcal{N}^\tau\overline{\partial} = \overline{\partial}\mathcal{N}^\tau$ on $\mathtt{Dom}(\overline{\partial})$.

Now, given that $\overline{\partial}\phi = 0$, applying (iii) leads to

$$\overline{\partial}\mathcal{N}^\tau\phi = \mathcal{N}^\tau\overline{\partial}\phi = 0.$$

Applying (ii) yields

$$\phi = \overline{\partial}\,\overline{\partial}^*_\tau\mathcal{N}^\tau\phi,$$

which implies that $u = \overline{\partial}^*_\tau\mathcal{N}^\tau\phi$ solves $\overline{\partial}u = \phi$.

Finally, since $\mathrm{Range}(\mathcal{N}^\tau) \subset \mathtt{Dom}(\square^\tau)$, applying (3.6) to $\mathcal{N}^\tau\phi$ gives:

$$\|\mathcal{N}^\tau\phi\|_\tau \leq C_\tau\|\phi\|_\tau,$$

$$\|\overline{\partial}\mathcal{N}^\tau\phi\|_\tau + \|\overline{\partial}^*_\tau\mathcal{N}^\tau\phi\|_\tau \leq 2\sqrt{C_\tau}\|\phi\|_\tau.$$

Thus, all stated properties are proved. □

Theorem 3.2. *Assume the hypotheses of Theorem 2 are satisfied. Let $\phi \in L^2_{p,q}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$ be a form such that $\mathrm{supp}(\phi) \subset \overline{\Omega}$, with $q \geq 1$, and $\overline{\partial}\phi = 0$ in the sense of distributions on X. Then, there exists a form $u \in L^2_{p,q-1}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$, supported in $\overline{\Omega}$, such that*

$$\overline{\partial}u = \phi$$

in the sense of distributions on X.

Proof. Let $\phi \in L^2_{p,q}(\mathcal{X}, \mathcal{F}^{\otimes\tau})$ with $\mathrm{supp}(\phi) \subset \overline{\Omega}$. Clearly, ϕ can be viewed as an element of $L^2_{p,q}(\Omega, \mathcal{F}^{\otimes\tau})$. By Theorem 2, the solution operator $\mathcal{N}^\tau_{n-p,n-q}$ is well-defined for $n - q \geq 1$. Define the form u on Ω by

$$u = -\star\#_{\mathcal{F}^{\otimes\tau}}\overline{\partial}\mathcal{N}^\tau_{n-p,n-q}\#_{\mathcal{F}^{\otimes\tau}}\star\phi. \tag{3.9}$$

Extend u to X by setting $u = 0$ on $X \setminus \overline{\Omega}$. Our goal is to prove that u satisfies $\overline{\partial}u = \phi$ distributionally on X.

First, we establish that $\overline{\partial}u = \phi$ on Ω in the distribution sense. Let η belong to $\mathtt{Dom}(\overline{\partial}, \mathcal{F}^{*\otimes\tau})$. Then

$$\langle\overline{\partial}\eta, \#_{\mathcal{F}^{\otimes\tau}} \star \phi\rangle_{\tau,\Omega} = (-1)^{p+q}\langle\phi, \#_{\mathcal{F}^{*\otimes\tau}} \star \overline{\partial}\eta\rangle_{\tau,\Omega}.$$

Due to the density of $\mathscr{B}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ in $\mathtt{Dom}(\overline{\partial}, \mathcal{F}^{\otimes\tau}) \cap \mathtt{Dom}(\overline{\partial}^*, \mathcal{F}^{\otimes\tau})$ (see Proposition 1) and because ϑ^τ coincides with $\overline{\partial}^*_\tau$ on $\mathscr{B}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ when acting distributionally, we infer

$$\langle\overline{\partial}\eta, \#_{\mathcal{F}^{\otimes\tau}} \star \phi\rangle_{\tau,\Omega} = \langle\phi, \overline{\partial}^*_\tau \#_{\mathcal{F}^{*\otimes\tau}} \star \eta\rangle_{\tau,\Omega}.$$

Given that $\mathrm{supp}(\phi) \subset \overline{\Omega}$ and using the distributional assumption $\overline{\partial}\phi = 0$, it follows that

$$\langle\overline{\partial}\eta, \#_{\mathcal{F}^{\otimes\tau}} \star \phi\rangle_{\tau,\Omega} = \langle\overline{\partial}\phi, \#_{\mathcal{F}^{*\otimes\tau}} \star \eta\rangle_{\tau,X} = 0,$$

implying

$$\overline{\partial}^*_\tau(\#_{\mathcal{F}^{\otimes\tau}} \star \phi) = 0 \quad \text{on} \quad \Omega$$

in the distributional sense. Applying Theorem 2(iv), we obtain

$$\overline{\partial}^*_\tau \mathcal{N}^\tau_{n-p,n-q}(\#_{\mathcal{F}^{\otimes\tau}} \star \phi) = \mathcal{N}^\tau_{n-r,n-s-1}\overline{\partial}^*_\tau(\#_{\mathcal{F}^{\otimes\tau}} \star \phi) = 0. \tag{3.10}$$

Now, compute $\overline{\partial}u$ on Ω using (3.9), (3.10), and standard properties of $\star$ and #:

$$\begin{aligned}
\overline{\partial}u &= -\overline{\partial} \star \#_{\mathcal{F}^{*\otimes\tau}}\overline{\partial}\mathcal{N}^\tau_{n-p,n-q}\#_{\mathcal{F}^{\otimes\tau}} \star \phi \\
&= (-1)^{p+q} \star \#_{\mathcal{F}^{*\otimes\tau}}\overline{\partial}^*_\tau\overline{\partial}\mathcal{N}^\tau_{n-p,n-q}\#_{\mathcal{F}^{\otimes\tau}} \star \phi \\
&= (-1)^{p+q} \star \#_{\mathcal{F}^{*\otimes\tau}}(\overline{\partial}^*_\tau\overline{\partial} + \overline{\partial}\overline{\partial}^*_\tau)\mathcal{N}^\tau_{n-p,n-q}\#_{\mathcal{F}^{\otimes\tau}} \star \phi \\
&= (-1)^{p+q} \star \#_{\mathcal{F}^{*\otimes\tau}}\#_{\mathcal{F}^{\otimes\tau}} \star \phi \\
&= \phi.
\end{aligned}$$

Since u vanishes outside $\overline{\Omega}$, we verify the distributional identity on X as follows. Let $\eta \in \mathtt{Dom}(\overline{\partial}^*_\tau, \mathcal{F}^{\otimes\tau})$. Then:

$$\begin{aligned}
\langle u, \overline{\partial}^*_\tau\eta\rangle_{\tau,X} &= \langle u, \overline{\partial}^*_\tau\eta\rangle_{\tau,\Omega} \\
&= \langle\#_{\mathcal{F}^{\otimes\tau}} \star \overline{\partial}^*_\tau\eta, \#_{\mathcal{F}^{\otimes\tau}} \star u\rangle_{\tau,\Omega}.
\end{aligned}$$

Since

$$\#_{\mathcal{F}^{\otimes\tau}} \star u = (-1)^{r+s+1}\overline{\partial}\mathcal{N}^\tau_{n-p,n-q}\#_{\mathcal{F}^{\otimes\tau}} \star \phi \in \mathtt{Dom}(\overline{\partial}^*_\tau, \mathcal{F}^{*\otimes\tau}),$$

we can apply integration by parts to get

$$\begin{aligned}
\langle u, \overline{\partial}^*_\tau\eta\rangle_{\tau,X} &= (-1)^{p+q}\langle\overline{\partial}\#_{\mathcal{F}^{\otimes\tau}} \star \eta, \#_{\mathcal{F}^{\otimes\tau}} \star u\rangle_{\tau,\Omega} \\
&= \langle\#_{\mathcal{F}^{\otimes\tau}} \star \eta, \#_{\mathcal{F}^{\otimes\tau}} \star \overline{\partial}u\rangle_{\tau,\Omega} \\
&= \langle\overline{\partial}u, \eta\rangle_{\tau,\Omega}.
\end{aligned}$$

Using the previous calculation that $\overline{\partial}u = \phi$, we find

$$\langle u, \overline{\partial}^*_\tau\eta\rangle_{\tau,X} = \langle\phi, \eta\rangle_{\tau,\Omega} = \langle\phi, \eta\rangle_{\tau,X}.$$

Hence, $\overline{\partial}u = \phi$ in the distribution sense on X, completing the proof. □

4. On the Solvability of the $\overline{\partial}_b$-Equation

In this section, we present several results related to the existence of solutions for the $\overline{\partial}_b$-problem.

Theorem 4.1. *Let X be a Kähler manifold of complex dimension $n \geq 2$, and let $\Omega \Subset \mathcal{X}$ denote a relatively compact domain with smooth boundary, assumed to satisfy the weak $Z(q)$ condition. Let E be a holomorphic line bundle over X, and denote by $\mathcal{F}^{\otimes\tau}$ its m-fold tensor product for some positive integer m. Suppose that a strongly plurisubharmonic function exists in an open neighborhood of $b\Omega$. Then for any $f \in C^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau})$ with $1 \leq q \leq n-2$ and $\overline{\partial}_b f = 0$, there exists an extension $F \in C^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ satisfying $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$.*

Proof. The proof follows the arguments of Theorem 4.1 in Saber [48]. □

Theorem 4.2. *Assume the same setup as in the previous theorem. Given $f \in C^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau})$ for $1 \leq q \leq n-2$ with $\overline{\partial}_b f = 0$, there exists a function $u \in C^{\infty}_{p,q-1}(b\Omega, \mathcal{F}^{\otimes\tau})$ such that $\overline{\partial}_b u = f$.*

Proof. Let $f \in C^{\infty}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau})$ satisfy $\overline{\partial}_b f = 0$. From Theorem 5.1, there is an extension $F \in C^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ with $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$. Using Theorem 3, we find $U \in C^{\infty}_{p,q-1}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ such that $\overline{\partial} U = F$ in Ω. Setting $u = U|_{b\Omega}$ yields $\overline{\partial}_b u = f$. □

Corollary 4.1. *Let X be a Kähler manifold of complex dimension $n \geq 2$, and let $\Omega \Subset \mathcal{X}$ be a smoothly bounded domain that is weakly q-concave. Let E be a holomorphic line bundle on X and $\mathcal{F}^{\otimes\tau}$ its m-fold tensor product. Assume a strongly plurisubharmonic function exists in a neighborhood of $b\Omega$. If $H^{p,q}(\mathcal{X}, \mathcal{F}^{\otimes\tau}) = 0$, then for each $f \in C^{\infty}_{p,q}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ with $\overline{\partial} f = 0$ and $1 \leq q \leq n-2$, there exists $u \in C^{\infty}_{p,q-1}(\overline{\Omega}, \mathcal{F}^{\otimes\tau})$ such that $\overline{\partial} u = f$.*

Proof. The proof follows the methodology of Corollary 4.3 in Saber [48]. □

Finally, we summarize a necessary and sufficient condition for the solvability of the $\bar{\partial}$-problem with boundary data in a fractional Sobolev space.

Theorem 4.3. *Let X, Ω, and E be as in Theorem 5.1. Suppose $f \in W^{1/2}_{p,q}(b\Omega, \mathcal{F}^{\otimes\tau})$ with $0 \leq p \leq n$ and $1 \leq q \leq n-2$, satisfying $\overline{\partial}_b f = 0$. Then there exists a function $F \in L^2_{p,q-1}(\Omega, \mathcal{F}^{\otimes\tau})$ such that $F|_{b\Omega} = f$ and $\overline{\partial} F = 0$ in Ω.*

Proof. This result is obtained following the proof of Theorem 4.4 in Saber [48]. □

5. Extension of Forms from the Boundary

Let X be a connected complex manifold of complex dimension $n \geq 2$, and let $\Omega \Subset \mathcal{X}$ be an open subset with a $\mathscr{C}^{\infty}$-smooth boundary. Suppose E is a holomorphic vector bundle over X. In this section, we establish several extension results.

Lemma 5.1. *Given any $\phi \in C^{\infty}_{p,q}(b\Omega, E)$ satisfying $\overline{\partial}_b \phi = 0$, there exists an extension $\widetilde{\phi} \in C^{\infty}_{p,q}(\overline{\Omega}, E)$ such that $\widetilde{\phi}|_{b\Omega} = \phi$ and $\overline{\partial}\widetilde{\phi}$ vanishes to infinite order along $b\Omega$.*

Proof. The argument follows similarly to that presented in Ohsawa [14, 15]. □

Using foundational results from the theory of Kodaira, Andreotti, and Vesentini (see Kodaira [17] and Andreotti-Vesentini [22]), we derive the following sufficient condition for smooth extension up to a given order.

Lemma 5.2. *Let X be a connected Kähler manifold of dimension n, and let $\Omega \Subset \mathcal{X}$ be a relatively compact domain with $\mathscr{C}^\infty$-smooth boundary satisfying the weak $Z(q)$ condition. Suppose E is a holomorphic vector bundle over X. Assume that Ω admits a $\mathscr{C}^\infty$ defining function ρ such that*

$$\partial\overline{\partial}\Big(-\log(-\rho)\Big) \geq c\Big(\partial(-\log(-\rho)) \otimes \overline{\partial}(-\log(-\rho)) + \omega\Big)$$

for some positive constant c on Ω. Then, for any $\psi \in C^\infty_{p,q}(b\Omega, E)$ with $\overline{\partial}_b\psi = 0$ and $q < n-1$, and for any nonnegative integer k, there exists a $\overline{\partial}$-closed E-valued (p,q)-form Ψ_k of class C^k on $\overline{\Omega}$ satisfying $\Psi_k|_{b\Omega} = \psi$.

Proof. The proof strategy parallels that of Ohsawa [14, 15]. □

Funding. The research work was funded by Umm Al-Qura University, Saudi Arabia under grant number: 25UQU4220004GSSR04.

Acknowledgment. The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number: 25UQU4220004GSSR04.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M. Derridj, Regularité pour $\bar{\partial}$ dans Quelques Domaines Faiblement Pseudo-Convexes, J. Differ. Geom. 13 (1978), 559–576. https://doi.org/10.4310/jdg/1214434708.

[2] M. Shaw, Local Existence Theorems with Estimates for $\bar{\partial}_b$ on Weakly Pseudo-convex CR Manifolds, Math. Ann. 294 (1992), 677–700. https://doi.org/10.1007/bf01934348.

[3] J. Cao, M.C. Shaw, L. Wang, Estimates for the $\overline{\partial}$-Neumann Problem and Nonexistence of C^2 Levi-Flat Hypersurfaces in $\mathbb{P}^n$, Math. Z. 248 (2004), 183–221. https://doi.org/10.1007/s00209-004-0661-0.

[4] S. Sambou, Résolution du $\overline{\partial}$ pour les Courants Prolongeables Définis dans un Anneau, Ann. Fac. Sci. Toulouse Math. 11 (2002), 105–129. https://doi.org/10.5802/afst.1020.

[5] O. Abdelkader, S. Saber, The $\overline{\partial}$-Neumann Operator on Strongly Pseudoconvex Domain with Piecewise Smooth Boundary, Math. Slovaca 55 (2005), 317–328. https://eudml.org/doc/32121.

[6] O. Abdelkader, S. Saber, Vanishing Theorems on Strongly q-Convex Manifolds, Int. J. Geom. Methods Mod. Phys. 02 (2005), 467–483. https://doi.org/10.1142/s0219887805000569.

[7] O. Abdelkader, S. Saber, Solution to $\overline{\partial}$-Equations with Exact Support on Pseudo-Convex Manifolds, Int. J. Geom. Methods Mod. Phys. 04 (2007), 339–348. https://doi.org/10.1142/s0219887807002090.

[8] S. Saber, Solution to $\overline{\partial}$ Problem with Exact Support and Regularity for the $\overline{\partial}$-Neumann Operator on Weakly q-Convex Domains, Int. J. Geom. Methods Mod. Phys. 07 (2010), 135–142. https://doi.org/10.1142/s0219887810003963.

[9] S. Saber, The $\overline{\partial}$ -Neumann Operator on Lipschitz q-Pseudoconvex Domains, Czechoslov. Math. J. 61 (2011), 721–731. https://doi.org/10.1007/s10587-011-0021-2.

[10] S. Saber, The L^2 $\overline{\partial}$-Cauchy Problem on Weakly q-pseudoconvex Domains in Stein Manifolds, Czechoslov. Math. J. 65 (2015), 739–745. https://doi.org/10.1007/s10587-015-0205-2.

[11] S. Saber, The L^2 $\overline{\partial}$-Cauchy Problem on Pseudoconvex Domains and Applications, Asian-Eur. J. Math. 11 (2018), 1850025. https://doi.org/10.1142/s1793557118500250.

[12] L. Hörmander, L^2 Estimates and Existence Theorems for the $\bar{\partial}$ Operator, Acta Math. 113 (1965), 89–152. https://doi.org/10.1007/bf02391775.

[13] G.B. Folland, J.J. Kohn, The Neumann Problem for the Cauchy-Riemann Complex, Princeton University Press, 1972.

[14] T. Ohsawa, On the Extension L^2 Holomorphic Functions III: Negligible Weights, Math. Z. 219 (1995), 215–225. https://doi.org/10.1007/bf02572360.

[15] T. Ohsawa, Pseudoconvex Domains in $\mathbb{P}^n$: A Question on the 1-Convex Boundary Points, in: G. Komatsu, M. Kuranishi (Eds.), Analysis and Geometry in Several Complex Variables, Birkhäuser Boston, Boston, MA, 1999: pp. 239–252. https://doi.org/10.1007/978-1-4612-2166-1_11.

[16] K. Kodaira, On Kähler Varieties of Restricted Type (An Intrinsic Characterization of Algebraic Varieties), Ann. Math. 60 (1954), 28–48.

[17] K. Kodaira, Complex Manifolds and Deformation of Complex Structures, Springer, 1986.

[18] E. Vesentini, Lectures on Levi Convexity of Complex Manifolds and Cohomology Vanishing Theorems, Tata Institute of Fundamental Research, Bombay, 1967.

[19] P.A. Griffiths, The Extension Problem in Complex Analysis II; Embeddings with Positive Normal Bundle, Amer. J. Math. 88 (1966), 366–446. https://doi.org/10.2307/2373200.

[20] H. Grauert, I. Lieb, Das Ramirezsche Integral und die Lösung der Gleichung $\overline{\partial} f = \alpha$ im Bereich der Beschränkten Formen, Rice Inst. Pamph. - Rice Univ. Stud. 56 (1970), 29–50. https://hdl.handle.net/1911/63010.

[21] X. Yang, RC-Positivity, Rational Connectedness and Yau's Conjecture, Cambridge J. Math. 6 (2018), 183–212. https://doi.org/10.4310/CJM.2018.v6.n2.a2.

[22] A. Andreotti, E. Vesentini, Carleman Estimates for the Laplace-beltrami Equation on Complex Manifolds, Publ. Math. Inst. Hautes Études Sci. 25 (1965), 81–130. https://doi.org/10.1007/bf02684398.

[23] L. Ho, The $\overline{\partial}$-Problem on Weak $Z(q)$ Domains, Math. Ann. 290 (1991), 3–18.

[24] S. Saber, Solvability of the Tangential Cauchy-Riemann Equations on Boundaries of Strictly q-Convex Domains, Lobachevskii J. Math. 32 (2011), 189–193. https://doi.org/10.1134/S1995080211030115.

[25] S. Saber, Global Boundary Regularity for the $\overline{\partial}$-Problem on Strictly Q-convex and Q-concave Domains, Complex Anal. Oper. Theory 6 (2010), 1157–1165. https://doi.org/10.1007/s11785-010-0114-1.

[26] S. Saber, Solution to $\overline{\partial}$ Problem for Smooth Forms and Currents on Strictly q-Convex Domains, Int. J. Geom. Methods Mod. Phys. 9 (2012), 1220002. https://doi.org/10.1142/S0219887812200022.

[27] S. Saber, The $\bar{\partial}$-Problem on q-Pseudoconvex Domains with Applications, Math. Slovaca 63 (2013), 521–530. https://doi.org/10.2478/s12175-013-0115-4.

[28] S. Saber, The L^2 $\overline{\partial}$-Cauchy Problem on Weakly q-Pseudoconvex Domains in Stein Manifolds, Czech. Math. J. 65 (2015), 739–745. https://doi.org/10.1007/s10587-015-0205-2.

[29] S. Saber, Global Regularity for $\overline{\partial}$ on an Annulus between Two Weakly Convex Domains, Boll. Unione Mat. Ital. 11 (2018), 309–314. https://doi.org/10.1007/s40574-017-0135-z.

[30] S. Saber, The $\overline{\partial}$-Problem With Support Conditions and Pseudoconvexity of General Order in Kähler Manifolds, J. Korean Math. Soc. 53 (2016), 1211–1223. https://doi.org/10.4134/JKMS.J140768.

[31] S. Saber, Global Solution for the $\overline{\partial}$-Problem on Non Pseudoconvex Domains in Stein Manifolds, J. Korean Math. Soc. 54 (2017), 1787–1799. https://doi.org/10.4134/JKMS.J160668.

[32] S. Saber, Sobolev Regularity of the Bergman Projection on Certain Pseudoconvex Domains, Trans. A. Razmadze Math. Inst. 171 (2017), 90–102. https://doi.org/10.1016/j.trmi.2016.10.004.

[33] S. Saber, The L^2 $\overline{\partial}$-Cauchy Problem on Pseudoconvex Domains and Applications, Asian-Eur. J. Math. 11 (2018), 1850025. https://doi.org/10.1142/S1793557118500250.

[34] S. Saber, Global Regularity for $\overline{\partial}$ on an Annulus between Two Weakly Convex Domains, Boll. Unione Mat. Ital. 11 (2018), 309–314. https://doi.org/10.1007/s40574-017-0135-z.

[35] S. Saber, Solution to $\overline{\partial}$-Problem with Support Conditions in Weakly q-Convex Domains, Commun. Korean Math. Soc. 33 (2018), 409–421. https://doi.org/10.4134/CKMS.C170022.

[36] S. Saber, Compactness of the Canonical Solution Operator on Lipschitz q-Pseudoconvex Boundaries, Electron. J. Differ. Equ. 2019 (2019), 48.

[37] S. Saber, Compactness of the Complex Green Operator in a Stein Manifold, U.P.B. Sci. Bull. Ser. A 81 (2019), 185–200.

[38] S. Saber, Compactness of the Weighted dbar-Neumann Operator and Commutators of the Bergman Projection with Continuous Functions, J. Geom. Phys. 138 (2019), 194–205. https://doi.org/10.1016/j.geomphys.2018.12.022.

[39] S. Saber, Compactness of the Commutators of Toeplitz Operators on q-Pseudoconvex Domains, Electron. J. Differ. Equ. 2018 (2018), 111.

[40] S. Saber, Global Solvability and Regularity for $\overline{\partial}$ on an Annulus between Two Weakly Convex Domains Which Satisfy Property (P), Asian-Eur. J. Math. 12 (2019), 1950041. https://doi.org/10.1142/S1793557119500414.

[41] S. Saber, L^2 Estimates and Existence Theorems for $\overline{\partial}_b$ on Lipschitz Boundaries of Q-Pseudoconvex Domains, Comptes Rendus. Mathématique 358 (2020), 435–458. https://doi.org/10.5802/crmath.43.

[42] S. Saber, The $\overline{\partial}$-Cauchy Problem on Weakly q-Convex Domains in $\mathbb{CP}^n$, Kragujevac J. Math. 44 (2020), 581–591. https://doi.org/10.46793/KgJMat2004.581S.

[43] S. Saber, A. Alahmari, Global Regularity of $\overline{\partial}$ on Certain Pseudoconvexity, Trans. A. Razmadze Math. Inst. 175 (2021), 417–427.

[44] S. Saber, On the Applications of Bochner-Kodaira-Morrey-Kohn Identity, Kragujevac J. Math. 45 (2021), 881–896. https://doi.org/10.46793/KgJMat2106.881S.

[45] H.D.S. Adam, K.I.A. Ahmed, S. Saber, M. Marin, Sobolev Estimates for the $\overline{\partial}$ and the $\overline{\partial}$-Neumann Operator on Pseudoconvex Manifolds, Mathematics 11 (2023), 4138. https://doi.org/10.3390/math11194138.

[46] H.D.S. Adam, K.I. Adam, S. Saber, G. Farid, Existence Theorems for the dbar Equation and Sobolev Estimates on q-Convex Domains, AIMS Math. 8 (2023), 31141–31157. https://doi.org/10.3934/math.20231594.

[47] S. Saber, A. Alahmari, Compactness Estimate for the $\overline{\partial}$-Neumann Problem on a q-Pseudoconvex Domain in a Stein Manifold, Kragujevac J. Math. 47 (2023), 627–636.

[48] S. Saber, M. Youssif, Y. Arko, et al. Subellipticity, Compactness, H^ϵ Estimates and Regularity for $\bar{\partial}$ on Weakly q-Pseudoconvex/Concave Domains, Rend. Semin. Mat. Univ. Padova (2024). https://doi.org/10.4171/rsmup/160.

[49] S. Saber, A.A. Alahmari, Generalization of Kodaira's Embedding Theorem for Compact Kähler Manifolds with Semi-positive Chern Class, Int. J. Anal. Appl. 23 (2025), 72. https://doi.org/10.28924/2291-8639-23-2025-72.

[50] G.M. Henkin, A. Iordan, Regularity of $\overline{\partial}$ on Pseudoconcave Compacts and Applications, Asian J. Math. 4 (2000), 855–884. https://doi.org/10.4310/ajm.2000.v4.n4.a9.

[51] K. Takegoshi, On Weakly 1-complete Surfaces Without Non-constant Holomorphic Functions, Publ. Res. Inst. Math. Sci. 18 (1982), 1175–1183. https://doi.org/10.2977/prims/1195183302.

[52] K. Takegoshi, Representation Theorems of Cohomology on Weakly 1-complete Manifolds, Publ. Res. Inst. Math. Sci. 18 (1982), 131–186. https://doi.org/10.2977/prims/1195183572.

[53] K. Takegoshi, Global Regularity and Spectra of Laplace-Beltrami Operators on Pseudoconvex Domains, Publ. Res. Inst. Math. Sci. 19 (1983), 275–304.