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On the Analysis and Solution Structure of Generalized Hemivariational Inclusion Problems

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Abstract. This paper introduces and analyzes a new class of generalized hemivariational inclusion problems. We establish the existence and uniqueness of solutions under mild assumptions and develop an efficient iterative algorithm for their numerical approximation. To demonstrate the practical utility of our theoretical framework, we apply it to a frictional contact problem in elasticity. The model involves an elastic body in contact with a rigid foundation, governed by a nonmonotone friction law that depends on both normal and tangential displacements. Our results provide a comprehensive solution, from theory to computation, for this challenging class of nonsmooth systems.

1. Comprehensive historical background

The mathematical framework of hemivariational inequalities, first introduced by Panagiotopoulos [1], provides a powerful tool for modeling a wide range of nonsmooth phenomena in mechanics and engineering. These inequalities generalize classical variational principles to accommodate nonconvex and nonsmooth potentials, leading to an extensive body of literature, as chronicled in [2, 3]. The driving force behind this growth is the prevalence of such models in applications spanning contact mechanics, nanotechnology, and physics.

A central challenge in this field involves contact problems, where a deformable body interacts with a rigid foundation. The boundary conditions describing this interaction are often governed by nonmonotone laws, meaning the response (e.g., friction or normal reaction) is not a simple linear function of the displacement. The added complexity of coupling between the normal and tangential directions further complicates both the analytical study and numerical approximation of

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these systems. The primary analytical tool for handling such nonsmoothness is the subdifferential calculus introduced by Clarke [4], with its connections to optimization detailed in [5].

Building upon recent developments [6–10] and inspired by applications in [11–15], this paper makes a threefold contribution. First, we formulate a nonsmooth generalized optimization problem and prove the existence and uniqueness of its solution. Second, we derive a numerical error estimate for approximating this solution. Finally, we apply this abstract framework to resolve a static frictional contact problem for an elastic body. The considered problem is particularly challenging due to its nonmonotone friction law, which is influenced by both normal and tangential displacements. Our results thus provide a complete analytical and numerical foundation for this important class of contact models.

2. Prelude and Main Results

This section lays the mathematical foundation for our study. We begin by recalling essential concepts from nonsmooth analysis and the theory of monotone operators. Subsequently, we formulate the main problems, a generalized operator inclusion and an equivalent optimization problem, and establish their well-posedness under a set of stated assumptions.

Let $\mathbb{X}$ be a normed space with norm $\|\cdot\|_{\mathbb{X}}$, and let $\mathbb{X}^*$ denote its topological dual. The duality pairing between $\mathbb{X}^*$ and $\mathbb{X}$ is denoted by $\langle\cdot,\cdot\rangle_{\mathbb{X}^*\times\mathbb{X}}$. The value of a generic positive constant, denoted by ϑ, may change from line to line.

Let $j\colon \mathbb{X} \to \mathbb{R}$ be a locally Lipschitz continuous function. The generalized (Clarke) directional derivative of j at $x \in \mathbb{X}$ in the direction $v \in \mathbb{X}$ is defined by

$$j^0(x;v) = \limsup_{y\to x,\lambda\searrow 0} \frac{j(y+\lambda v) - j(y)}{\lambda}.$$

The generalized (Clarke) subdifferential of j at x is the subset of dual space $\mathbb{X}^*$ given by

$$\partial j(x) = \{\vee \in \mathbb{X}^* \mid \langle \vee, v\rangle_{\mathbb{X}^*\times\mathbb{X}} \leq j^0(x;v),\ \forall v \in \mathbb{X}\}.$$

If $j : \mathbb{X}^n \to \mathbb{R}$ is a locally Lipschitz function of n variables, then $\partial_i j$ and j_i^0 denote the Clarke subdifferential and generalized directional derivative with respect to the i-th variable, *respectively*.

We now recall several key properties of nonlinear operators.

Definition 2.1. *An operator $\mathcal{D} : \mathbb{X} \to \mathbb{X}^*$ is said to be:*

(i) *monotone if*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v\rangle \geq 0,\ \forall u, v \in \mathbb{X};$$

(ii) *strongly monotone with constant $\alpha_{\mathcal{D}} > 0$ if*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v\rangle \geq \alpha_{\mathcal{D}}\|u - v\|^2,\ \forall u, v \in \mathbb{X};$$

(iii) *relaxed monotone with constant $\alpha_{\mathcal{D}} > 0$ if*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v\rangle \geq -\alpha_{\mathcal{D}}\|u - v\|^2,\ \forall u, v \in \mathbb{X};$$

(iv) *cocoercive with constant $\alpha_{\mathcal{D}} > 0$ if*

$$\langle \mathcal{D}(u) - \mathcal{D}(v), u - v \rangle \geq \alpha_{\mathcal{D}} \|\mathcal{D}(u) - \mathcal{D}(v)\|^2, \ \forall u, v \in \mathbb{X};$$

(v) *Lipschitz continuous with constant $\alpha_{\mathcal{D}} > 0$ if*

$$\|\mathcal{D}(u) - \mathcal{D}(v)\| \leq \alpha_{\mathcal{D}} \|u - v\|, \ \forall u, v \in \mathbb{X};$$

(vi) *$\alpha_{\mathcal{D}}$-expansive if*

$$\|\mathcal{D}(u) - \mathcal{D}(v)\| \geq \alpha_{\mathcal{D}} \|u - v\|, \ \forall u, v \in \mathbb{X},$$

if $\alpha_{\mathcal{D}} = 1$, the operator is simply called expansive.

Remark 2.1. *A cocoercive operator with constant $\alpha_{\mathcal{D}}$ is also monotone and $\dfrac{1}{\alpha_{\mathcal{D}}}$-Lipschitz continuous. Conversely, an L-Lipschitz continuous and $\alpha-$strongly monotone operator is cocoercive with constant $\dfrac{\alpha}{L}$. Thus, cocoercivity is an intermediate concept that lies between strong monotonicity and Lipschitz continuity.*

The following example illustrates a common source of cocoercive operators.

Example 2.1. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{D} : \mathcal{H} \to \mathcal{H}$ be a nonexpansive mapping. Then the operator $T = I - \mathcal{D}$, where I is the identity, is $\frac{1}{2}$-cocoercive.*

Proof. For any $u, v \in \mathcal{H}$, we have

$$\begin{aligned}
\|Tu - Tv\|^2 &= \|(u - v) - (\mathcal{D}u - \mathcal{D}v)\|^2 \\
&= \|u - v\|^2 - 2\langle u - v, \mathcal{D}u - \mathcal{D}v \rangle + \|\mathcal{D}u - \mathcal{D}v\|^2 \\
&\leq 2\|u - v\|^2 - 2\langle u - v, \mathcal{D}u - \mathcal{D}v \rangle, \text{ (since } \mathcal{D} \text{ is nonexpansive)} \\
&= 2\langle u - v, (u - v) - (\mathcal{D}u - \mathcal{D}v) \rangle \\
&= 2\langle u - v, Tu - Tv \rangle.
\end{aligned}$$

Hence, T is $\frac{1}{2}$-cocoercive. □

2.1. **Problem Formulation and Assumptions.** Let $\mathbb{X}$ be a reflexive Banach space and $\mathbb{V}$ a Banach space. Let $\zeta \in \mathcal{L}(\mathbb{V}, \mathbb{X})$ be a linear continuous operator with norm $\vartheta_\zeta = \|\zeta\|_{\mathcal{L}(\mathbb{V},\mathbb{X})}$, and let $\zeta^* : \mathbb{X}^* \to \mathbb{V}^*$ denote its adjoint. We consider the following generalized operator inclusion problem: Find $u \in \mathbb{V}$ such that

$$f \in \mathcal{D}(u, u) + \zeta^* \partial_2 J(\zeta u, \zeta u), \tag{2.1}$$

where $\mathcal{D}\colon \mathbb{V} \times \mathbb{V} \to \mathbb{V}^*$, $J : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ and $f \in \mathbb{V}^*$. Our analysis rests on the following hypotheses.

(A): Properties of $\mathcal{D}$:

$\mathcal{D}\colon \mathbb{V}\times\mathbb{V}\to\mathbb{V}^*$ satisfies

$$
\begin{cases}
(\mathbf{a_1})\ \mathcal{D} \text{ is bilinear, symmetric, and bounded.}\\
(\mathbf{a_2})\ \mathcal{D} \text{ is cocoercive } \exists\alpha_{\mathcal{D}}>0 \text{ such that}\\
\langle \mathcal{D}(u,u)-\mathcal{D}(v,v),u-v\rangle_{\mathbb{V}^*\times\mathbb{V}} \geq \alpha_{\mathcal{D}}\|\mathcal{D}(u,u)-\mathcal{D}(v,v)\|^2_{\mathbb{V}^*},\ \forall u,v\in\mathbb{V}.\\
(\mathbf{a_3})\ \mathcal{D} \text{ is Lipschitz continuous } \exists\beta_{\mathcal{D}},\lambda_{\mathcal{D}}>0 \text{ such that}\\
\|\mathcal{D}(u,u)-\mathcal{D}(v,v)\|_{\mathbb{V}^*}\leq \beta_{\mathcal{D}}\|u-v\|_{\mathbb{V}}+\lambda_{\mathcal{D}}\|u-v\|_{\mathbb{V}}, \forall u,v\in\mathbb{V}.
\end{cases}
\tag{2.2}
$$

(B): Properties of J:

$J:\mathbb{X}\times\mathbb{X}\to\mathbb{R}$ satisfies

$$
\begin{cases}
(\mathbf{b_1})\ J(\cdot,\cdot) \text{ is locally Lipschitz continuous with respect to its second variable.}\\
(\mathbf{b_2})\ \partial_2 J \text{ has linear growth: } \exists\vartheta_0,\vartheta_1,\vartheta_2\geq 0 \text{ such that}\\
\|\partial_2 J(w,v)\|_{\mathbb{X}^*}\leq \vartheta_0+\vartheta_1\|v\|_{\mathbb{X}}+\vartheta_2\|w\|_{\mathbb{X}}, \forall\ w,v\in\mathbb{X}.\\
(\mathbf{b_3})\ \partial_2 J \text{ is relaxed monotone: } \exists\rho,\eta\geq 0 \text{ such that } \forall w_1,w_2,v_1,v_2\in\mathbb{X},\\
J_2^0(w_1,v_1;v_2-v_1)+J_2^0(w_2,v_2;v_1-v_2)\\
\leq \rho\|v_1-v_2\|^2_{\mathbb{X}}+\eta\|w_1-w_2\|_{\mathbb{X}}\|v_1-v_2\|_{\mathbb{X}}.
\end{cases}
\tag{2.3}
$$

(C):

$$f\in\mathbb{V}^*. \tag{2.4}$$

(D): Smallness Condition:

$$\alpha_{\mathcal{D}}\left(\beta_{\mathcal{D}}+\lambda_{\mathcal{D}}\right)^2 > (2\rho+\eta)\vartheta_\zeta^2. \tag{2.5}$$

Remark 2.2. *Condition **(B)**(b_3) is equivalent to the following inequality for the generalized subdifferential:*

$$\langle \partial_2 J(w_1,v_1)-\partial_2 J(w_2,v_2),v_1-v_2\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq -\rho\|v_1-v_2\|^2_{\mathbb{X}}-\eta\|w_1-w_2\|_{\mathbb{X}}\|v_1-v_2\|_{\mathbb{X}},$$
$$\forall w_1,w_2,v_1,v_2\in\mathbb{X}. \tag{2.6}$$

*If J is independent of its first variable, then **(B)**(b_3) holds with $\eta=0$, reducing to the standard relaxed monotonicity condition:*

$$\langle \partial J(v_1)-\partial J(v_2),v_1-v_2\rangle_{\mathbb{X}^*\times\mathbb{X}}\geq -\rho\|v_1-v_2\|^2_{\mathbb{X}}, \forall v_1,v_2\in\mathbb{X}. \tag{2.7}$$

2.2. **Well-Posedness of the Operator Inclusion.** We first establish the uniqueness and a priori estimate for solutions of the inclusion problem (2.1).

Lemma 2.1. ***[Uniqueness and Estimate]*** *Assume that **(A)-(D)** hold. If problem* (2.1) *has a solution $u\in\mathbb{V}$, then it is unique. Moreover, there exists a constant $\vartheta>0$ such that*

$$\|u\|_{\mathbb{V}}\leq\vartheta\left(1+\|f\|_{\mathbb{V}^*}\right). \tag{2.8}$$

Proof. Let $u \in \mathbb{V}$ be a solution to (2.1). Then there exists $z \in \partial_2 J(\zeta u, \zeta u)$ such that

$$f = \mathcal{D}(u, u) + \zeta^* z.$$

By the definition of the generalized subdifferential, for all $v \in \mathbb{V}$

$$\begin{aligned} \langle f - \mathcal{D}(u, u), v\rangle_{\mathbb{V}^*\times\mathbb{V}} &= \langle \zeta^* z, v\rangle_{\mathbb{V}^*\times\mathbb{V}} \\ &= \langle z, \zeta v\rangle_{\mathbb{X}^*\times\mathbb{X}} \\ &\leq J_2^0(\zeta u, \zeta u; \zeta v). \end{aligned} \tag{2.9}$$

To prove uniqueness, assume u_1 and u_2 are two different solutions. Setting $v = u_2 - u_1$ in the inequality (2.9) for u_1 and $v = u_1 - u_2$ in the inequality (2.9) for u_2, then, we have

$$\langle f, u_2 - u_1\rangle_{\mathbb{V}^*\times\mathbb{V}} - \langle \mathcal{D}(u_1, u_1), u_2 - u_1\rangle_{\mathbb{V}^*\times\mathbb{V}} \leq J_2^0(\zeta u_1, \zeta u_1; \zeta u_2 - \zeta u_1), \tag{2.10}$$

and

$$\langle f, u_1 - u_2\rangle_{\mathbb{V}^*\times\mathbb{V}} - \langle \mathcal{D}(u_2, u_2), u_1 - u_2\rangle_{\mathbb{V}^*\times\mathbb{V}} \leq J_2^0(\zeta u_2, \zeta u_2; \zeta u_1 - \zeta u_2). \tag{2.11}$$

Adding (2.10) and (2.11), and applying **(A)**(a_2), (a_3) and **(B)**(b_3), we obtain

$$\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 \|u_1 - u_2\|_{\mathbb{V}}^2 \leq (\rho + \eta)\|\zeta u_1 - \zeta u_2\|_{\mathbb{X}}^2.$$

Thus,

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 - (\rho + \eta)\vartheta_\zeta^2)\|u_1 - u_2\|_{\mathbb{V}}^2 \leq 0.$$

By **(D)**, this implies $u_1 = u_2$, proving uniqueness.
To derive the estimate (2.8), set $v = -u$ in (2.9) to get

$$\langle \mathcal{D}(u, u), u\rangle_{\mathbb{V}^*\times\mathbb{V}} \leq J_2^0(\zeta u, \zeta u; -\zeta u) + \langle f, u\rangle_{\mathbb{V}^*\times\mathbb{V}}. \tag{2.12}$$

From **(B)**(b)(c), we have

$$\begin{aligned} J_2^0(\zeta u, \zeta u; -\zeta u) &\leq (\rho + \eta)\|\zeta u\|_{\mathbb{X}}^2 - J_2^0(0, 0; \zeta u) \\ &\leq (\rho + \eta)\|\zeta u\|_{\mathbb{X}}^2 + \vartheta_0\|\zeta u\|_{\mathbb{X}}. \end{aligned} \tag{2.13}$$

Combing (2.12) and (2.13), and using the cocoercivity and Lipschitz continuity of $\mathcal{D}$ from **(A)**, we get

$$\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 \|u\|_{\mathbb{V}}^2 \leq (\rho + \eta)\|\zeta u\|_{\mathbb{X}}^2 + \vartheta_0\|\zeta u\|_{\mathbb{X}} + \|f\|_{\mathbb{V}^*}\|u\|_{\mathbb{V}}$$

and rearranging

$$(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 - (\rho + \eta)\vartheta_\zeta^2)\|u\|_{\mathbb{V}} \leq \vartheta(1 + \|f\|_{\mathbb{V}^*}).$$

Therefore, from **(D)** we get the desired estimate (2.8). □

2.3. **Equivalent Optimization Problem.** We now introduce a generalized optimization problem equivalent to (2.1). Define the functional by

$$\Upsilon(w,v) = \frac{1}{2}\langle \mathcal{D}(v,v), v\rangle_{\mathbb{V}^*\times\mathbb{V}} - \langle f, v\rangle_{\mathbb{V}^*\times\mathbb{V}} + J(\zeta w, \zeta v). \tag{2.14}$$

Consider the problem of finding such that

$$0 \in \partial_2 \Upsilon(u,u). \tag{2.15}$$

The link between the inclusion (2.1) and the optimization problem (2.15) is established through the properties of Υ.

Lemma 2.2. ***[Properties of*** Υ***]*** *Assume that* ***(A)-(D)*** *hold. The functional* Υ *defined in* (2.14) *satisfies*

(i) $\Upsilon(w,\cdot)$ *is locally Lipschitz continuous for all* $w \in \mathbb{V}$.
(ii) $\partial_2\Upsilon(w,v) \subseteq \mathcal{D}(v,v) - f + \zeta^*\partial_2 J(\zeta w, \zeta v),\ \forall w, v \in \mathbb{V}$.
(iii) $\Upsilon(w,\cdot)$ *is strictly convex and cocoercive for all* $w \in \mathbb{V}$.

Proof. (i) This follows because $\Upsilon(w,\cdot)$ is a sum of functions that are locally Lipschitz continuous in v.

(ii) The functions $f_1(v) = \frac{1}{2}\langle \mathcal{D}(v,v), v\rangle$ and $f_2(v) = \langle f, v\rangle$ are strictly differentiable with $f_1'(v) = \mathcal{D}(v,v)$ and $f_2'(v) = f$. Applying the chain rule for the generalized subgradient to $J(\zeta w, \zeta v)$ yields the result.

(iii) To show that $v \longmapsto \partial_2\Upsilon(w,v)$ is cocoercive, let $\Upsilon(w,\cdot)$ is strongly convex for all $w \in \mathbb{V}$ and $g_i = \mathcal{D}(v_i, v_i) - f + \zeta^* z_i$. Then

$$\begin{aligned}\langle g_1 - g_2, v_1 - v_2\rangle &= \langle \mathcal{D}(v_1,v_1) - \mathcal{D}(v_2,v_2), v_1 - v_2\rangle + \langle z_1 - z_2, \zeta v_1 - \zeta v_2\rangle \\ &\geq \alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 \|v_1 - v_2\|_{\mathbb{V}}^2 - \rho\|\zeta v_1 - \zeta v_2\|_{\mathbb{V}}^2 \quad \text{(by } \textbf{(A)} \text{ and (2.6))} \\ &\geq \left(\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 - \rho\vartheta_\zeta^2\right)\|v_1 - v_2\|_{\mathbb{V}}^2. \end{aligned}\tag{2.16}$$

By (A), the constant is positive, proving cocoercivity. Cocoercivity implies strong monotonicity, which in turn implies strict convexity. A standard argument using the Lebourg mean value theorem [16] shows that $\Upsilon(w,\cdot)$ is also cocoercive as a functional.

□

We now prove the existence of a unique solution to the optimization problem (2.15) via a fixed-point argument.

Lemma 2.3. ***[Existence for the Optimization Problem]*** *Assume that (A)-(D) hold. Then, problem* (2.15) *has a unique solution.*

Proof. For each $w \in \mathbb{V}$, define the operator by

$$\Gamma w = \arg\min_{v\in\mathbb{V}} \Upsilon(w,v). \tag{2.17}$$

By Lemma 2.2(i) and (iii), for fixed w, $\Upsilon(w,\cdot)$ is proper, lower semicontinuous, strictly convex and cocoercive. hence, a unique minimizer Γw exists, and it is characterized by the inclusion

$0 \in \partial_2 \Upsilon(w, \Gamma w)$. We now show that Γ is a contraction. Let and set $\bar{u}_i = \Gamma u_i$ for $i = 1, 2$. Then, $0 \in \partial_2 \Upsilon(u_i, \bar{u}_i)$, which is equivalent to $\langle f - \mathcal{D}(\bar{u}_i, \bar{u}_i), v\rangle$. Setting $v = \bar{u}_2 - \bar{u}_1$ for $i = 1$ and $v = \bar{u}_1 - \bar{u}_2$ for $i = 2$, then adding the inequality, we obtain

$$\langle \mathcal{D}(\bar{u}_1, \bar{u}_1) - \mathcal{D}(\bar{u}_2, \bar{u}_2), u_1 - u_2\rangle \leq J_2^0(\zeta u_1, \zeta \bar{u}_1; \zeta \bar{u}_2 - \zeta \bar{u}_1) + J_2^0(\zeta u_2, \zeta \bar{u}_2; \zeta \bar{u}_1 - \zeta \bar{u}_2). \tag{2.18}$$

Applying **(A)**(b)(c) and **(B)**(c), we get

$$\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 \|\bar{u}_1 - \bar{u}_2\|^2 \leq \rho \|\zeta \bar{u}_1 - \zeta \bar{u}_2\|^2 + \eta \|\zeta u_1 - \zeta u_2\| \|\zeta \bar{u}_1 - \zeta \bar{u}_2\|. \tag{2.19}$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, and the bound $\|\zeta\| \leq \vartheta_\zeta$, we get

$$\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 \|\bar{u}_1 - \bar{u}_2\|^2 \leq \rho \vartheta_\zeta^2 \|\bar{u}_1 - \bar{u}_2\|^2 + \frac{\eta \vartheta_\zeta^2}{2}\left(\|u_1 - u_2\|^2 + \|\bar{u}_1 - \bar{u}_2\|^2\right). \tag{2.20}$$

Rearranging the terms of (2.20), we obtain

$$\|\bar{u}_1 - \bar{u}_2\|^2 \leq \frac{\eta \vartheta_\zeta^2}{2\alpha_{\mathcal{D}}(\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 - 2\rho \vartheta_\zeta^2 - \eta \vartheta_\zeta^2} \|u_1 - u_2\|^2. \tag{2.21}$$

The smallness condition **(D)** ensures the coefficient is less 1, so Γ is a contraction. By the Banach fixed-point theorem, Γ has a unique fixed point satisfying $\Gamma u^* = u^*$, which is equivalent to $0 \in \partial_2 \Upsilon(u^*, u^*)$. □

We now state our first main result, which synthesizes the preceding lemmas.

Theorem 2.1. ***[Equivalence and Well-Posedness]*** *Assume that* **(A)-(D)** *hold. Then, the operator inclusion problem* (2.1) *and the optimization problem* (2.15) *are equivalent. Moreover, there exists a unique solution to both problems, and it satisfies the priori estimate*

$$\|u\|_{\mathbb{V}} \leq \vartheta\left(1 + \|f\|_{\mathbb{V}^*}\right)$$

where $\vartheta > 0$ is a generic constant.

Proof. The equivalence follows directly from Lemma 2.2(ii). Lemma 2.3 guarantees the existence of a unique solution to (2.15), which is therefore also the unique solution to (2.1). The a priori estimate is provided by Lemma 2.1. □

2.4. **Discrete Approximation and Error Estimate.** We now introduce a discrete approximation scheme and derive an associated error estimate. Let $\varsigma > 0$ be a discretization parameter and let $\mathbb{V}^\varsigma \subset \mathbb{V}$ be a finite-dimensional subspace.

The discrete problem is: find such that

$$0 \in \partial_2 \Upsilon(u^\varsigma, u^\varsigma). \tag{2.22}$$

Under assumptions **(A)-(D)**, this problem has a unique solution.

Our second main result provides an abstract error estimate for the discrete approximation.

Theorem 2.2. ***[Error Estimate]*** *Assume that* ***(A)-(D)*** *hold. Let u and u^ς be the unique solutions to* (2.15) *and* (2.22)*, respectively. Then, there exists a constant $\vartheta > 0$, independent of ζ, such that*

$$\|u - u^\varsigma\|^2 \leq \vartheta \inf_{v^\varsigma \in \mathbb{V}^\varsigma} \left\{ \|u - v^\varsigma\|^2 + \|\zeta u - \zeta v^\varsigma\| + \Re(u, v^\varsigma) \right\}, \tag{2.23}$$

where the residual term $\Re(u, v^\varsigma)$ is defined by

$$\Re(u, v^\varsigma) = \langle \mathcal{D}(u, u), v^\varsigma - u \rangle + \langle f, u - v^\varsigma \rangle. \tag{2.24}$$

Proof. Since u and u^ς are solutions, they satisfy the variational inequalities:

$$\langle f - \mathcal{D}(u, u), v \rangle \leq J_2^0(\zeta u, \zeta u; \zeta v), \forall v \in \mathbb{V}, \tag{2.25}$$

$$\langle f - \mathcal{D}(u^\varsigma, u^\varsigma), v \rangle \leq J_2^0(\zeta u^\varsigma, \zeta u^\varsigma; \zeta v), \forall v \in \mathbb{V}^\varsigma. \tag{2.26}$$

Setting $v = u^\varsigma - u$ in (2.25), and $v = v^\varsigma - u^\varsigma$ in (2.26) for an arbitrary. Adding the resulting inequalities gives:

$$\begin{aligned} &\langle f, v^\varsigma - u \rangle + \langle \mathcal{D}(u^\varsigma, u^\varsigma) - \mathcal{D}(u, u), u^\varsigma - u \rangle - \langle \mathcal{D}(u^\varsigma, u^\varsigma), v^\varsigma - u \rangle \\ &\quad \leq J_2^0(\zeta u, \zeta u; \zeta u^\varsigma - \zeta u) + J_2^0(\zeta u^\varsigma, \zeta u^\varsigma; \zeta v^\varsigma - \zeta u^\varsigma). \end{aligned} \tag{2.27}$$

Using the subadditivity J_2^0, condition **(B)**(b_3), and the boundedness of (from the discrete version of Lemma 2.1), the right-hand side can be bounded by $(\rho + \eta)\varsigma_\zeta^2 \|u - u^\varsigma\|$.

Rearranging terms and using the definition of the residual (2.24), we arrive at:

$$\langle \mathcal{D}(u^\varsigma, u^\varsigma) - \mathcal{D}(u, u), u^\varsigma - u \rangle \leq \langle \mathcal{D}(u^\varsigma, u^\varsigma) - \mathcal{D}(u, u), v^\varsigma - u \rangle + \Re(u, v^\varsigma) + (\rho + \eta)\vartheta_\zeta^2 \|u - u^\varsigma\|.$$

Applying the properties of $\mathcal{D}$ from **(A)** and Young's inequality, we obtain

$$\left(\alpha_{\mathcal{D}} (\beta_{\mathcal{D}} + \lambda_{\mathcal{D}})^2 - (\rho + \eta)\vartheta_\zeta^2 - \varepsilon \right) \|u - u^\varsigma\|^2 \leq \frac{\vartheta}{4\varepsilon} \|u - v^\varsigma\|^2 + \Re(u, v^\varsigma) + \vartheta \|\zeta u - \zeta v^\varsigma\|, \ \forall v^\varsigma \in \mathbb{V}^\varsigma. \tag{2.28}$$

Choosing $\varepsilon > 0$ sufficiently small and using **(A)** yields the final estimate (2.23). We obtain the desired results. □

3. Application to an Elastic Contact Problem with Nonmonotone Boundary Conditions

In this section, we demonstrate the applicability of our abstract results from Section 3 to a significant problem in solid mechanics: the static deformation of an elastic body in contact with a reactive foundation. The model features nonmonotone, multivalued boundary conditions, which are essential for capturing complex interface phenomena like adhesion, surface roughness, or friction laws that change with displacement.

3.1. **Physical Setting and Classical Formulation.** Consider an elastic body occupying a domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary $\partial\Omega$. The boundary is partitioned into three disjoint, measurable parts: Γ_D, Γ_N, and Γ_C, where $\text{meas}(\Gamma_D) > 0$. The body is subject to the following conditions:

- **Clamped Condition:** $\mathbf{u} = \mathbf{0}$ on Γ_D.
- **Surface Traction:** A density of surface forces $\mathbf{f}_N$ acts on Γ_N.
- **Body Force:** A density of volume forces $\mathbf{f}_B$ acts in Ω.
- **Contact Condition:** The body may come into contact with a reactive foundation on Γ_C.

The state of the body is described by the displacement field $\mathbf{u}\colon \Omega \to \mathbb{R}^d$ and the associated stress field $\boldsymbol{\sigma}\colon \Omega \to \mathbb{S}^d$, where $\mathbb{S}^d$ denotes the space of symmetric $d \times d$ matrices. We assume a linear elastic constitutive law,

$$\boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}),$$

where $\mathbb{E}$ is the elasticity tensor and the linearized strain tensor $\boldsymbol{\varepsilon}(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ is defined by

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad i, j = 1, \cdots, d.$$

The system is in static equilibrium, governed by

$$\textbf{Div}\ \boldsymbol{\sigma} + \mathbf{f}_B = \mathbf{0} \text{ in } \Omega.$$

The contact conditions on Γ_C are described by subdifferential laws. For a vector $\mathbf{v}$ and tensor $\boldsymbol{\sigma}$ on the boundary, we define their normal and tangential components as:

$$v_\nu = \mathbf{v} \cdot \boldsymbol{\nu}, \qquad \mathbf{v}_\tau = \mathbf{v} - v_\nu \boldsymbol{\nu},$$
$$\sigma_\nu = \boldsymbol{\sigma\nu} \cdot \boldsymbol{\nu}, \qquad \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}.$$

The contact laws are:

(1) **Normal Direction:**

$$-\sigma_\nu \in \partial j_\nu(u_\nu) \quad \text{on } \Gamma_C. \tag{3.1}$$

(2) **Tangential Direction:**

$$-\boldsymbol{\sigma}_\tau \in \varsigma\tau(u_\nu), \partial j_\tau(\mathbf{u}\tau) \quad \text{on } \Gamma_C. \tag{3.2}$$

Here, $j\nu$ and j_τ are superpotentials whose generalized gradients model nonmonotone reactions, and ς_τ is a state-dependent friction bound.

In summary, the classical problem is to find $(\mathbf{u}, \boldsymbol{\sigma})$ satisfying:

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}) && \text{in } \Omega,\\
\textbf{Div}\ \boldsymbol{\sigma} + \mathbf{f}_B &= \mathbf{0} && \text{in } \Omega,\\
\mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D,\\
\boldsymbol{\sigma\nu} &= \mathbf{f}_N && \text{on } \Gamma_N,\\
-\sigma_\nu &\in \partial j_\nu(u_\nu) && \text{on } \Gamma_C,
\end{aligned}$$

$$-\boldsymbol{\sigma}_\tau \in \varsigma_\tau(u_\nu)\partial j_\tau(\mathbf{u}_\tau) \qquad \text{on } \Gamma_C.$$

3.2. **Weak Formulation.** We define the Hilbert space

$$\mathbb{V} = \{\mathbf{v} \in \mathbf{H}^1(\Omega)^d | \mathbf{v} = \mathbf{0} \quad \text{on } \Gamma_D\},$$

equipped with the inner product $(\mathbf{u}, \mathbf{v})_{\mathbb{V}} = (\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{L^2(\Omega)}$. Let $\gamma\colon \mathbb{V} \to L^2(\Gamma_C)^d$ be the trace operator with norm $|\gamma|$.

Using Green's formula and the properties of the generalized subdifferential, we obtain the weak formulation.

Problem 3.1. *Find* $\mathbf{u} \in \mathbb{V}$ *such that*

$$(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{L^2(\Omega)} + J^0(\gamma\mathbf{u}, \gamma\mathbf{u}; \gamma\mathbf{v}) \geq \langle \mathbf{f}, \mathbf{v} \rangle \quad \forall \mathbf{v} \in V, \tag{3.3}$$

where

$$\langle \mathbf{f}, \mathbf{v} \rangle = \int_\Omega \mathbf{f}_B \cdot v dx + \int_{\Gamma_N} \mathbf{f}_N \cdot \boldsymbol{\gamma}\mathbf{v} da,$$

and $J\colon L^2(\Gamma_C)^d \times L^2(\Gamma_C)^d \to \mathbb{R}$ *is defined by*

$$J(\mathbf{w}, \mathbf{v}) = \int_{\Gamma_C} [j_\nu(\mathbf{X}, v_\nu) + \zeta_\tau(\mathbf{X}, w_\nu) j_\tau(\mathbf{x}, \mathbf{v}_\tau)] da. \tag{3.4}$$

3.3. **Existence, Uniqueness, and Numerical Analysis.** We now verify that this contact problem fits our abstract framework.

Assumptions.

- **(H1)** $\mathbb{E} \in L^\infty(\Omega)$ is symmetric and uniformly positive definite.
- **(H2)** $j_\nu\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies:
  (1) $j_\nu(\cdot, \xi)$ is measurable, and $j_\nu(\cdot, 0) \in L^1(\Gamma_C)$.
  (2) $j_\nu(\mathbf{x}, \cdot)$ is locally Lipschitz.
  (3) $|\partial j_\nu(\mathbf{x}, \xi)| \leq \vartheta_0 + \vartheta_1 |\xi|$ for all $\xi \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
  (4) There exists $m_\nu \geq 0$ such that $j_\nu^0(\mathbf{x}, \xi_1; \xi_2 - \xi_1) + j_\nu^0(\mathbf{x}, \xi_2; \xi_1 - \xi_2) \leq m_\nu |\xi_1 - \xi_2|^2$ for all $\xi_1, \xi_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
- **(H3)** $j_\tau\colon \Gamma_C \times \mathbb{R}^d \to \mathbb{R}$ satisfies analogous conditions to **(H2)**.
- **(H4)** $\varsigma_\tau\colon \Gamma_C \times \mathbb{R} \to \mathbb{R}$ satisfies:
  (1) $\varsigma_\tau(\cdot, \delta)$ is measurable.
  (2) $0 \leq \varsigma_\tau(\mathbf{x}, \delta) \leq \overline{\varsigma}$ for all $\delta \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
  (3) $|\varsigma_\tau(\mathbf{x}, \delta_1) - \varsigma_\tau(\mathbf{x}, \delta_2)| \leq L_\varsigma |\delta_1 - \delta_2|$ for all $\delta_1, \delta_2 \in \mathbb{R}$, a.e. $\mathbf{x} \in \Gamma_C$.
- **(H5)** $\mathbf{f}_B \in L^2(\Omega)^d$, $\mathbf{f}_N \in L^2(\Gamma_N)^d$.

Lemma 3.1. *Under assumptions **(H2)-(H4)**, the functional J defined in* (3.4) *satisfies hypothesis **(H2)** of Theorem 2.2.*

Proof. The result follows from direct calculations using the assumptions and properties of the Clarke subdifferential. □

Theorem 3.1. ***(Existence and Uniqueness)*** *Under assumptions* ***(H1)-(H5)****, if the elasticity tensor* $\mathbb{E}$ *is sufficiently positive definite relative to the constants in* ***(H2)-(H4)****, then Problem 3.1 has a unique solution* $\mathbf{u} \in \mathbb{V}$.

Proof. Apply Theorem 2.2 with $\mathcal{A}(\mathbf{u}, \mathbf{v}) = (\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))$ and J defined in (3.4). □

For numerical approximation, let $\mathbb{V}^h \subset \mathbb{V}$ be a finite element space with mesh parameter h. The discrete problem is:

Problem 3.2. *Find* $\mathbf{u}^h \in \mathbb{V}^h$ *such that*

$$(\mathbb{E}\boldsymbol{\varepsilon}(\mathbf{u}^h), \boldsymbol{\varepsilon}(\mathbf{v}^h))_{L^2(\Omega)} + J^0(\gamma\mathbf{u}^h, \gamma\mathbf{u}^h; \gamma\mathbf{v}^h) \geq \langle \mathbf{f}, \mathbf{v}^h \rangle \quad \forall \mathbf{v}^h \in \mathbb{V}^h. \tag{3.5}$$

Theorem 3.2. ***(Error Estimate).*** *Under the assumptions of Theorem 3.1, if the solution* $\mathbf{u}$ *satisfies* $\mathbf{u} \in H^2(\Omega)^d$, $\gamma\mathbf{u} \in H^2(\Gamma_C)^d$, *and* $\boldsymbol{\sigma\nu}|_{\Gamma_C} \in L^2(\Gamma_C)^d$, *then there exists* $C > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}^h\|_{\mathbb{V}} \leq Ch.$$

Proof. The proof follows standard finite element techniques using the strong monotonicity of $\mathcal{A}$ and the properties of J, combined with interpolation estimates. □

This application demonstrates the power and versatility of our abstract framework in handling complex mechanical contact problems with nonmonotone boundary conditions.

4. Conclusions

This paper has established a comprehensive theoretical and numerical framework for generalized hemivariational inequalities (GHVIs) through the lens of nonsmooth generalized optimization. Our main contributions are threefold:

- **Theoretical Innovation:** We introduced a novel problem class that generalizes many existing inclusion and inequality problems, providing a more versatile tool for modeling complex systems.
- **Numerical Analysis:** We developed a robust discretization scheme using a polygonal domain, continuous piecewise affine finite elements, and a specific quadrature rule. For this scheme, we successfully established the existence and uniqueness of a solution and laid the groundwork for sample error estimation.
- **Practical Application:** The theory was validated through a mathematically formulated static contact problem, modeling the frictional interaction between an elastic body and a foundation with a nonmonotone law.

The successful application of concepts like cocoercivity and Lipschitz continuity underscores the robustness of our approach, opening avenues for future research in computational nonsmooth mechanics.

Conflicts of Interest: The author declares that there are no conflicts of interest regarding the publication of this paper.

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