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# A Bicubic trigonometric B-Spline Approach for Solving the Nonlinear Generalized 2D Burger's Equation

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ABSTRACT. Nonlinear reaction-diffusion problems, such as the nonlinear generalized two-dimensional Burgers' equation, play a crucial role in various fields, including developmental biology, population dynamics, engineering, and physics. This study focuses on the numerical solution of the two-dimensional Burgers' equation using a collocation method based on bicubic trigonometric B-spline functions combined with a θ-weighted scheme. The spatial and temporal domains are discretized using bicubic trigonometric B-spline functions and a finite difference approach, respectively. The nonlinear terms in the equation are handled through quasilinearization. The effectiveness of the proposed method is demonstrated by simulating some test problems with different initial and boundary conditions. The influence of various reaction terms is analyzed and presented in both tabular and graphical formats. Moreover, using the Von Neumann stability analysis, the proposed scheme is shown to be conditionally stable. The results indicate that the present method is highly effective for solving nonlinear partial differential equations arising in a wide range of scientific and engineering applications.

## 1. Introduction

Diverse physical phenomena including fluid flow, mass transfer, air pollution, acoustic waves, shock waves, groundwater movement, chemical separation, nuclear reactor theory, and the logistic growth of populations—are frequently modeled using nonlinear partial differential equations [1-7]. Solving these equations presents a major challenge for mathematicians, engineers, and numerical scientists. As a result, a wide range of techniques has been employed by researchers, including the finite difference method, finite volume method, finite element

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method, variational iteration, Adomian decomposition, residual power series, differential transform, B-spline approximation, non-polynomial splines, polynomial differential quadrature, among others [8-14].

Burgers' equation is a nonlinear advection-diffusion equation used in the analysis of shock waves, wave propagation in thermoelastic media, dispersion and pollution transport, and the formation of structures in cosmological adhesion models. It reflects the interaction between diffusive viscous processes and nonlinear convective phenomena in two spatial dimensions.

This paper presents an investigation of the two-dimensional Burgers' equation, which is expressed as:

$$u_t(x,y,t) + u(x,y,t)\left(u_x(x,y,t) + u_y(x,y,t)\right) = \delta\left(u_{xx}(x,y,t) + u_{yy}(x,y,t)\right) \quad (1.1)$$

$$\text{for } a \le x \le b, \quad c \le y \le d, \quad \text{and } 0 \le t \le T,$$

subject to the initial condition

$$u(x,y,0) = p(x,t), \quad (1.2)$$

and Dirichlet boundary conditions:

$$\begin{cases} u(a,y,t) = p_1(y,t), & u(b,y,t) = p_2(y,t), \\ u(x,c,t) = p_3(x,t), & u(x,d,t) = p_4(x,t), \end{cases} \quad (1.3)$$

here, $u(x,y,t)$ is the unknown function, $t$ denotes time, and $x, y$ are spatial coordinates. The term $u_{xx}(x,y,t) + u_{yy}(x,y,t)$ represents diffusion in two spatial dimensions, and $\delta$ is the Reynolds number, which is used to predict the transition from laminar to turbulent flow. Laminar flow occurs at low Reynolds numbers, while turbulence appears at high Reynolds numbers. For large Reynolds numbers, the Burgers' equation behaves like a hyperbolic PDE, and the solution tends to exhibit multiple features due to the formation of sharp, shock-like wave fronts [15].

Due to its importance in modeling a wide range of physical processes including turbulence, gas dynamics, traffic flow, and shock wave generation Burgers' equation has received considerable attention from researchers in recent years. However, the nonlinear nature of Burgers' equation makes it difficult to obtain analytical solutions. Consequently, various numerical techniques and algorithms have been developed to efficiently and accurately approximate its solutions.

Wazwaz [16] applied the Adomian Decomposition Method (ADM) to Burgers' equation and generated analytical approximations that effectively handle the nonlinear convective terms and exhibit rapid convergence. He [17] used the Variational Iteration Method (VIM) to solve Burgers' equation, demonstrating the method's simplicity and efficiency in producing accurate solutions without requiring linearization or discretization. Zhao and Li [18] used the space-time continuous Galerkin method to solve the 2D Burgers' equation, achieving high-order accuracy and unconditional stability without mesh ratio restrictions. Parand et al. [19] developed a spectral collocation method for solving one- and two-dimensional Burgers' equations, combining the

Jacobian-free Newton–Krylov method with Bessel functions. This method provides high precision and stability, especially for problems involving sharp gradients. Based on differential quadrature, Arora and Kumar [20] proposed the modified cubic-B-spline differential quadrature method to find the approximate solution of the Burgers' equation. Kırlı and Irk [21] solved Burgers' equation using Crank–Nicolson time-stepping, B-spline collocation, and Galerkin finite element methods, focusing on precision and stability in the presence of shock waves and steep gradients. Furthermore, Zaman et al. [22] applied the Haar Wavelet Collocation Method to solve Burgers' equation, emphasizing the method's computational efficiency and ability to handle discontinuities and nonlinearities in the solution.

Based on the authors' knowledge, the two-dimensional Burgers' equation has not been previously solved using the bicubic trigonometric B-spline interpolation method. The primary objective of this study is to solve Equation (1.1) using bicubic trigonometric B-spline functions. The proposed scheme employs a collocation approach with bicubic trigonometric B-spline functions and their derivatives for the spatial variables, combined with a finite difference method for the temporal variable. The Crank–Nicolson method is applied to ensure numerical stability.

The paper is organized as follows: Section 2 presents the mathematical foundation of the bicubic trigonometric B-spline interpolation method. Section 3 details the derivation of the proposed numerical scheme for solving the two-dimensional Burgers' equation, incorporating suitable initial and boundary conditions. Section 4 discusses the stability analysis of the method. In Section 5, the scheme is applied to several test problems, and the $L2$ and $L\infty$ error norms are calculated to evaluate accuracy. Finally, Section 6 concludes the study by summarizing the key findings.

## 2. Description of the bicubic trigonometric B-spline interpolation

A Trigonometric B-spline surface is constructed from a linear combination of recursive functions, called trigonometric B-spline basis functions. The derivation of trigonometric B-spline basis and its properties are discussed in [23].

Suppose that $\{x_l\}$ represents a uniform partition of an interval along the $x$-axis, where $x_{l+1} = x_l + \Delta x$, $l \in Z$, and $\Delta x$ represents the step size of the partition. The trigonometric B-spline basis of order $k$, with degree $k-1$, is defined as follows:

$$T_l^k(x) = \frac{\sin\left(\frac{x - x_l}{2}\right)}{\sin\left(\frac{x_{l+k-1} - x_l}{2}\right)} T_l^{k-1}(x) + \frac{\sin\left(\frac{x_{l+k} - x}{2}\right)}{\sin\left(\frac{x_{l+k} - x_{l+1}}{2}\right)} T_{l+1}^{k-1}(x), \tag{2.1}$$

with the initial function defined by:

$$T_l^1(x) = \begin{cases} 1, & x \in [x_l, x_{l+1}] \\ 0, & otherwise \end{cases}$$

In this work, a trigonometric B-spline of degree three is employed. Evaluating (2.1) up to $k = 4$ gives:

$$T_l^4(x)=\frac{1}{\theta}\begin{cases}\sigma^3(x_l), & x\in[x_l,x_{l+1}]\\ \sigma(x_l)[\sigma(x_l)\varsigma(x_{l+2})+\varsigma(x_{l+3})\sigma(x_{l+1})]+\varsigma(x_{l+4})\sigma^2(x_{i+1}), & x\in[x_{l+1},x_{l+2}]\\ \sigma(x_l)\varsigma^2(x_{l+3})+\varsigma(x_{l+4})[\sigma(x_{l+1})\varsigma(x_{l+3})+\sigma(x_{l+2})\varsigma(x_{l+4})], & x\in[x_{l+2},x_{l+3}]\\ \varsigma^3(x_{l+4}), & x\in[x_{l+3},x_{l+4}]\end{cases} \tag{2.2}$$

where

$$\sigma(x_l)=\sin\left(\frac{x-x_l}{2}\right),\ \ \varsigma(x_l)=\sin\left(\frac{x_l-x}{2}\right),\theta=\sin\left(\frac{\Delta x}{2}\right)\sin(\Delta x)\sin\left(\frac{3\Delta x}{2}\right)$$

Since the basis $T_l^4(x)$ is a piecewise trigonometric function of degree 3, it is called a cubic trigonometric B-spline basis. The basis has second-order parametric continuity.

When evaluating cubic trigonometric B-spline basis in (2.2) at $x_l$, there are three nonzero bases functions, namely $T_{l-3}^4(x_l), T_{l-2}^4(x_l)$, and $T_{l-1}^4(x_l)$. The nonzero values are given by:

$$\begin{aligned}T_{l-3}^4(x_l)&=sin^2\left(\frac{\Delta x}{2}\right)\csc(\Delta x)\csc\left(\frac{3\Delta x}{2}\right),\\ T_{l-2}^4(x_l)&=\frac{2}{1+2\cos(\Delta x)},\\ T_{l-1}^4(x_l)&=sin^2\left(\frac{\Delta x}{2}\right)\csc(\Delta x)\csc\left(\frac{3\Delta x}{2}\right)\end{aligned} \tag{2.3}$$

the first and second derivatives of the basis with respect to $x$ are also considered. The first derivative $\frac{d}{dx}[T_l^4(x)]$ is continuous. At $x_l$ the nonzero values are:

$$\frac{d}{dx}[T_{l-3}^4(x_l)]=-\frac{3\csc\left(\frac{\Delta x}{2}\right)}{4(1+2\cos(\Delta x))}\ \ ,\frac{d}{dx}[T_{l-2}^4(x_l)]=0\ \ ,\frac{d}{dx}[T_{l-1}^4(x_l)]=\frac{3\csc\left(\frac{\Delta x}{2}\right)}{4(1+2\cos(\Delta x))} \tag{2.4}$$

The second derivative of the cubic trigonometric B-spline basis at $x_l$ is given by:

$$\begin{aligned}\frac{d^2}{dx^2}[T_{l-3}^4(x_l)]&=\frac{3(1+3\cos(\Delta x))csc^2\left(\frac{\Delta x}{2}\right)}{16\left(2\cos\left(\frac{\Delta x}{2}\right)+\cos\left(\frac{3\Delta x}{2}\right)\right)},\\ \frac{d^2}{dx^2}[T_{l-2}^4(x_l)]&=-\frac{3cot^2\left(\frac{\Delta x}{2}\right)}{2+4\cos(\Delta x)},\\ \frac{d^2}{dx^2}[T_{l-1}^4(x_l)]&=\frac{3(1+3\cos(\Delta x))csc^2\left(\frac{\Delta x}{2}\right)}{16(2\cos\left(\frac{\Delta x}{2}\right)+\cos\left(\frac{3\Delta x}{2}\right)}\end{aligned} \tag{2.5}$$

Now, consider a uniform partition $\{y_s\}$ of an interval along the $y$-axis, where $y_{s+1}=y_s+\Delta y,\ s\in Z$, and $\Delta y$ represents the grid spacing.

The trigonometric B-spline basis of order $k$ (degree $k-1$) is defined recursively as:

$$T_s^k(y)=\frac{\sin\left(\frac{y-y_s}{2}\right)}{\sin\left(\frac{y_{s+k-1}-y_s}{2}\right)}T_s^{k-1}(y)+\frac{\sin\left(\frac{y_{s+k}-y}{2}\right)}{\sin\left(\frac{y_{s+k}-y_{s+1}}{2}\right)}T_{s+1}^{k-1}(y), \tag{2.6}$$

with the base case:

$$T_s^1(y)=\begin{cases}1, & y\in[y_s,y_{s+1}]\\ 0, & otherwise\end{cases}$$

The corresponding cubic trigonometric B-spline basis is given by:

$$T_s^4(y) = \frac{1}{\theta}\begin{cases} \sigma^3(y_s), & y \in [y_s, y_{s+1}] \\ \sigma(y_s)[\sigma(y_s)\varsigma(y_{s+2}) + \varsigma(y_{s+3})\sigma(y_{s+1})] + \varsigma(y_{s+4})\sigma^2(y_{s+1}), & y \in [y_{s+1}, y_{s+2}] \\ \sigma(y_s)\varsigma^2(y_{s+3}) + \varsigma(y_{s+4})[\sigma(y_{s+1})\varsigma(y_{s+3}) + \sigma(y_{s+2})\varsigma(y_{s+4})], & y \in [y_{s+2}, y_{s+3}] \\ \varsigma^3(y_{s+4}), & y \in [y_{s+3}, y_{s+4}] \end{cases} \quad (2.7)$$

where

$$\sigma(y_s) = \sin\left(\frac{y - y_s}{2}\right), \quad \varsigma(y_s) = \sin\left(\frac{y_s - y}{2}\right), \theta = \sin\left(\frac{\Delta y}{2}\right)\sin(\Delta y)\sin\left(\frac{3\Delta y}{2}\right)$$

The corresponding nonzero values when evaluating $T_s^4(y)$ and its derivatives at $y_s$ are:

$$\begin{aligned} T_{s-3}^4(y_s) &= sin^2\left(\frac{\Delta y}{2}\right)\csc(\Delta y)\csc\left(\frac{3\Delta y}{2}\right), \\ T_{s-2}^4(y_s) &= \frac{2}{1 + 2\cos(\Delta y)}. \\ T_{s-1}^4(y_s) &= sin^2\left(\frac{\Delta y}{2}\right)\csc(\Delta y)\csc\left(\frac{3\Delta y}{2}\right) \end{aligned} \quad (2.8)$$

First derivatives:

$$\frac{d}{dy}[T_{s-3}^4(y_s)] = -\frac{3\csc\left(\frac{\Delta y}{2}\right)}{4(1 + 2\cos(\Delta y))}, \frac{d}{dy}[T_{s-2}^4(y_s)] = 0 \ , \frac{d}{dy}[T_{s-1}^4(y_s)] = \frac{3\csc\left(\frac{\Delta y}{2}\right)}{4(1 + 2\cos(\Delta y))} \quad (2.9)$$

Second derivatives:

$$\begin{aligned} \frac{d^2}{dy^2}[T_{s-3}^4(y_s)] &= \frac{3(1 + 3\cos(\Delta y))csc^2\left(\frac{\Delta y}{2}\right)}{16\left(2\cos\left(\frac{\Delta y}{2}\right) + \cos\left(\frac{3\Delta y}{2}\right)\right)}, \\ \frac{d^2}{dy^2}[T_{s-2}^4(y_s)] &= -\frac{3cot^2\left(\frac{\Delta y}{2}\right)}{2 + 4\cos(\Delta y)}, \\ \frac{d^2}{dy^2}[T_{s-1}^4(y_s)] &= \frac{3(1 + 3\cos(\Delta y))csc^2\left(\frac{\Delta y}{2}\right)}{16\left(2\cos\left(\frac{\Delta y}{2}\right) + \cos\left(\frac{3\Delta y}{2}\right)\right)} \end{aligned} \quad (2.10)$$

An arbitrary trigonometric B-spline surface equation $U(x, y)$, can be formed using the bases in (2.2) and (2.7):

$$U(x, y) = \sum_{l=-3}^{m-1}\sum_{s=-3}^{n-1}\Psi_{l,s}\, T_l^4(x)T_s^4(y), \quad x \in [x_0, x_m], y \in [y_0, y_n], \quad m, n \geq 1, \quad (2.11)$$

Here, $\Psi_{l,s}$ are unknown coefficients. As the surface is constructed using two cubic trigonometric B-spline basis functions, it is referred to as a bicubic trigonometric B-spline surface.

Evaluating $U(x, y)$ at $(x_l, y_s)$ and using the simplifications from Equation (2.11) in Equations (2.3) and (2.8), yields:

$$\begin{aligned} U(x_l, y_s) = {} & b_1\big(b_3\Psi_{l-3,s-3} + b_4\Psi_{l-3,s-2} + b_3\Psi_{l-3,s-1}\big) + b_2\big(b_3\Psi_{l-2,s-3} + b_4\Psi_{l-2,s-2} + b_3\Psi_{l-2,s-1}\big) \\ & + b_1\big(b_3\Psi_{l-1,s-3} + b_4\Psi_{l-1,s-2} + b_3\Psi_{l-1,s-1}\big), \end{aligned} \quad (2.12)$$

where

$b_1 = sin^2\left(\frac{\Delta x}{2}\right)\csc(\Delta x)csc\left(\frac{3\Delta x}{2}\right)$, $b_2 = \frac{2}{1+2\cos(\Delta x)}$, $b_3 = sin^2\left(\frac{\Delta y}{2}\right)\csc(\Delta y)csc\left(\frac{3\Delta y}{2}\right)$, and $b_4 = \frac{2}{1+2\cos(\Delta y)}$

By taking the first derivatives of $U(x_l, y_s)$ with respect to $x$ and $y$ and evaluating them at $(x_l, y_s)$ using Equations (2.4) and (2.9), the following equations are obtained, respectively

$$\frac{d}{dx}U(x_l, y_s) = -b_x\left(b_1\Psi_{l-3,s-3} + b_2\Psi_{l-3,s-2} + b_1\Psi_{l-3,s-1}\right) + b_x\left(b_1\Psi_{l-1,s-3} + b_2\Psi_{l-1,s-2} + b_1\Psi_{l-1,s-1}\right), \quad (2.13)$$

$$\frac{d}{dy}U(x_l, y_s) = b_1\left(-b_y\Psi_{l-3,s-3} + b_y\Psi_{l-3,s-1}\right) + b_2\left(-b_y\Psi_{l-2,s-3} + b_y\Psi_{l-2,s-1}\right) + b_1\left(-b_y\Psi_{l-1,s-3} + b_y\Psi_{l-1,s-1}\right), \quad (2.14)$$

where $b_x = \frac{3\csc\left(\frac{\Delta x}{2}\right)}{4(1+2\cos(\Delta x))}$ and $b_y = \frac{3\csc\left(\frac{\Delta y}{2}\right)}{4(1+2\cos(\Delta y))}$

By taking the second derivatives of $U(x, y)$ with respect to $x$ and $y$ and evaluating them at $(x_l, y_s)$ using Equations (2.5) and (2.10), the following equations are obtained:

$$\frac{d^2}{dx^2}U(x_l, y_s) = b_{xx1}\left(b_1\Psi_{l-3,s-3} + b_2\Psi_{l-3,s-2} + b_1\Psi_{l-3,s-1}\right) + b_{xx2}\left(b_1\Psi_{l-2,s-3} + b_2\Psi_{l-2,s-2} + b_1\Psi_{l-2,s-1}\right) + b_{xx1}\left(b_1\Psi_{l-1,s-3} + b_2\Psi_{l-1,s-2} + b_1\Psi_{l-1,s-1}\right), \quad (2.15)$$

$$\frac{d^2}{dy^2}U(x_l, y_s) = b_1\left(b_{yy1}\Psi_{l-3,s-3} + b_{yy2}\Psi_{l-3,s-2} + b_{yy1}\Psi_{l-3,s-1}\right) + b_2\left(b_{yy1}\Psi_{l-2,s-3} + b_{yy2}\Psi_{l-2,s-2} + b_{yy1}\Psi_{l-2,s-1}\right) + b_1\left(b_{yy1}\Psi_{l-1,s-3} + b_{yy2}\Psi_{l-1,s-2} + b_{yy1}\Psi_{l-1,s-1}\right), \quad (2.16)$$

where

$b_{xx1} = \frac{3(1+3\cos(\Delta x))csc^2\left(\frac{\Delta x}{2}\right)}{16\left(2\cos\left(\frac{\Delta x}{2}\right)+\cos\left(\frac{3\Delta x}{2}\right)\right)}$, $b_{xx2} = -\frac{3cot^2\left(\frac{\Delta x}{2}\right)}{2+4\cos(\Delta x)}$, $b_{yy1} = \frac{3(1+3\cos(\Delta y))csc^2\left(\frac{\Delta y}{2}\right)}{16\left(2\cos\left(\frac{\Delta y}{2}\right)+\cos\left(\frac{3\Delta y}{2}\right)\right)}$, $b_{yy2} = -\frac{3cot^2\left(\frac{\Delta y}{2}\right)}{2+4\cos(\Delta y)}$

The simplifications of the bicubic trigonometric B-spline basis and its derivatives at $(x_l, y_s)$ are extensively used in solving two-dimensional PDEs using bicubic trigonometric B-spline.

## 3. Analysis of the method

To begin, the domain of the problem is discretized as follows:

$$\begin{aligned} &x_l = l\Delta x, \quad \Delta x = \frac{b-a}{m}, \quad m > 1, \quad l \in \mathbb{Z} \\ &y_s = s\Delta y, \quad \Delta y = \frac{d-c}{n}, \quad n > 1, \quad s \in \mathbb{Z} \\ &t_k = k\Delta t, \quad \Delta t \text{ is the time-step}, \quad k \in \mathbb{N} \end{aligned} \quad (3.1)$$

Next, the bicubic trigonometric B-spline surface $U(x, y)$, as given in Equation (2.11), is presumed to be the solution of Equation (1.1). Evaluating it at $(x_l, y_s)$, Equation (1.1) becomes

$$\frac{\partial}{\partial t}U(x_l,y_s)+U(x_l,y_s)\left(\frac{\partial}{\partial x}U(x_l,y_s)+\frac{\partial}{\partial y}U(x_l,y_s)\right)=\delta\left(\frac{\partial^2}{\partial x^2}U(x_l,y_s)+\frac{\partial^2}{\partial y^2}U(x_l,y_s)\right), \quad (3.2)$$

where $l=0,1,\dots,m$ $\;$ *and* $s=0,1,\dots,n$.

By simplifying the bicubic trigonometric B-spline surface function and its derivatives at the points $x_l$ and $y_s$, and substituting the simplified forms (2.15) and (2.16) into Equation (3.2), and then rearranging the terms, we obtain the following equation:

$$G_{l,s}=\delta\left(\frac{\partial^2}{\partial x^2}U(x_l,y_s)+\frac{\partial^2}{\partial y^2}U(x_l,y_s)\right)$$

$$G_{l,s}=\mathcal{B}_1\Psi_{l-3,s-3}+\mathcal{B}_2\Psi_{l-3,s-2}+\mathcal{B}_1\Psi_{l-3,s-1}+\mathcal{B}_3\Psi_{l-2,s-3}+\mathcal{B}_4\Psi_{l-2,s-2}+\mathcal{B}_3\Psi_{l-2,s-1}+\mathcal{B}_1\Psi_{l-1,s-3}+\mathcal{B}_2\Psi_{l-1,s-2}+\mathcal{B}_1\Psi_{l-1,s-1}, \quad (3.3)$$

where

$$\mathcal{B}_1=\delta(b_{xx1}b_1+b_1b_{yy1}),\ \mathcal{B}_2=\delta(b_{xx1}b_2+b_1b_{yy2}), \mathcal{B}_3=\delta(b_{xx2}b_1+b_2b_{yy1}) \text{ and } \mathcal{B}_4=\delta(b_{xx2}b_2+b_2b_{yy2}),$$

by introducing the time level $k$, the following equation is obtained:

$$(u_t)_{l,s}^k=G_{l,s}^k-\left(u_{l,s}\left((u_x)_{l,s}+(u_y)_{l,s}\right)\right)^k, \quad (3.4)$$

here, $u_{l,s}^k$ represents $u(x_l,y_s,t_k)$. A $\theta$-weighted scheme is used for time discretization, where $0\le\theta\le 1$ . Thus, Equation (3.4) becomes

$$(u_t)_{l,s}^k=\theta F^{(k+1)}+(1-\theta)F^k, \quad (3.5)$$

where $F^k=G_{l,s}^k-\left(u_{l,s}((u_x)_{l,s}+(u_y)_{l,s})\right)^k$.

The Crank-Nicolson method is adopted by setting $\theta=\frac{1}{2}$ , ensuring numerical stability [27]. Applying the forward difference scheme

$$(u_t)_{l,s}^k=\frac{u_{l,s}^{(k+1)}-u_{l,s}^k}{\Delta t}$$

into Equation (3.5) and based on the time level yields:

$$u_{l,s}^{(k+1)}=u_{l,s}^k+\frac{\Delta t}{2}\left[F^{(k+1)}+F^k\right] \quad (3.6)$$

Expanding $F^{(k+1)}$ and $F^k$, we get:

$$u_{l,s}^{k+1}=u_{l,s}^k+\frac{\Delta t}{2}\left[G_{l,s}^{(k+1)}-\left(u_{l,s}\left((u_x)_{l,s}+(u_y)_{l,s}\right)\right)^{(k+1)}+G_{l,s}^k-\left(u_{l,s}((u_x)_{l,s}+(u_y)_{l,s})\right)^k\right], \quad (3.7)$$

The nonlinear terms in Equation (3.7) are linearized using the quasilinearization technique, resulting in:

$$\left(u_{l,s}\left((u_x)_{l,s}+(u_y)_{l,s}\right)\right)^{(k+1)}=\left(\left((u_x)_{l,s}^k+(u_y)_{l,s}^k\right)\right)u_{l,s}^{(k+1)}-\left(\left((u_x)_{l,s}^k+(u_y)_{l,s}^k\right)\right)u_{l,s}^k+\left(u_{l,s}^k\left((u_x)_{l,s}^k+(u_y)_{l,s}^k\right)\right) \quad (3.8)$$

Substituting Equation (3.8) into Equation (3.7), we obtain:

$$u_{l,s}^{(k+1)}+\frac{\Delta t}{2}\Big(\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)u_{l,s}^{(k+1)}-\frac{\Delta t}{2}G_{l,s}^{(k+1)}=u_{l,s}^k+\frac{\Delta t}{2}\Big(\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)u_{l,s}^k$$
$$-\frac{\Delta t}{2}\Big(2u_{l,s}^k\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)+\frac{\Delta t}{2}G_{l,s}^k$$

Rearranging the terms, we get:

$$\left(1+\frac{\Delta t}{2}\Big(\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)\right)u_{l,s}^{(k+1)}-\frac{\Delta t}{2}G_{l,s}^{(k+1)}$$
$$=\left(1+\frac{\Delta t}{2}\Big(\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)-\frac{\Delta t}{2}\Big(2\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)\right)u_{l,s}^k+\frac{\Delta t}{2}G_{l,s}^k \tag{3.9}$$

Let $E_{l,s}^k=\left(1+\frac{\Delta t}{2}\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\right)$

and $D_{l,s}^k=\left(1+\frac{\Delta t}{2}\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)-\frac{\Delta t}{2}\Big(2\big((u_x)_{l,s}^k+(u_y)_{l,s}^k\big)\Big)\right)$

Equation (3.9) becomes

$$E_{l,s}^k*u_{l,s}^{(k+1)}-\frac{\Delta t}{2}G_{l,s}^{(k+1)}=D_{l,s}^k*u_{l,s}^k+\frac{\Delta t}{2}G_{l,s}^k,\quad l=0,1,\dots,m \quad and\ s=0,1,\dots,n \tag{3.10}$$

The boundary conditions (1.3) are discretized as follows:

$$\begin{cases}U^k(x_0,y_s)=p_1(y_s,t^k), & U(x_m,y_s)=p_2(y_s,t^k). \quad s=0,1,\dots,n\\ U^k(x_l,y_0)=p_3(x_l,t^k), & U(x_l,y_n)=p_4(x_l,t^k). \quad l=1,2,\dots,m-1.\end{cases} \tag{3.11}$$

The boundary conditions in Equation (3.11) are not computed at $l=0,m$ since these values were already determined when $s=0,n$. The simplified form of the bicubic trigonometric B-spline surface from Equation (2.12) is substituted into Equation (3.11). For the first set conditions, $l$ is evaluated at 0 and $m$, and for the second set, $s$ is evaluated at 0 and $n$.

Based on the formulation of the bicubic trigonometric B-spline surface given in (2.11), there are$(m+3)\ (n+3)$ unknowns, denoted as $\Psi_{l,s}$, that need to be determined. The differential equation in (3.10) provides a total of $(m+1)\ (n+1)$ equations. Furthermore, an additional $(2(m+1)+2(n+1)-4)$ equations are derived from the boundary conditions in (3.11). Given that

$$(m+3)(n+3)-(m+1)(n+1)-(2(m+1)+2(n+1)-4)\ =\ 8$$

As a result, an underdetermined system of linear equations with 8 independent variables is formed.

To proceed to the next time level, the values $\Psi_{l,s}^0$ must be determined. These values are computed from the discretized initial condition (1.2)

$$u_{l,s}^0=p(x_l,y_s),\quad l=0,1,\dots,m\,,s=0,1,\dots,n\,. \tag{3.12}$$

In total, $(m+3)(n+3)$ values must be determined, while Equation (3.12) provides only $(m+1)(n+1)$ linear equations. Thus, the resulting underdetermined system is solved using MATLAB's built-in function *lsqminnorm* to compute $\Psi_{l,s}^0$.

For $k \geq 1$, the values of $\Psi_{l,s}^{0}$ are used along with Equations (3.10) and (3.11), leading in an underdetermined system of linear equations with 8 independent variables at each time step $k$. The least squares method is employed to solve this system. The computed values of $\Psi_{l,s}^{k}$ are then substituted back into Equation (2.11), yielding an approximate solution for the Burger's equation at time step $k$.

## 4. Stability analysis for the numerical scheme

In this section, we analyze the stability of the numerical scheme using the Fourier stability principle, assuming that the solution to Equation (3.10) has a specific form. By applying this principle, we assess the method's stability and performance, providing important insights into its accuracy and reliability. According to Fourier stability analysis, leading to

$$\Psi_{l,s}^{k} = \lambda^{k} e^{i\beta l\Delta x} e^{i\gamma s\Delta y} \tag{4.1}$$

where $\lambda$ is the amplification factor and $i = \sqrt{-1}$. The parameters $\beta$ and $\gamma$ are the mode numbers [24, 25]. The spline-difference equation is given by:

$$E_{l,s}^{k} * u_{l,s}^{k+1} - \frac{\Delta t}{2} G_{l,s}^{k+1} = D_{l,s}^{k} * u_{l,s}^{k} + \frac{\Delta t}{2} G_{l,s}^{k} \tag{4.2}$$

Substituted The terms $U$ and $G$ from Equations (2.12) and (3.3) into the left-hand side of Equation (4.2). and expressing it in terms of the coefficients $\Psi_{l,s}^{k}$, we obtain:

$$\begin{aligned} E_{l,s}^{k} * u_{l,s}^{k+1} - \frac{\Delta t}{2} G_{l,s}^{k+1} &= z_1 \Psi_{l-3,s-3}^{k+1} + z_2 \Psi_{l-3,s-2}^{k+1} + z_1 \Psi_{l-3,s-1}^{k+1} + z_3 \Psi_{l-2,s-3}^{k+1} + z_4 \Psi_{l-2,s-2}^{k+1} + z_3 \Psi_{l-2,s-1}^{k+1} \\ &\quad + z_1 \Psi_{l-1,s-3}^{k+1} + z_2 \Psi_{l-1,s-2}^{k+1} + z_1 \Psi_{l-1,s-1}^{k+1} \end{aligned}$$

where

$$z_1 = \left(E_{l,s}^{k} b_1 b_3 - \frac{\Delta t}{2} \mathcal{B}_1\right), \quad z_2 = \left(E_{l,s}^{k} b_1 b_4 - \frac{\Delta t}{2} \mathcal{B}_2\right), \; z_3 = \left(E_{l,s}^{k} b_2 b_3 - \frac{\Delta t}{2} \mathcal{B}_3\right) \; z_4 = \left(E_{l,s}^{k} b_2 b_4 - \frac{\Delta t}{2} \mathcal{B}_4\right)$$

The Fourier harmonics from Equation (4.1) are applied to the equation, and the terms are then simplified as follows:

$$\begin{aligned} E_{l,s}^{k} * u_{l,s}^{k+1} - \frac{\Delta t}{2} G_{l,s}^{k+1} &= z_1\left(\Psi_{l-3,s-3}^{k+1} + \Psi_{l-3,s-1}^{k+1} + \Psi_{l-1,s-3}^{k+1} + \Psi_{l-1,s-1}^{k+1}\right) + z_2\left(\Psi_{l-3,s-2}^{k+1} + \Psi_{l-1,s-2}^{k+1}\right) \\ &\quad + z_3\left(\Psi_{l-2,s-3}^{k+1} + \Psi_{l-2,s-1}^{k+1}\right) + z_4 \Psi_{l-2,s-2}^{k+1} \\ &= \lambda^{k+1}\Big[z_1\left(e^{i\beta(l-3)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\beta(l-3)\Delta x} e^{i\gamma(s-1)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-1)\Delta y}\right) \\ &\quad + z_2\left(e^{i\beta(l-3)\Delta x} e^{i\gamma(s-2)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-2)\Delta y}\right) \\ &\quad + z_3\left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\beta(l-2)\Delta x} e^{i\gamma(s-1)\Delta y}\right) + z_4\left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-2)\Delta y}\right)\Big] \end{aligned}$$

$$\begin{aligned} E_{l,s}^{k} * u_{l,s}^{k+1} - \frac{\Delta t}{2} G_{l,s}^{k+1} &= \lambda^{k+1}\left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-2)\Delta y}\right) \times \\ &\quad \left[z_1(2\cos(\beta\Delta x))(2\cos(\gamma\Delta y)) + z_2(2\cos(\beta\Delta x)) + z_3(2\cos(\gamma\Delta y)) + z_4\right] \end{aligned}$$

Similarly, the terms $U$ and $G$ from Equations (2.12) and (3.3) are substituted into the right-hand side of Equation (4.2). The equation is then rearranged in terms of the coefficients $\Psi_{l,s}^{k}$ to yield:

$$D_{l,s}^{k} * u_{l,s}^{k} + \frac{\Delta t}{2} G_{l,s}^{k} = z_5 \Psi_{l-3,s-3}^{k} + z_6 \Psi_{l-3,s-2}^{k} + z_5 \Psi_{l-3,s-1}^{k} + z_7 \Psi_{l-2,s-3}^{k} + z_8 \Psi_{l-2,s-2}^{k} + z_7 \Psi_{l-2,s-1}^{k} + z_5 \Psi_{l-1,s-3}^{k} + z_6 \Psi_{l-1,s-2}^{k+1} + z_5 \Psi_{l-1,s-1}^{k+1}$$

where

$z_5 = \left(D_{l,s}^{k} b_1 b_3 + \frac{\Delta t}{2}\mathcal{B}_1\right)$, $z_6 = \left(D_{l,s}^{k} b_1 b_4 + \frac{\Delta t}{2}\mathcal{B}_2\right)$, $z_7 = \left(D_{l,s}^{k} b_2 b_3 + \frac{\Delta t}{2}\mathcal{B}_3\right)$, and

$z_8 = \left(D_{l,s}^{k} b_2 b_4 + \frac{\Delta t}{2}\mathcal{B}_4\right)$

The Fourier harmonics from Equation (4.1) are substituted into the equation, and the terms are simplified as follows:

$$D_{l,s}^{k} * u_{l,s}^{k} + \frac{\Delta t}{2} G_{l,s}^{k} = z_5\left(\Psi_{l-3,s-3}^{k} + \Psi_{l-3,s-1}^{k} + \Psi_{l-1,s-3}^{k} + \Psi_{l-1,s-1}^{k+1}\right) + z_6\left(\Psi_{l-3,s-2}^{k} + \Psi_{l-1,s-2}^{k+1}\right) + z_7\left(\Psi_{l-2,s-3}^{k} + \Psi_{l-2,s-1}^{k}\right) + z_8\left(\Psi_{l-2,s-2}^{k}\right)$$

$$= \lambda^k \Big[ z_5\left(e^{i\beta(l-3)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\beta(l-3)\Delta x} e^{i\gamma(s-1)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-1)\Delta y}\right) + z_6\left(e^{i\beta(l-3)\Delta x} e^{i\gamma(s-2)\Delta y} + e^{i\beta(l-1)\Delta x} e^{i\gamma(s-2)\Delta y}\right) + z_7\left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-3)\Delta y} + e^{i\gamma(s-1)\Delta y}\right) + z_8\left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-2)\Delta y}\right)\Big]$$

$$D_{l,s}^{k} * u_{l,s}^{k} + \frac{\Delta t}{2} G_{i,s}^{k} = \lambda^k \left(e^{i\beta(l-2)\Delta x} e^{i\gamma(s-2)\Delta y}\right) \times \left[z_5(2\cos(\beta\Delta x))(2\cos(\gamma\Delta y)) + z_6(2\cos(\beta\Delta x)) + z_7(2\cos(\gamma\Delta y)) + z_8\right]$$

Hence, the amplification factor $\lambda$ is given by:

$$\lambda = \frac{z_5(2\cos(\beta\Delta x))(2\cos(\gamma\Delta y)) + z_6(2\cos(\beta\Delta x)) + z_7(2\cos(\gamma\Delta y)) + z_8}{z_1(2\cos(\beta\Delta x))(2\cos(\gamma\Delta y)) + z_2(2\cos(\beta\Delta x)) + z_3(2\cos(\gamma\Delta y)) + z_4}$$

Since the cosine terms are bounded as:

$$-1 \le \cos(\beta\Delta x) \le 1 \quad \text{and} \quad -1 \le \cos(\gamma\Delta y) \le 1$$

The maximum possible values occur when $\cos(\beta\Delta x) = 1$ and $\cos(\gamma\Delta y) = 1$, leading to:

$$\lambda = \frac{4z_5 + 2z_6 + 2z_7 + z_8}{4z_1 + 2z_2 + 2z_3 + z_4},$$

To ensure $|\lambda| \le 1$the coefficients must satisfy the condition:

$$4z_5 + 2z_6 + 2z_7 + z_8 \le 4z_1 + 2z_2 + 2z_3 + z_4$$

This ensures that the numerical scheme defined in Equation (3.10) is conditionally stable.

## 5. Numerical tests and discussion

In this section, we present the numerical results obtained using the proposed algorithm for two test problems. We evaluate the accuracy of the proposed technique using the $L_2$ and $L_\infty$ error norms, which are calculated as follows:

$$L_2 = \sqrt{h\sum_{j=1}^{n}\left|U_j - u_j\right|^2}, \qquad L_\infty = \max_{1\le j\le M}\left|U_j - u_j\right|,$$

where, $U_j$ and $u_j$ represent the approximate and exact solutions, respectively. The numerical results are presented graphically and in tabular form using MATLAB software.

Problem 5.1.

Consider the two-dimensional Burgers' equation as follows [27]:

$$u_t + u(u_x + u_y) = \delta(u_{xx} + u_{yy}) \qquad ,0 < x < 1 \,, 0 < y < 1 \,, t \geq 0, \tag{5.1}$$

where the exact solution to the problem is given by

$$u(x,y,t) = \frac{1}{1 + e^{\frac{(x+y-t)}{2\delta}}} \tag{5.2}$$

and $\delta > 0$ is the Reynolds number. The initial and boundary conditions are derived from the exact solutions. Error norms are recorded for different time level and are shown in Table 1. It is observed that the accuracy of the scheme, in terms of error norms, decreases as $m$ and $n$ increase. The 2D plots of the numerical and exact solutions at different time stages, showing the effect of time on $u(x, y, t)$ are displayed in Fig. 1. The 3D graphics of the exact and numerical solutions are shown in Figs. 2-5, clearly demonstrating good agreement with the exact solution.

Table 1. Accuracy test for the two-dimensional Burgers' Equation (5.1) with the exact solution (5.2) and $\delta = 4$.

| $m \times n$ | $t = 0.25$ | | $t = 0.5$ | | $t = 0.75$ | | $t = 1$ | |
|---|---|---|---|---|---|---|---|---|
| | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
| $2 \times 2$ | 1.3094e-03 | 9.2589e-04 | 1.3754e-03 | 9.7256e-04 | 1.4111e-03 | 9.9780e-04 | 1.4504e-03 | 1.0256e-03 |
| $4 \times 4$ | 2.0025e-04 | 2.5945e-04 | 2.0119e-04 | 2.6573e-04 | 2.1246e-04 | 2.7382e-04 | 2.3799e-04 | 2.8453e-04 |
| $6 \times 6$ | 3.0619e-04 | 7.0916e-04 | 1.0176e-04 | 1.3669e-04 | 1.0571e-04 | 1.4212e-04 | 1.1136e-04 | 1.5103e-04 |
| $8 \times 8$ | 5.1409e-05 | 8.4568e-05 | 5.1766e-05 | 8.7228e-05 | 5.5239e-05 | 9.2168e-05 | 8.8460e-04 | 1.0307e-04 |
| $12 \times 12$ | 2.3005e-05 | 4.5342e-05 | 2.3309e-05 | 4.7479e-05 | 7.9740e-05 | 2.3783e-04 | 1.0409e-04 | 2.3927e-04 |
| $16 \times 16$ | 1.2926e-05 | 2.9335e-05 | 1.9312e-05 | 3.1821e-05 | 6.1085e-05 | 4.7079e-05 | 2.0237e-04 | 1.2013e-04 |
| $20 \times 20$ | 8.2628e-05 | 2.0996e-05 | 2.1403e-05 | 2.4691e-05 | 9.3544e-05 | 5.7781e-05 | 4.2404e-04 | 2.3514e-04 |
| $24 \times 24$ | 1.9267e-05 | 8.2035e-05 | 2.4361e-05 | 8.2148e-05 | 1.4241e-05 | 1.1244e-05 | 9.5010e-04 | 5.1065e-04 |

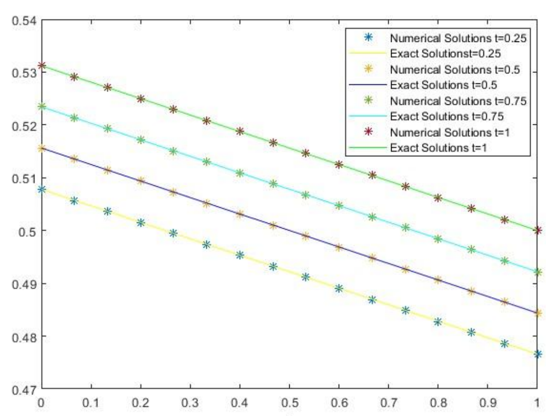


Fig. 1. Physical behavior of the numerical and exact solutions for Problem 5.1

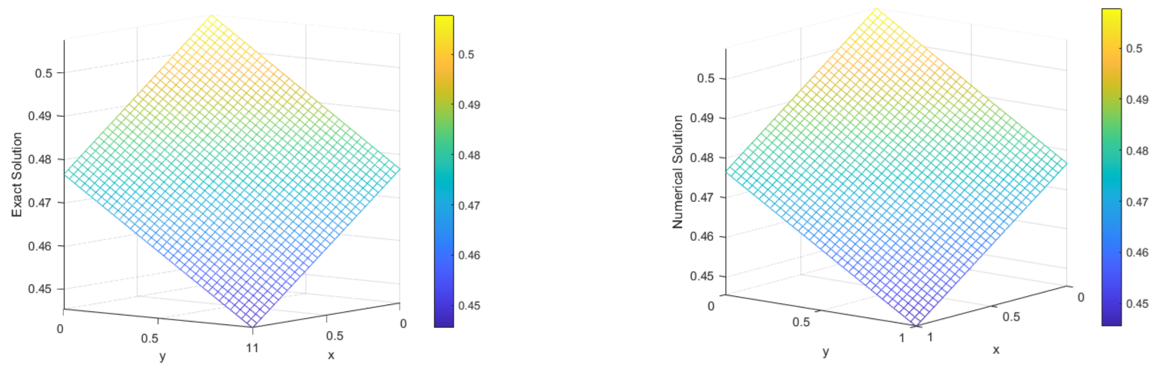


Fig. 2. Exact solutions vs numerical solutions for problem 5.1 at $t = 0.25$

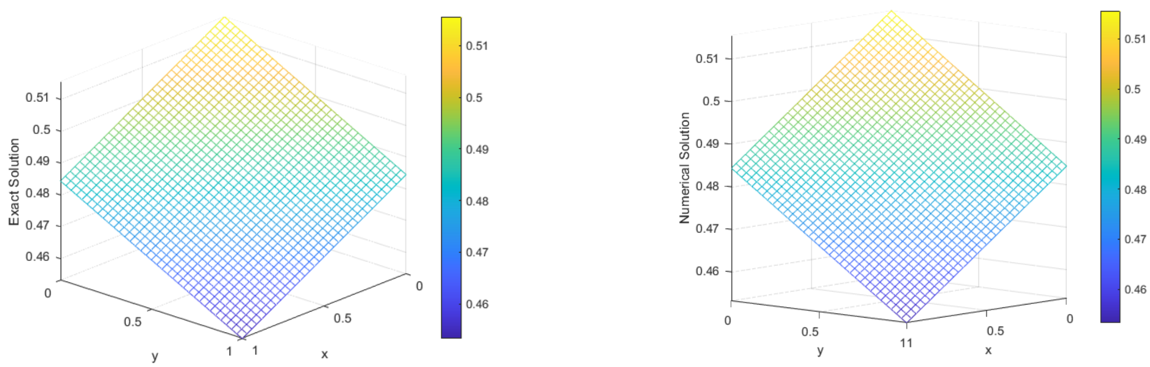


Fig. 3. Exact solutions vs numerical solutions for problem 5.1 at $t = 0.5$

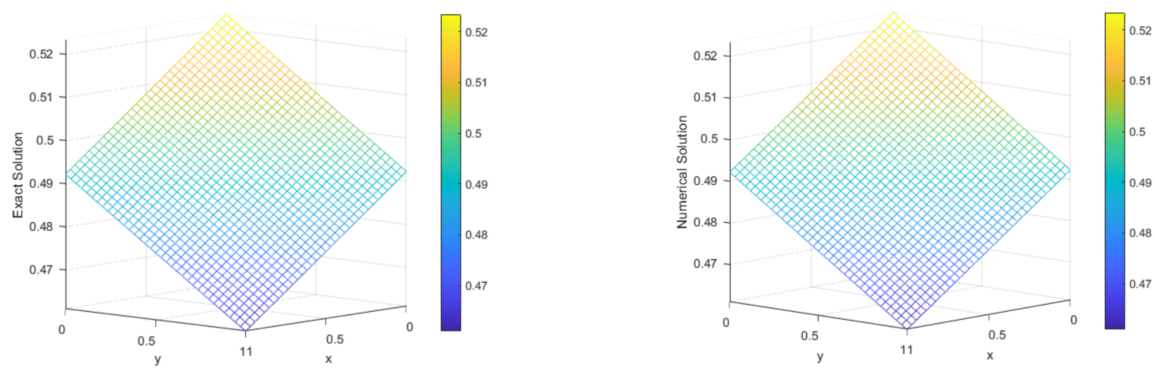


Fig. 4. Exact solutions vs numerical solutions for problem 5.1 at $t = 0.75$

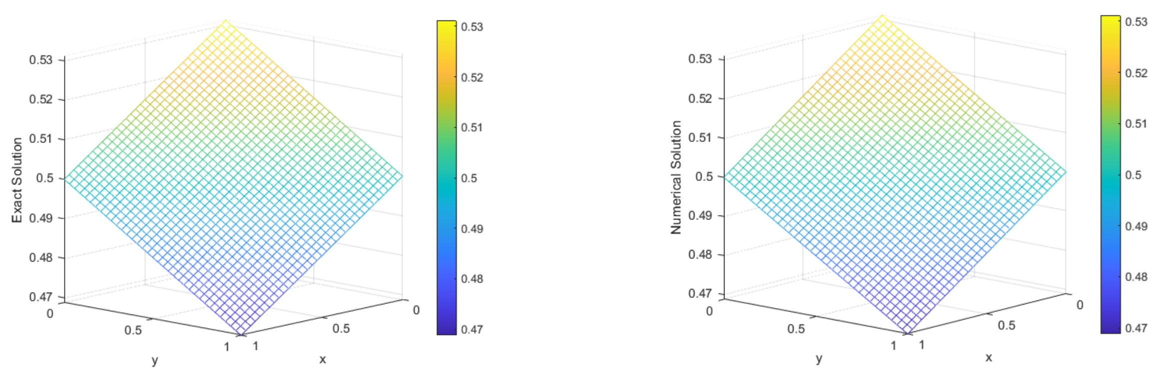


Fig. 5. Exact solutions vs numerical solutions for problem 5.1 at $t = 1$

Problem 5.2

Consider the two-dimensional Burgers' equation as follows [28]:

$$u_t + u(u_x + u_y) = \delta(u_{xx} + u_{yy}) \qquad , -0.5 < x, y < 0.5\, , t \geq 0 \tag{5.3}$$

with the exact solution

$$u(x, y, t) = \frac{1}{2} - \tanh\left(\frac{(x + y - t)}{2\delta}\right), \tag{5.4}$$

where $\delta > 0$ is the Reynolds number. The initial and boundary conditions are derived from the exact solutions. The $L_\infty$ and $L_2$ error norms are calculated for different time values and summarized in Table 2. It is observed that the accuracy of the scheme, in terms of error norms, decreases as $m$ and $n$ increase. Figure 6 presents the 2D plots of the numerical and exact solutions at various time stages, highlighting the effect of time on $u(x, y, t)$. The 3D representations of the exact and numerical solutions, shown in Figs. 7-10, demonstrate excellent agreement with the exact solution.

Table 2. Accuracy test for the two-dimensional Burgers' Equation (5.3) with the exact solution (5.4) and $\delta = 4$.

| $m \times n$ | $t = 0.25$ | | $t = 0.5$ | | $t = 0.75$ | | $t = 1$ | |
|---|---|---|---|---|---|---|---|---|
| | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ | $L_\infty$ | $L_2$ |
| $2 \times 2$ | 1.9489e-03 | 1.3781e-03 | 2.3381e-03 | 1.6533e-03 | 2.5381e-03 | 1.7947e-03 | 2.7679e-03 | 1.9572e-03 |
| $4 \times 4$ | 3.7158e-04 | 3.6059e-04 | 4.6426e-04 | 4.0494e-04 | 6.0232e-04 | 4.7419e-04 | 1.3843e-03 | 1.0370e-03 |
| $6 \times 6$ | 9.9058e-04 | 2.5516e-03 | 2.2847e-04 | 2.0487e-04 | 3.7598e-04 | 2.7204e-04 | 1.3843e-03 | 1.0370e-03 |
| $8 \times 8$ | 8.6883e-05 | 1.0697e-04 | 1.6732e-04 | 1.3747e-04 | 3.4759e-04 | 2.1894e-04 | 1.3753e-03 | 5.4129e-04 |
| $12 \times 12$ | 4.9509e-05 | 5.7588e-05 | 1.4652e-04 | 9.4656e-05 | 3.8468e-04 | 8.3991e-04 | 7.6515e-04 | 4.2456e-04 |
| $16 \times 16$ | 3.8538e-05 | 3.8687e-05 | 1.7203e-04 | 9.2764e-05 | 8.3903e-04 | 3.8863e-04 | 8.1254e-04 | 5.8516e-04 |
| $20 \times 20$ | 3.5017e-05 | 2.9633e-05 | 2.3306e-04 | 1.1267e-05 | 1.7102e-04 | 7.3182e-04 | 6.5285e-04 | 4.1049e-04 |
| $24 \times 24$ | 6.7180e-05 | 2.8827e-05 | 2.8933e-05 | 3.1012e-05 | 3.7162e-04 | 1.5004e-04 | 4.2404e-04 | 2.3514e-04 |

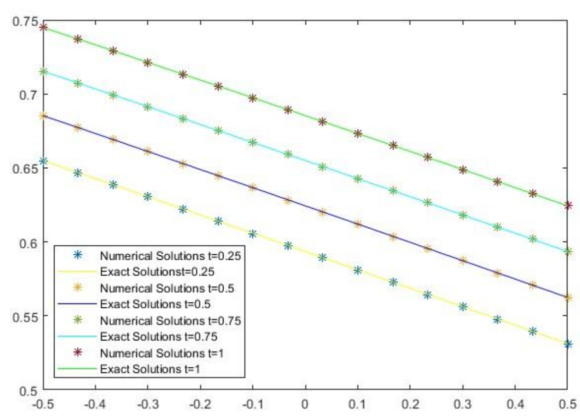


Fig. 6. Physical behavior of the numerical and exact solutions for Problem 5.2

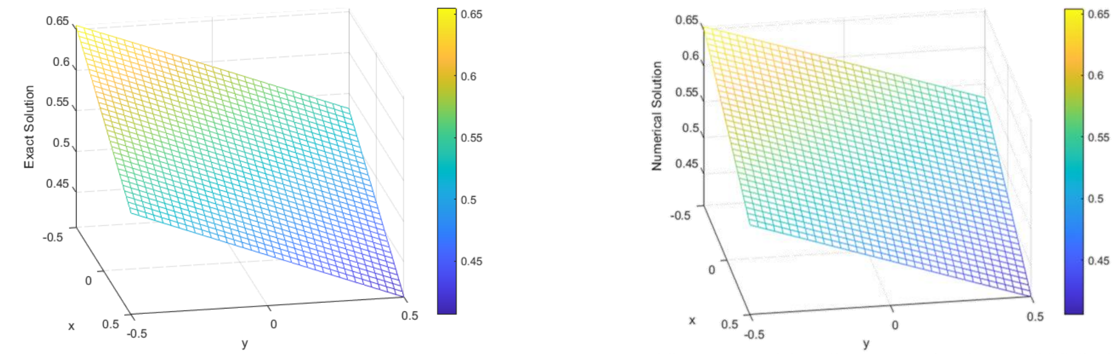


Fig. 7. Exact solutions vs numerical solutions for problem 5.2 at $t = 0.25$

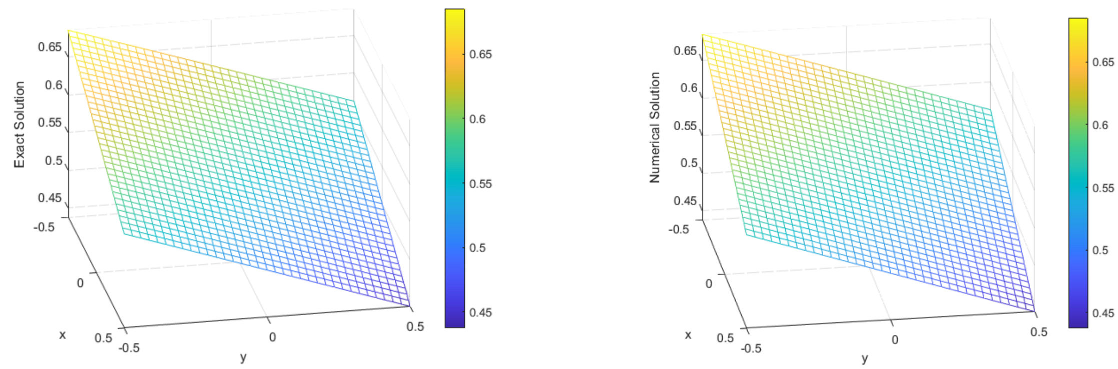


Fig. 8. Exact solutions vs numerical solutions for problem 5.2 at $t = 0.5$

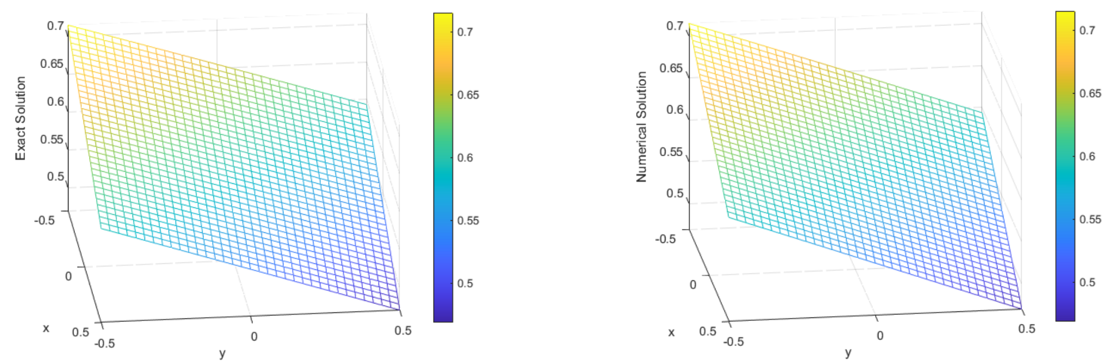


Fig. 9. Exact solutions vs numerical solutions for problem 5.2 at $t = 0.75$

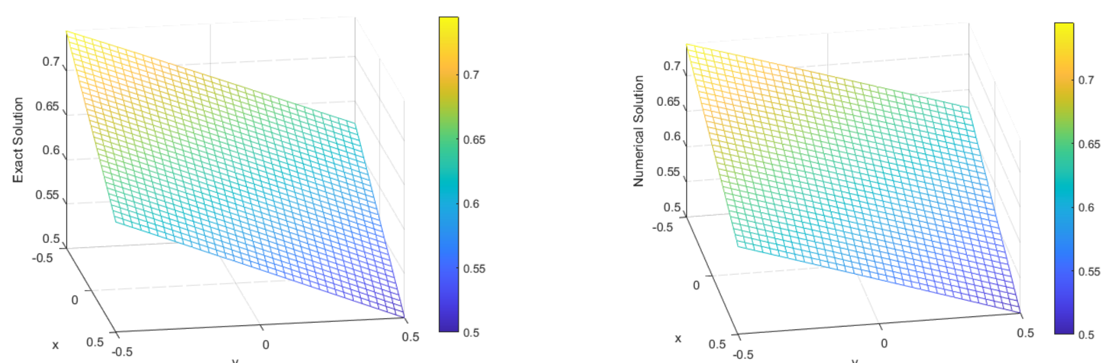


Fig. 10. Exact solutions vs numerical solutions for problem 5.2 at $t = 1$

## 6. Conclusions

In this study, a bicubic trigonometric B-spline interpolation technique was employed to solve the nonlinear reaction-diffusion equation, specifically the well-known two-dimensional burger's equation, using a collocation approach that incorporates bicubic trigonometric B-spline functions and a θ-weighted scheme. For discretization along spatial and temporal grids, the proposed numerical technique uses bicubic trigonometric B-spline functions and finite difference approach, respectively. To the authors' knowledge, this method has not been previously used to solve the two-dimensional Burgers' equation. However, it is worth noting that the trigonometric B-spline method was recently proposed for solving the Burgers. The conditional stability of the proposed scheme is also analyzed in this study. The method's accuracy was validated through several problems, with numerical results presented in tables and graphs, demonstrating good agreement with the exact solutions. The numerical results further show that only a few numbers of grid points ($m$ and $n$) are required to achieve a high degree of accuracy, making the method computationally efficient. It was concluded that the proposed method is well suited for solving the two-dimensional burger's equation, as it provides accurate solutions while reducing computational effort and time.

Therefore, it is suggested that this method can serve as an alternative for solving higher-dimensional problems. The feasibility of extending the present approach may prove useful in addressing higher-dimensional partial differential equations that arise in various applications in science and engineering.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

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