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A Novel Study on the Non-Negative Solution of an Eighth-Order $\mathbb{BVP}$

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Abstract. In this article, we investigate the existence of non-negative solutions for a boundary value problem associated with an eighth-order differential equation $\lambda^{(8)}(\varpi) = \psi(\varpi, \lambda(\varpi), \lambda^{(1)}(\varpi), \cdots, \lambda^{(7)}(\varpi))$ for $0 < \varpi < 1$, under initial values $\lambda(0) = \lambda'(0) = \lambda''(0) = \lambda'''(0) = 0$ and $\lambda^{(4)}(1) = \lambda^{(5)}(1) = \lambda^{(6)}(1) = \lambda^{(7)}(1) = 0$, where ψ is non-negative continuous function. For this study, we use the nonlinear Leray-Schauder alternative and the Leray-Schauder fixed-point theorem to prove the existence of at least one non-negative solution. As a numerical application, we present an example to confirm the utility of the achieved results.

1. Introduction

Nowadays, study on the boundary value problems ($\mathbb{BVP}$s) is one of the interesting research area for many researchers [1–4]. Recently, several novel research studies have been come to the literature discussing qualitative and quantitative analyses of $\mathbb{BVP}$s [6–12]. In advanced studies, Eighth order boundary value problems arise in various fields of modern mathematics, chemical-physical events, etc. The hydrodynamic stability problems of the eighth degree $\mathbb{BVP}$s have been discussed in [13]. Some novel studies of problems with higher order $\mathbb{BVP}$s to explore the existence of a solutions have been discussed [14–17].

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In [18], the authors have analysed three positive solutions for higher-order m-point $\mathbb{BVP}$s. In 2019, Thenmozhi *et al.* derived some novel results of the existence of non-negative solution for the eighth-order differential equations by employing fixed point theorem. Moreover, these problems have been numerically solved using different numerical methods [19]. Many authors have relied to provide approximate solutions to differential equations of the eighth order [20–23]. In [24], a numerical solution of the eighth order $\mathbb{BVP}$s using Petrov-Galerkin scheme with quintic B-splines and septic B-splines functions has been derived. In [25], a numerical solution of eighth order $\mathbb{BVP}$s using variational iteration method has been derived. The authors numerically solved an eighth order $\mathbb{BVP}$s using Galerkin scheme with septic B-splines [26].

The aim of this work is to explore the possibility of the existence of non-negative solutions to the $\mathbb{BVP}$

$$\begin{cases} \lambda^{(8)}(\varpi) = \psi\Big(\varpi, \lambda(\varpi), \lambda^{(1)}(\varpi), \cdots, \lambda^{(7)}(\varpi)\Big), \quad 0 < \varpi < 1, \\ \lambda(0) = \lambda'(0) = \lambda''(0) = \lambda'''(0) = 0, \\ \lambda^{(4)}(1) = \lambda^{(5)}(1) = \lambda^{(6)}(1) = \lambda^{(7)}(1) = 0. \end{cases} \tag{1.1}$$

where $\psi \in C\Big(\mathcal{A}, [0,\infty)\Big)$ is non-negative continuous function, here $\bar{\mathrm{I}} = [0,1]$ and

$$\mathcal{A} = \bar{\mathrm{I}} \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times [0,\infty) \times (-\infty, 0] \times [0,\infty) \times (-\infty, 0].$$

This paper is arranged as follows. Section 2 is limited to the definition and the theorem which shall be employed to verify the results. In Section 3, we derive our fundamental results consisting the existence theorems of at least one non-negative solution for (1.1) by setting some sufficient criteria applying the nonlinear Leray-Schauder alternative and the fixed-point Leray-Schauder theorem. In Section 3.1, as a numerical application, we yield an example to check the desired outcomes. The conclusion is given in Section 4.

2. Preliminaries

In this part, we provide a definition and fundamental theorem of the fixed point of Leray-Schauder to further determine the non-negative solution of the problem (1.1).

Definition 2.1. *Assume $\mathbb{H}$ is Banach space containing a cone Δ. The application Ξ is a positive continuous concave operator on Δ equipped $\Xi : \Delta \to [0,\infty)$ is continuous and*

$$\Xi\Big(\varrho\lambda + (1-\varrho)\upsilon\Big) \geq \varrho\, \Xi(\lambda) + (1-\varrho)\Xi(\upsilon), \qquad \forall\, \lambda, \upsilon \in \Delta,\ \varrho \in \bar{\mathrm{I}}.$$

Likewise, we can put that the application Θ is a positive continuous convex operator on Δ equipped $\Theta : \Delta \to [0,\infty)$ is continuous and $\Theta\Big(\varrho\lambda + (1-\varrho)\upsilon\Big) \leq \varrho\Theta(\lambda) + (1-\varrho)\Theta(\upsilon)$, for each $\lambda, \upsilon \in \Delta$, and $\varrho \in \bar{\mathrm{I}}$.

Theorem 2.1 ([27,28]). *Let $\mathbb{H}$ is a Banach space and $\Upsilon \subset \mathbb{H}$ be a bounded open subset, $0 \in \Upsilon$. $\Pi : \overline{\Upsilon} \to \mathbb{H}$ be a completely continuous operator. Then, either* i): *There exists $\lambda \in \partial\Upsilon$ and $\omega > 1$ such that $\Pi(\lambda) = \omega\lambda$, or* ii): *$\exists \lambda^* \in \overline{\Upsilon}$ a fixed point.*

3. Fundamental outcomes

In this part, first we will derive all csufficient conditions for the function ψ, which help us to apply the nonlinear Leray-Schauder alternative, and the Leray-Schauder fixed point theorem to prove the existence of at least one non-negative solution for (1.1).

Lemma 3.1. *Assume that* $\mathbb{H}$ *be a Banach space*

$$\mathbb{H} = \Big\{\lambda \in C^7(\bar{\mathrm{I}}) \,:\, \lambda(0) = \lambda^{'}(0) = \lambda^{''}(0) = \lambda^{'''}(0) = 0,\ \lambda^{(4)}(1) = \lambda^{(5)}(1) = \lambda^{(6)}(1) = 0\Big\},$$

equipped the norm

$$\|\lambda\| = \max\Big\{|\lambda|_0, |\lambda^{'}|_0, |\lambda^{''}|_0, |\lambda^{'''}|_0, |\lambda^{(4)}|_0, |\lambda^{(5)}|_0, |\lambda^{(6)}|_0, |\lambda^{(7)}|_0\Big\},$$

where $|\lambda|_0 = \max_{0\le\varpi\le1} |\lambda(\varpi)|$. *So for everything* $\lambda \in \mathbb{H}$, *we have* $\|\lambda\| = |\lambda^{(7)}|_0$ *and* $|\lambda|_0 \le \frac{1}{252}\|\lambda\|$, $\|\lambda^{'}|_0 \le \frac{1}{72}\|\lambda\|$, $|\lambda^{''}|_0 \le \frac{1}{30}\|\lambda\|$, $|\lambda^{'''}|_0 \le \frac{1}{24}\|\lambda\|$, $|\lambda^{(4)}|_0 \le \frac{1}{6}\|\lambda\|$, $|\lambda^{(5)}|_0 \le \frac{1}{2}\|\lambda\|$, $|\lambda^{(6)}|_0 \le \|\lambda\|$.

Proof. Consider the homogeneous $\mathbb{BVP}$ of the seventh-order $-\lambda^{(7)}(\varpi) = 0$, for $0 < \varpi < 1$, with boundary conditions (1.1). Then its Green function $\aleph(\varpi, \xi)$ is given by

$$\aleph(\varpi,\xi) = \frac{1}{720}\begin{cases} -\xi^6 + 6\xi^5\varpi - 15\xi^4\varpi^2 + 20\xi^3\varpi^3, & 0 \le \xi \le \varpi \le 1, \\ \varpi^6 - 6\xi\varpi^5 + 15\xi^2\varpi^4, & 0 \le \varpi \le \xi \le 1. \end{cases} \tag{3.1}$$

From (3.1) we can simply know that $\aleph(\varpi, \xi) \ge 0$ and

$$\tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi} \ge 0,\ \tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2} \ge 0,\ \tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3} \ge 0,\ \tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4} \ge 0,\ \tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5} \le 0,\ \tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6} \ge 0, \tag{3.2}$$

and we integrate

$$\begin{aligned}
\int_0^1 |\aleph(\varpi,\xi)|\,\mathrm{d}\xi &= \int_0^1 \aleph(\varpi,\xi)\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-\tfrac{1}{7}\varpi^7 + \varpi^6 - 3\varpi^5 + 5\varpi^4\Big),\\
\int_0^1 \Big|\tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\Big|\,\mathrm{d}\xi &= \int_0^1 \tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-\varpi^6 + 6\varpi^5 - 15\varpi^4 + 20\varpi^3\Big),\\
\int_0^1 \Big|\tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\Big|\,\mathrm{d}\xi &= \int_0^1 \tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-6\varpi^5 + 30\varpi^4 - 60\varpi^3 + 60\varpi^2\Big),\\
\int_0^1 \Big|\tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\Big|\,\mathrm{d}\xi &= \int_0^1 \tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-30\varpi^4 + 120\varpi^3 - 180\varpi^2 + 120\varpi\Big),\\
\int_0^1 \Big|\tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\Big|\,\mathrm{d}\xi &= \int_0^1 \tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-120\varpi^3 + 360\varpi^2 - 360\varpi + 120\Big),\\
\int_0^1 \Big|\tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\Big|\,\mathrm{d}\xi &= -\int_0^1 \tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(360\varpi^2 - 720\varpi + 360\Big),\\
\int_0^1 \Big|\tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\Big|\,\mathrm{d}\xi &= \int_0^1 \tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\,\mathrm{d}\xi = \tfrac{1}{720}\Big(-720\varpi + 720\Big).
\end{aligned}$$

By integration, we find

$$\max_{\varpi\in\bar{I}}\int_0^1 |\aleph(\varpi,\xi)|\,\mathrm{d}\xi = \tfrac{1}{252}, \qquad \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\right|\mathrm{d}\xi = \tfrac{1}{72}, \qquad \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\right|\mathrm{d}\xi = \tfrac{1}{30},$$
$$\max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\right|\mathrm{d}\xi = \tfrac{1}{24}, \qquad \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\right|\mathrm{d}\xi = \tfrac{1}{6}, \qquad \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\right|\mathrm{d}\xi = \tfrac{1}{2},$$
$$\max_{\varpi\bar{I}}\int_0^1 \left|\tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\right|\mathrm{d}\xi = 1.$$

We have $\lambda \in \mathbb{H}$ and $\|\lambda\| = \mu$. So

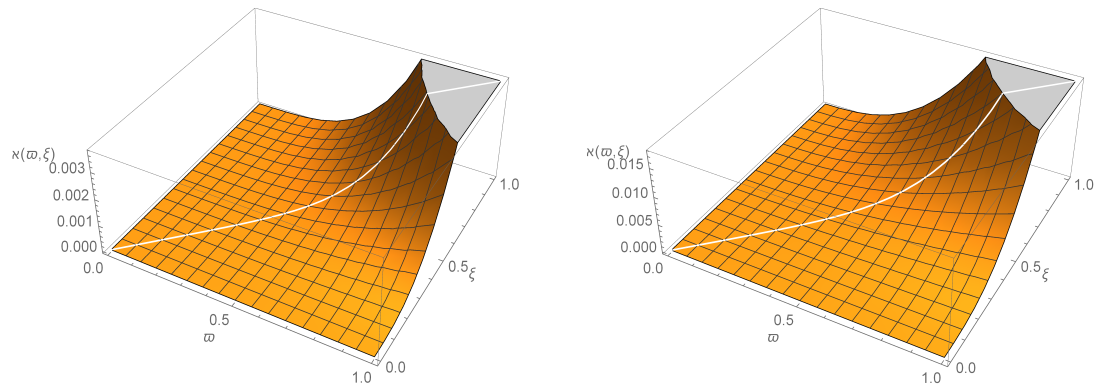
$$\lambda(\varpi) = \int_0^1 \aleph(\varpi,\xi)[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi, \qquad \lambda'(\varpi) = \int_0^1 \tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi,$$
$$\lambda''(\varpi) = \int_0^1 \tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi, \qquad \lambda'''(\varpi) = \int_0^1 \tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi,$$
$$\lambda^{(4)}(\varpi) = \int_0^1 \tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi, \qquad \lambda^{(5)}(\varpi) = \int_0^1 \tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi,$$

and $\lambda^{(6)}(\varpi) = \int_0^1 \frac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}[\lambda^{(7)}(\xi)]\,\mathrm{d}\xi$. Therefore

$$|\lambda|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 |\aleph(\varpi,\xi)||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 |\aleph(\varpi,\xi)|\,\mathrm{d}\xi = \tfrac{1}{252}|\lambda^{(7)}|_0,$$
$$|\lambda'|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\right|\mathrm{d}\xi = \tfrac{1}{72}|\lambda^{(7)}|_0,$$
$$|\lambda''|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\right|\mathrm{d}\xi = \tfrac{1}{30}|\lambda^{(7)}|_0,$$
$$|\lambda'''|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\right|\mathrm{d}\xi = \tfrac{1}{24}|\lambda^{(7)}|_0,$$
$$|\lambda^{(4)}|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\right|\mathrm{d}\xi = \tfrac{1}{6}|\lambda^{(7)}|_0,$$
$$|\lambda^{(5)}|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\right|\mathrm{d}\xi = \tfrac{1}{2}|\lambda^{(7)}|_0,$$
$$|\lambda^{(6)}|_0 \le \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\right||\lambda^{(7)}(\xi)|\,\mathrm{d}\xi \le |\lambda^{(7)}|_0 \max_{\varpi\in\bar{I}}\int_0^1 \left|\tfrac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\right|\mathrm{d}\xi = |\lambda^{(7)}|_0,$$

Then $|\lambda^{(7)}|_0 = \|\lambda\| = \mu$, here the proof is finished. □

According to the proposed Green function $\aleph(\varpi,\xi)$ in Eq. (3.1), the following plots, Figures 1a, 1b, 2a, 2b, 3a, and 3b, of the function are observed to verify the constraints (3.2).



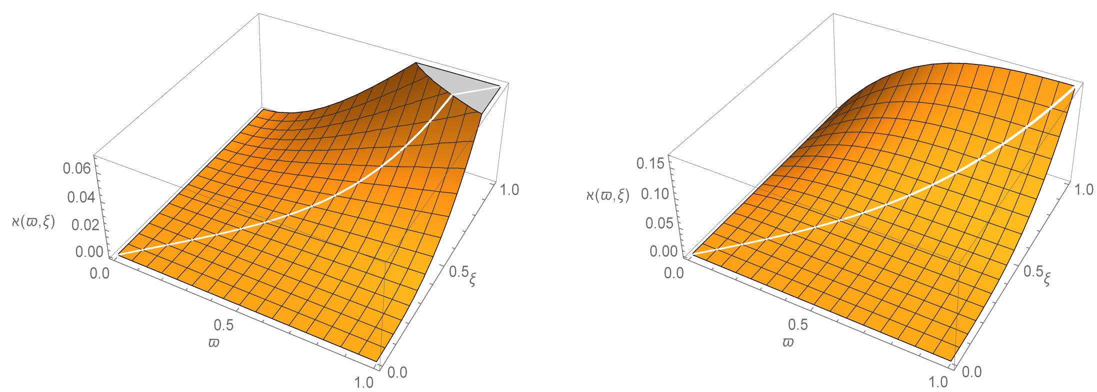
(A) The first Green function (B) The second Green function

Figure 1. The first and second Green functions.

- The first Green function $\aleph(\varpi, \xi) \geq 0$,

$$\aleph(\varpi, \xi) = \frac{1}{720} \begin{cases} -\xi^6 + 6\xi^5\varpi - 15\xi^4\varpi^2 + 20\xi^3\varpi^3, & 0 \leq \xi \leq \varpi \leq 1, \\ \varpi^6 - 6\xi\varpi^5 + 15\xi^2\varpi^4, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.3}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 1a.



(A) The third Green function (B) The fourth Green function

Figure 2. The third and fourth Green functions.

- The second Green function $\frac{\partial \aleph(\varpi,\xi)}{\partial \varpi} \geq 0$,

$$\frac{\partial \aleph(\varpi,\xi)}{\partial \varpi} = \frac{1}{720} \begin{cases} 6\xi^5 - 30\xi^4\varpi + 60\xi^3\varpi^2, & 0 \leq \xi \leq \varpi \leq 1, \\ 6\varpi^5 - 30\xi\varpi^4 + 60\xi^2\varpi^3, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.4}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 1b.

- The third Green function $\frac{\partial^2 \aleph(\varpi,\xi)}{\partial^2 \varpi} \geq 0$,

$$\frac{\partial^2 \aleph(\varpi,\xi)}{\partial^2 \varpi} = \frac{1}{720} \begin{cases} -30\xi^4 + 120\xi^3\varpi, & 0 \leq \xi \leq \varpi \leq 1, \\ 30\varpi^4 - 120\xi\varpi^3 + 180\xi^2\varpi^2, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.5}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 2a.

- The fourth Green function $\frac{\partial^3 \aleph(\varpi,\xi)}{\partial^3 \varpi} \geq 0$,

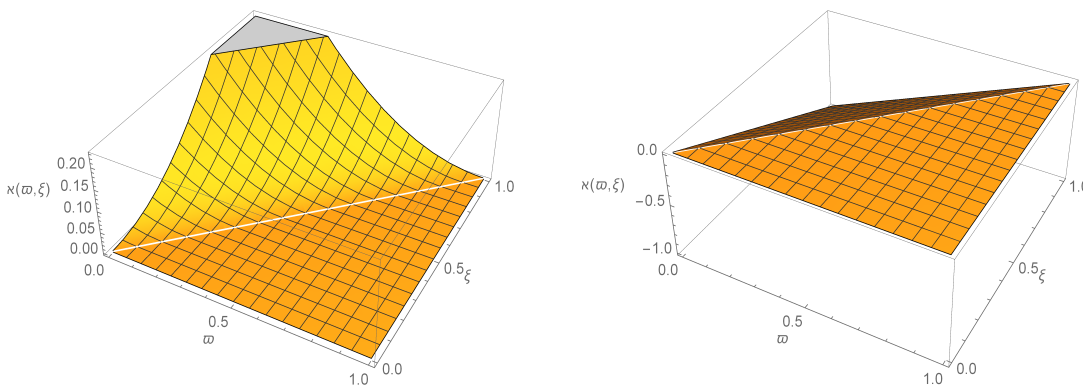
$$\frac{\partial^3 \aleph(\varpi,\xi)}{\partial^3 \varpi} = \frac{1}{720} \begin{cases} 120\xi^3, & 0 \leq \xi \leq \varpi \leq 1, \\ 120\varpi^3 - 360\xi\varpi^2 + 360\xi^2\varpi, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.6}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 2b.

- The fifth Green function $\frac{\partial^4 \aleph(\varpi,\xi)}{\partial^4 \varpi} \geq 0$:

$$\frac{\partial^4 \aleph(\varpi,\xi)}{\partial^4 \varpi} = \frac{1}{720} \begin{cases} 0, & 0 \leq \xi \leq \varpi \leq 1, \\ 360\varpi^2 - 720\xi\varpi + 360\xi^2, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.7}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 3a.



(A) The fifth Green function

(B) The sixth Green function

FIGURE 3. The fifth and sixth Green functions.

- The sixth Green function $\frac{\partial^5 \aleph(\varpi,\xi)}{\partial^5 \varpi} \leq 0$,

$$\frac{\partial^5 \aleph(\varpi,\xi)}{\partial^5 \varpi} = \frac{1}{720} \begin{cases} 0, & 0 \leq \xi \leq \varpi \leq 1, \\ 720\varpi - 720\xi, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.8}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 3b.

- The seventh Green function $\frac{\partial^6 \aleph(\varpi,\xi)}{\partial^6 \varpi} \geq 0$,

$$\frac{\partial^6 \aleph(\varpi,\xi)}{\partial^6 \varpi} = \frac{1}{720} \begin{cases} 0, & 0 \leq \xi \leq \varpi \leq 1, \\ 720, & 0 \leq \varpi \leq \xi \leq 1. \end{cases} \tag{3.9}$$

for any $\varpi, \xi \in \bar{\mathrm{I}}$. The plot of the function is given in Fig. 4.

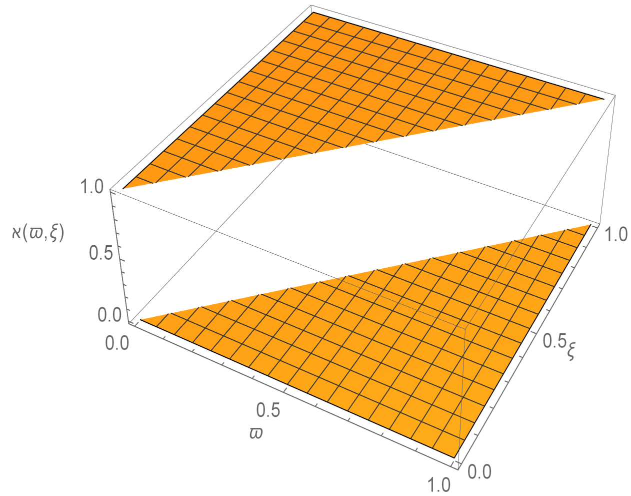


Figure 4. The seventh Green function.

Theorem 3.1. *Consider $\psi \in C(\mathcal{A}, [0,\infty))$ and $\psi(\varpi, 0,0,0,0,0,0,0,0) \neq 0$, $\varpi \in \bar{\mathrm{I}}$. Assume that there exist non-negative functions $\zeta_\jmath \in L^1(\bar{\mathrm{I}})$, $\jmath = 0,1,2,3,4,5,6,7,8$ such that*

$$\Sigma = \tfrac{1}{252}\int_0^1 \zeta_0(\xi)\,\mathrm{d}\xi + \tfrac{1}{72}\int_0^1 \zeta_1(\xi)\,\mathrm{d}\xi + \tfrac{1}{30}\int_0^1 \zeta_2(\xi)\,\mathrm{d}\xi + \tfrac{1}{24}\int_0^1 \zeta_3(\xi)\,\mathrm{d}\xi + \tfrac{1}{6}\int_0^1 \zeta_4(\xi)\,\mathrm{d}\xi + \tfrac{1}{2}\int_0^1 \zeta_5(\xi)\,\mathrm{d}\xi + \int_0^1 \zeta_6(\xi)\,\mathrm{d}\xi + \int_0^1 \zeta_7(\xi)\,\mathrm{d}\xi < 1, \tag{3.10}$$

and for every

$$(\varpi, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \in \bar{\mathrm{I}} \times \left[0, \tfrac{1}{252}\nu\right] \times \left[0, \tfrac{1}{72}\nu\right] \times \left[0, \tfrac{1}{30}\nu\right] \times \left[0, \tfrac{1}{24}\nu\right] \times \left[0, \tfrac{1}{6}\nu\right] \times \left[-\tfrac{1}{2}\nu, 0\right] \times [0,\nu] \times [-\nu, 0], \tag{3.11}$$

ψ specify

$$\psi(\varpi, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \leq \zeta_0(\varpi)\lambda_0 + \zeta_1(\varpi)\lambda_1 + \zeta_2(\varpi)\lambda_2 + \zeta_3(\varpi)\lambda_3 + \zeta_4(\varpi)\lambda_4 - \zeta_5(\varpi)\lambda_5 + \zeta_6(\varpi)\lambda_6 - \zeta_7(\varpi)\lambda_7 + \zeta_8(\varpi), \tag{3.12}$$

where $\nu = \Lambda(1-\Sigma)^{-1}$, $\Lambda = \int_0^1 \zeta_8(\xi) d\xi$. Therefore the problem (1.1) *accepts at least one non-negative solution $\lambda^* \in C^8(\bar{\mathrm{I}})$ as verifies*

$$252 \max_{\varpi\in\bar{\mathrm{I}}} \lambda^*(\varpi) \leq 72 \max_{\varpi\in\bar{\mathrm{I}}} (\lambda^*)'(\varpi) \leq 30 \max_{\varpi\in\bar{\mathrm{I}}} (\lambda^*)''(\varpi) \leq 24 \max_{\varpi\in\bar{\mathrm{I}}} (\lambda^*)'''(\varpi) \leq 6 \max_{\varpi\in\bar{\mathrm{I}}} (\lambda^*)^{(4)}(\varpi) \leq 2 \max_{\varpi\in\bar{\mathrm{I}}} [-(\lambda^*)^{(5)}(\varpi)] \leq \max_{\varpi\in\bar{\mathrm{I}}} (\lambda^*)^{(6)}(\varpi) \leq \max_{\varpi\in\bar{\mathrm{I}}} [-(\lambda^*)^{(7)}(\varpi)] \leq \nu.$$

Proof. As $\psi(\varpi,0,0,0,0,0,0,0,0) \neq 0$ and $|\psi(\varpi,0,0,0,0,0,0,0,0)| \leq \zeta_8(\varpi)$, $\varpi \in [0,1]$, we have $\Lambda = \int_0^1 \zeta_8(\xi)\,\mathrm{d}\xi > 0$, then according to (3.10) that $\nu > 0$. From the problem (1.1), we deduce $\lambda^{(7)}(\varpi) = -\int_\varpi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa$, where

$$\widehat{\psi}_\lambda(\kappa) = \psi\Big(\kappa, \lambda(\kappa), \lambda'(\kappa), \lambda''(\kappa), \lambda'''(\kappa), \lambda^{(4)}(\kappa), \lambda^{(5)}(\kappa), \lambda^{(6)}(\kappa), \lambda^{(7)}(\kappa)\Big),$$

which means that

$$\lambda(\varpi) = \int_0^1 \aleph(\varpi,\xi)\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi,$$

where $\aleph(\varpi,\xi)$ is given in the form (3.1). We have $\Upsilon_\nu = \{\lambda \in \mathbb{H} : \|\lambda\| \leq \nu\}$, so Υ_ν is a bounded closed convex set of $\mathbb{H}$ and $0 \in \Upsilon_\nu$. For any $\lambda \in \Upsilon_\nu$, we give the operator form Π as

$$(\Pi\lambda)(\varpi) = \int_0^1 \aleph(\varpi,\xi)\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi. \tag{3.13}$$

Hence

$$(\Pi\lambda)'(\varpi) = \int_0^1 \frac{\partial\aleph(\varpi,\xi)}{\partial\varpi}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi, \qquad (\Pi\lambda)''(\varpi) = \int_0^1 \frac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi,$$

$$(\Pi\lambda)'''(\varpi) = \int_0^1 \frac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi, \qquad (\Pi\lambda)^{(4)}(\varpi) = \int_0^1 \frac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi,$$

$$(\Pi\lambda)^{(5)}(\varpi) = \int_0^1 \frac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi, \qquad (\Pi\lambda)^{(6)}(\varpi) = \int_0^1 \frac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6}\left(\int_\xi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa\right)\mathrm{d}\xi,$$

and $(\Pi\lambda)^{(7)}(\varpi) = -\int_\varpi^1 \widehat{\psi}_\lambda(\kappa)\,\mathrm{d}\kappa$. Then, $(\Pi\lambda)(0) = (\Pi\lambda)'(0) = (\Pi\lambda)''(0) = (\Pi\lambda)'''(0) = 0$, $(\Pi\lambda)^{(4)}(1) = (\Pi\lambda)^{(5)}(1) = (\Pi\lambda)^{(6)}(1) = (\Pi\lambda)^{(7)}(1) = 0$. Thus, $\Pi : \Upsilon_\nu \to \mathbb{H}$. According to the Ascoli-Arzelá Theorem, we can say this operator $\Pi : \Upsilon_\nu \to \mathbb{H}$ is a completely continuous. Hence, the problem (1.1) has a solution $\lambda = \lambda(\varpi)$ if and only if λ verifies the equality $\Pi\lambda = \lambda$. Consider $\exists\ \lambda \in \partial\Upsilon_\nu$, $\omega > 1$ such that $\Pi\lambda = \omega\lambda$. We have $\|\lambda\| = \nu$, according to lemma 3.1 that $|\lambda|_0 \leq \frac{1}{252}\nu$, $|\lambda'|_0 \leq \frac{1}{72}\nu$, $|\lambda''|_0 \leq \frac{1}{30}\nu$, $|\lambda'''|_0 \leq \frac{1}{24}\nu$, $|\lambda^{(4)}|_0 \leq \frac{1}{6}\nu$, $|\lambda^{(5)}|_0 = \frac{1}{2}\nu$, $|\lambda^{(6)}|_0 = \nu$. By (3.10), (3.12) and (3.13) we get

$$\begin{aligned}
\omega\nu = \omega\|\lambda\| = \|\Pi\lambda\| &= \max_{\varpi\in\bar{I}} |\lambda^{(7)}(\varpi)| = \max_{\varpi\bar{I}}\left|-\int_\varpi^1 \widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi\right| = \max_{\varpi\bar{I}}\int_\varpi^1 \widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \\
&\leq \max_{\varpi\in\bar{I}}\int_0^1 \widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq \int_0^1 \Big[\zeta_0(\xi)\lambda(\xi) + \zeta_1(\xi)\lambda'(\xi) + \zeta_2(\xi)\lambda''(\xi) + \zeta_3(\xi)\lambda'''(\xi) \\
&\quad + \zeta_4(\xi)\lambda^{(4)}(\xi) - \zeta_5(\xi)\lambda^{(5)}(\xi) + \zeta_6(\xi)\lambda^{(6)}(\xi) - \zeta_7(\xi)\lambda^{(7)}(\xi) + \zeta_8(\xi)\Big]\mathrm{d}\xi \\
&\leq \int_0^1 \Big[\tfrac{1}{252}\zeta_0(\xi)\nu + \tfrac{1}{72}\zeta_1(\xi)\nu + \tfrac{1}{30}\zeta_2(\xi)\nu + \tfrac{1}{24}\zeta_3(\xi)\nu \\
&\quad + \tfrac{1}{6}\zeta_4(\xi)\nu + \tfrac{1}{2}\zeta_5(\xi)\nu + \zeta_6(\xi)\nu + \zeta_7(\xi)\nu + \zeta_8(\xi)\Big]\,\mathrm{d}\xi \\
&= \Big[\tfrac{1}{252}\int_0^1 \zeta_0(\xi)\,\mathrm{d}\xi + \tfrac{1}{72}\int_0^1 \zeta_1(\xi)\,\mathrm{d}\xi + \tfrac{1}{30}\int_0^1 \zeta_2(\xi)\,\mathrm{d}\xi + \tfrac{1}{24}\int_0^1 \zeta_3(\xi)\,\mathrm{d}\xi \\
&\quad + \tfrac{1}{6}\int_0^1 \zeta_4(\xi)\,\mathrm{d}\xi + \tfrac{1}{2}\int_0^1 \zeta_5(\xi)\,\mathrm{d}\xi + \int_0^1 \zeta_6(\xi)\,\mathrm{d}\xi + \int_0^1 \zeta_7(\xi)\,\mathrm{d}\xi\Big]\nu \\
&\quad + \int_0^1 \zeta_8(\xi)\,\mathrm{d}\xi = \Sigma\nu + \Lambda = \Sigma\nu + (1-\Sigma)\nu = \nu.
\end{aligned}$$

We obtain a contrariness. Thus, by Theorem 2.1, Π admits a fixed point $\lambda^* \in \mathbb{H}$. It is the solution to the problem (1.1). we conclude that $\psi(\varpi,0,0,0,0,0,0,0,0) \neq 0$, we affirm that trivial solution is not a solution of the (1.1), thus, $|\lambda^*|_0 > 0$. By (3.2) that $\lambda^*(\varpi)$ is nondecreasing and convex on $\bar{\mathrm{I}}$, then $\lambda^*(\varpi) \geq \varpi|\lambda^*|_0 > 0$, for each $\varpi \in \bar{\mathrm{I}}$, i.e., $\lambda^*(\varpi)$ is a non-negative solution of the problem (1.1). □

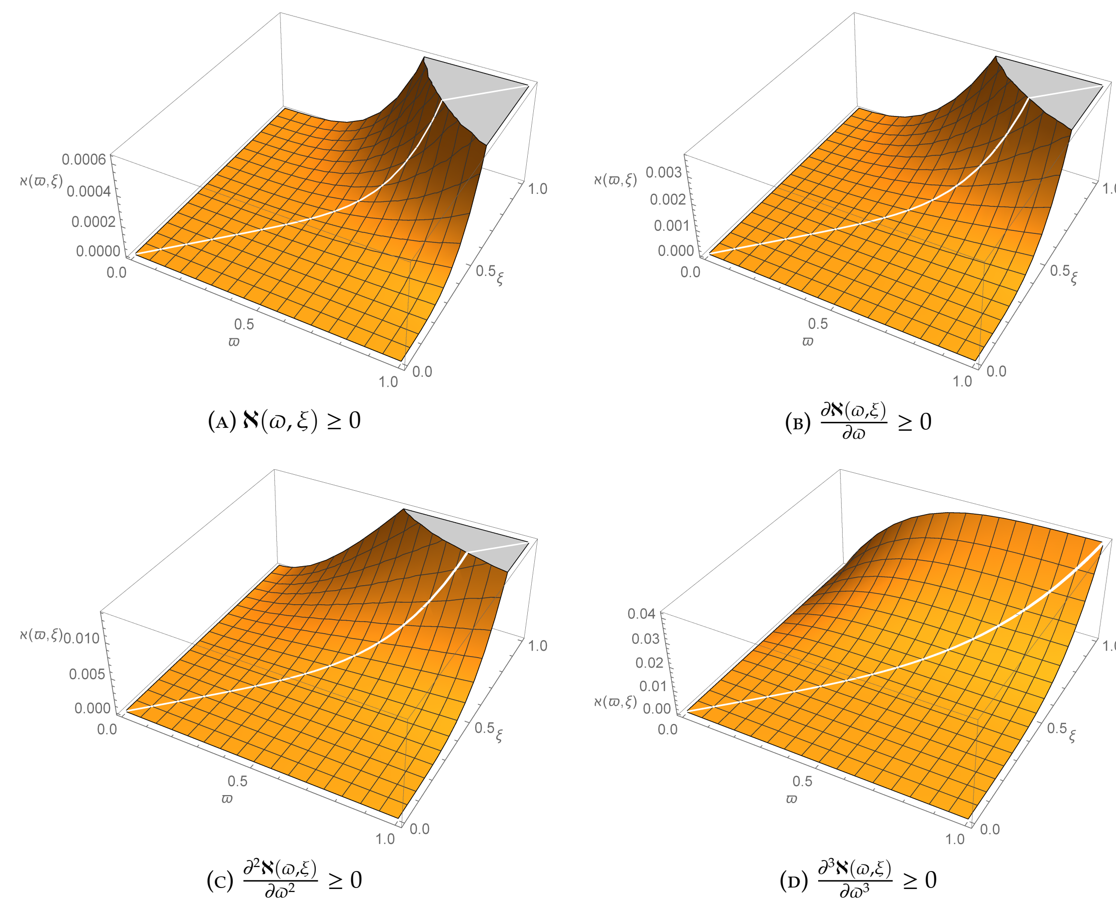
Lemma 3.2. *Consider Green's function of the eighth-order homogeneous problem $\lambda^{(8)}(\varpi) = 0$, for $\varpi \in \bar{\mathrm{I}}$, with boundary conditions* (1.1) *as*

$$\aleph(\varpi,\xi) = \frac{1}{5040}\begin{cases} \xi^4[(\varpi-\xi)^3 + 4\varpi(\varpi-\xi)^2 + 10\varpi^2(3\varpi-\xi)], \ 0 \leq \xi \leq \varpi \leq 1, \\ \varpi^4[(\xi-\varpi)^3 + 4\xi(\xi-\varpi)^2 + 10\xi^2(3\xi-\varpi)], \ 0 \leq \varpi \leq \xi \leq 1, \end{cases} \tag{3.14}$$

and for any $\varpi, \zeta \in \bar{\mathrm{I}}$,

$$\begin{aligned} &\aleph(\varpi,\xi) \geq 0, \quad \frac{\partial\aleph(\varpi,\xi)}{\partial\varpi} \geq 0, \quad \frac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2} \geq \quad 0, \frac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3} \geq 0, \\ &\frac{\partial^4\aleph(\varpi,\xi)}{\partial\varpi^4} \geq 0, \quad \frac{\partial^5\aleph(\varpi,\xi)}{\partial\varpi^5} \leq 0, \quad \frac{\partial^6\aleph(\varpi,\xi)}{\partial\varpi^6} \geq 0, \quad \frac{\partial^7\aleph(\varpi,\xi)}{\partial\varpi^7} \leq 0. \end{aligned} \tag{3.15}$$

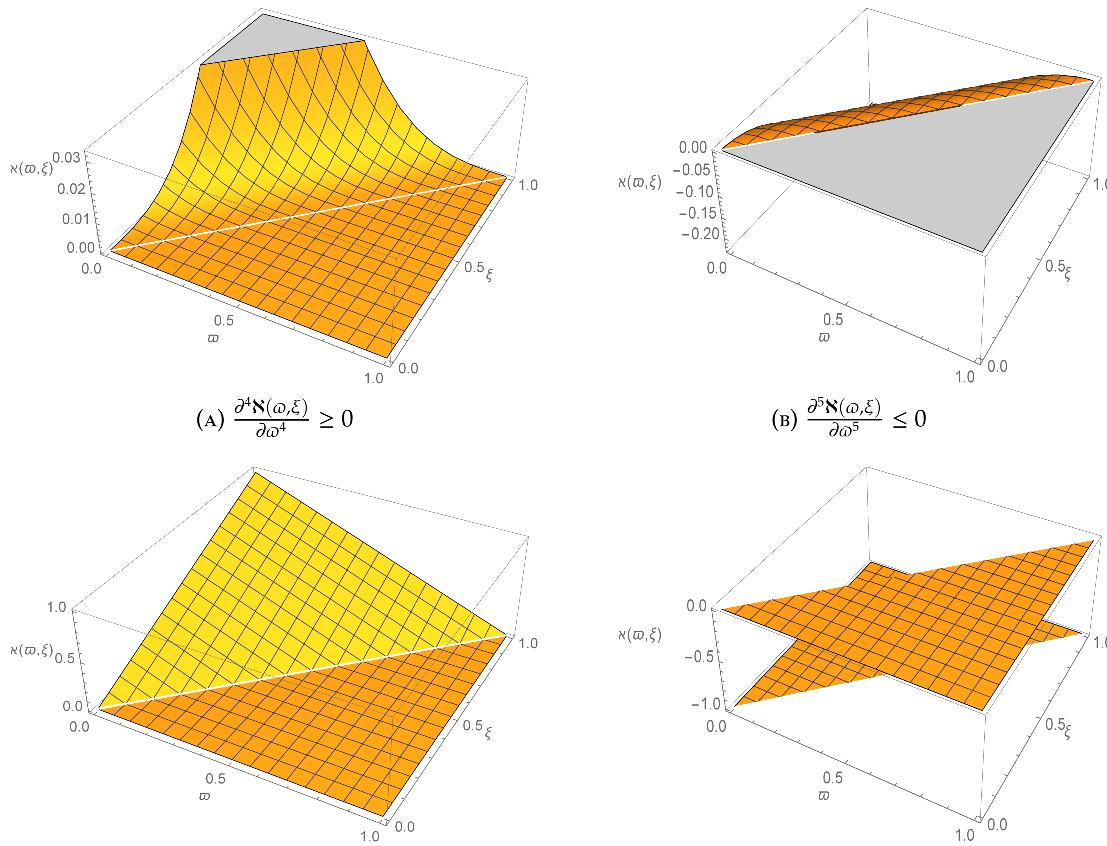
The plots of the above given Green function (3.14) with the conditions (3.15) are given in the group of Figures 5 and 6.



(A) $\aleph(\varpi,\xi) \geq 0$ (B) $\frac{\partial\aleph(\varpi,\xi)}{\partial\varpi} \geq 0$

(C) $\frac{\partial^2\aleph(\varpi,\xi)}{\partial\varpi^2} \geq 0$ (D) $\frac{\partial^3\aleph(\varpi,\xi)}{\partial\varpi^3} \geq 0$

FIGURE 5. Plots of the Green function (3.14) with the conditions (3.15).



(A) $\frac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4} \geq 0$ (B) $\frac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5} \leq 0$

(C) $\frac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6} \geq 0$ (D) $\frac{\partial^7 \aleph(\varpi,\xi)}{\partial \varpi^7} \leq 0$

Figure 6. Plots of the Green function (3.14) with the conditions (3.15).

Theorem 3.2. *Consider that* $\psi \in C(\mathcal{A}, [0, \infty))$ *and* $\psi(\varpi, 0, 0, 0, 0, 0, 0, 0, 0) \neq 0$, $\varpi \in \bar{\mathrm{I}}$. *Assume that there exists non-negative number* $\gamma > 0$ *such that*

$$\begin{aligned} \max\Big\{ &\psi(\varpi, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) : (\varpi, \lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) \\ &\in \bar{\mathrm{I}} \times [0, \gamma] \times \Big[0, \tfrac{24}{7}\gamma\Big] \times [0, 8\gamma] \times \Big[0, \tfrac{48}{5}\gamma\Big] \times [0, 48\gamma] \\ &\times [-192\gamma, 0] \times [0, 576\gamma] \times [-1152\gamma, 0] \Big\} \leq 1152\gamma. \end{aligned} \tag{3.16}$$

Then the problem (1.1) *exist at least one non-negative solution* $\lambda^* \in C^8(\bar{\mathrm{I}})$ *satisfies* $0 \leq \lambda(\varpi) \leq \gamma$, $0 \leq \lambda'(\varpi) \leq \frac{24}{7}\gamma$, $0 \leq \lambda''(\varpi) \leq 8\gamma$, $0 \leq \lambda'''(\varpi) \leq \frac{48}{5}\gamma$, $0 \leq \lambda^{(4)}(\varpi) \leq 48\gamma$, $-192\gamma \leq \lambda^{(5)}(\varpi) \leq 0$, $0 \leq \lambda^{(6)}(\varpi) \leq 576\gamma$, $-1152\gamma \leq \lambda^{(7)}(\varpi) \leq 0$.

Proof. By (3.15) and integration calculations, we get

$$\int_0^1 |\aleph(\varpi, \xi)| \, \mathrm{d}\xi = \int_0^1 \aleph(\varpi, \xi) \, \mathrm{d}\xi = \tfrac{1}{5040} \left(\tfrac{1}{8}\varpi^8 - \varpi^7 + \tfrac{7}{2}\varpi^6 - 7\varpi^5 + \tfrac{35}{4}\varpi^4 \right),$$

$$\int_0^1 \left| \tfrac{\partial \aleph(\varpi,\xi)}{\partial \varpi} \right| \mathrm{d}\xi = \int_0^1 \tfrac{\partial \aleph(\varpi,\xi)}{\partial \varpi} \, \mathrm{d}\xi = \tfrac{1}{5040} \left(\varpi^7 - 7\varpi^6 + 21\varpi^5 - 35\varpi^4 + 35\varpi^3 \right),$$

$$\int_0^1 \left|\frac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\right| \mathrm{d}\xi = \int_0^1 \frac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\,\mathrm{d}\xi = \tfrac{1}{5040}(7\varpi^6 - 42\varpi^5 + 105\varpi^4 - 140\varpi^3 + 105\varpi^2),$$
$$\int_0^1 \left|\frac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\right| \mathrm{d}\xi = \int_0^1 \frac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\,\mathrm{d}\xi = \tfrac{1}{5040}(42\varpi^5 - 210\varpi^4 + 420\varpi^3 - 420\varpi^2 + 210\varpi),$$
$$\int_0^1 \left|\frac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\right| \mathrm{d}\xi = \int_0^1 \frac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\,\mathrm{d}\xi = \tfrac{1}{5040}(210\varpi^4 - 840\varpi^3 + 1260\varpi^2 - 840\varpi + 210),$$
$$\int_0^1 \left|\frac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\right| \mathrm{d}\xi = -\int_0^1 \frac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\,\mathrm{d}\xi = \tfrac{1}{5040}(-840\varpi^3 + 2520\varpi^2 - 2520\varpi + 840),$$
$$\int_0^1 \left|\frac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6}\right| \mathrm{d}\xi = \int_0^1 \frac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6}\,\mathrm{d}\xi = \tfrac{1}{5040}(2520\varpi^2 - 5040\varpi + 2520),$$
$$\int_0^1 \left|\frac{\partial^7 \aleph(\varpi,\xi)}{\partial \varpi^7}\right| \mathrm{d}\xi = -\int_0^1 \frac{\partial^7 \aleph(\varpi,\xi)}{\partial \varpi^7}\,\mathrm{d}\xi = 1 - \varpi,$$

Then,

$$\max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 |\aleph(\varpi,\xi)| d\xi = \tfrac{1}{1152}, \qquad \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial \aleph(\varpi,\xi)}{\partial \varpi}\right| \mathrm{d}\xi = \tfrac{1}{336},$$
$$\max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\right| \mathrm{d}\xi = \tfrac{1}{144}, \qquad \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\right| \mathrm{d}\xi = \tfrac{1}{120},$$
$$\max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\right| \mathrm{d}\xi = \tfrac{1}{24}, \qquad \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\right| \mathrm{d}\xi = \tfrac{1}{6},$$
$$\max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6}\right| \mathrm{d}\xi = \tfrac{1}{2}, \qquad \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left|\frac{\partial^7 \aleph(\varpi,\xi)}{\partial \varpi^7}\right| \mathrm{d}\xi = 1.$$

Then, we suppose the Banach space $\mathbb{H} = C^8(\bar{\mathrm{I}})$ provider by the norm

$$\|\lambda\| = \max\left\{|\lambda|_0,\ \tfrac{7}{24}|\lambda'|_0,\ \tfrac{1}{8}|\lambda''|_0,\ \tfrac{5}{48}|\lambda'''|_0,\ \tfrac{1}{48}|\lambda^{(4)}|_0, \tfrac{1}{192}|\lambda^{(5)}|_0, \tfrac{1}{576}|\lambda^{(6)}|_0,\ \tfrac{1}{1152}|\lambda^{(7)}|_0\right\},$$

we have $|\lambda|_0 = \max_{\varpi\in\bar{\mathrm{I}}} |\lambda(\varpi)|$, for $\lambda \in \mathbb{H}$ determine the operator Π by

$$(\Pi\lambda)(\varpi) = \int_0^1 \aleph(\varpi,\xi)\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi. \tag{3.17}$$

Therefore

$$(\Pi\lambda)'(\varpi) = \int_0^1 \frac{\partial \aleph(\varpi,\xi)}{\partial \varpi}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi, \qquad (\Pi\lambda)''(\varpi) = \int_0^1 \frac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi,$$
$$(\Pi\lambda)'''(\varpi) = \int_0^1 \frac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi, \qquad (\Pi\lambda)^{(4)}(\varpi) = \int_0^1 \frac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi,$$
$$(\Pi\lambda)^{(5)}(\varpi) = \int_0^1 \frac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi, \qquad (\Pi\lambda)^{(6)}(\varpi) = \int_0^1 \frac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi,$$

and

$$(\Pi\lambda)^{(7)}(\varpi) = \int_0^1 \frac{\partial^7 \aleph(\varpi,\xi)}{\partial \varpi^7}\widehat{\psi_\lambda}(\xi)\,\mathrm{d}\xi.$$

Then, $(\Pi\lambda)(0) = (\Pi\lambda)'() = (\Pi\lambda)''(0) = (\Pi\lambda)'''(0) = 0$ and $(\Pi\lambda)^{(4)}(1) = (\Pi\lambda)^{(5)}(1) = (\Pi\lambda)^{(6)}(1) = (\Pi\lambda)^{(7)}(1) = 0$. So, according to Theorem of Arzelá-Ascoli, it is clear to know that this operator $\Pi : \mathbb{H} \to \mathbb{H}$ is a completely continuous operator. I.e. the problem (1.1) admits a solution $\lambda = \lambda(\varpi)$ iff λ is a fixed point of operator Π indicated as (3.17). We put

$$\begin{aligned} \Upsilon_\gamma = \Big\{ &\lambda \in \mathbb{H} : \|\lambda\| \leq \gamma,\ \lambda(\varpi) \geq 0,\ \lambda'(\varpi) \geq 0,\ \lambda''(\varpi) \geq 0,\ \lambda'''(\varpi) \geq 0, \\ &\lambda^{(4)}(\varpi) \geq 0, \lambda^{(5)}(\varpi) \leq 0,\ \lambda^{(6)}(\varpi) \geq 0,\ \lambda^{(7)}(\varpi) \leq 0,\ \varpi \in \bar{\mathrm{I}} \Big\}, \end{aligned}$$

and Υ_γ is a bounded closed convex set of $\mathbb{H}$. If $\lambda \in \Upsilon_\gamma$, according (3.16) we find $|\lambda|_0 \leq \gamma$ and $|\lambda'|_0 \leq \frac{24}{7}\gamma$, $|\lambda''|_0 \leq 8\gamma$, $|\lambda'''|_0 \leq \frac{48}{5}\gamma$, $|\lambda^{(4)}|_0 \leq 48\gamma$, $|\lambda^{(5)}|_0 \leq 192\gamma$, $|\lambda^{(6)}|_0 \leq 576\gamma$, $|\lambda^{(7)}|_0 \leq 1152\gamma$, implies that $0 \leq \lambda(\varpi) \leq \gamma$ and $0 \leq \lambda'(\varpi) \leq \frac{24}{7}\gamma$, $0 \leq \lambda''(\gamma) \leq 8\gamma$, $0 \leq \lambda'''(\varpi) \leq \frac{48}{5}\gamma$, $0 \leq \lambda^{(4)}(\varpi) \leq 48\gamma$, $-192\gamma \leq \lambda^{(5)}(\varpi) \leq 0$, $0 \leq \lambda^{(6)}(\varpi) \leq 576\gamma$, $-1152\gamma \leq \lambda^{(7)}(\varpi) \leq 0$. By (3.16) we conclude that $\widehat{\psi}_\lambda(\varpi) \leq 1152\gamma$, for $\varpi \in \bar{\mathrm{I}}$. Then,

$$\begin{aligned}
|(\Pi\lambda)|_0(\varpi) &= \max_{\varpi \bar{\mathrm{I}}} \left| \int_0^1 \aleph(\varpi,\xi)\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \bar{\mathrm{I}}} \int_0^1 \aleph(\varpi,\xi)\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \aleph(\varpi,\xi)\,\mathrm{d}\xi = \gamma, \\
|(\Pi\lambda)'|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial \aleph(\varpi,\xi)}{\partial \varpi}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial \aleph(\varpi,\xi)}{\partial \varpi}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial \aleph(\varpi,\xi)}{\partial \varpi}\,\mathrm{d}\xi = \tfrac{24}{7}\gamma, \\
|(\Pi\lambda)''|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^2 \aleph(\varpi,\xi)}{\partial \varpi^2}\,\mathrm{d}\xi = 8\gamma, \\
|(\Pi\lambda)'''|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^3 \aleph(\varpi,\xi)}{\partial \varpi^3}\,\mathrm{d}\xi = \tfrac{48}{5}\gamma, \\
|(\Pi\lambda)^{(4)}|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^4 \aleph(\varpi,\xi)}{\partial \varpi^4}\,\mathrm{d}\xi = 48\gamma, \\
|(\Pi\lambda)^{(5)}|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right| \\
&\leq \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \left| \tfrac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5}\widehat{\psi}_\lambda(\xi) \right| \mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi \in \bar{\mathrm{I}}} \int_0^1 \left| \tfrac{\partial^5 \aleph(\varpi,\xi)}{\partial \varpi^5} \right| \mathrm{d}\xi = 192\gamma, \\
|(\Pi\lambda)^{(6)}|_0(\varpi) &= \max_{\varpi \in \bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^6 \aleph(\varpi,\xi)}{\partial \varpi^6}\widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right|
\end{aligned}$$

$$\leq \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \tfrac{\partial^6 \aleph(\varpi,\xi)}{\partial\varpi^6} \widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi\in\widehat{\psi}_\lambda(\kappa)} \int_0^1 \tfrac{\partial^6 \aleph(\varpi,\xi)}{\partial\varpi^6}\,\mathrm{d}\xi = 576\gamma,$$

$$|(\Pi\lambda)^{(7)}|_0(\varpi) = \max_{\varpi\in\bar{\mathrm{I}}} \left| \int_0^1 \tfrac{\partial^7 \aleph(\varpi,\xi)}{\partial\varpi^7} \widehat{\psi}_\lambda(\xi)\,\mathrm{d}\xi \right|$$

$$\leq \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left| \tfrac{\partial^7 \aleph(\varpi,\xi)}{\partial\varpi^7} \widehat{\psi}_\lambda(\xi) \right| \mathrm{d}\xi \leq 1152\gamma \times \max_{\varpi\in\bar{\mathrm{I}}} \int_0^1 \left| \tfrac{\partial^7 \aleph(\varpi,\xi)}{\partial\varpi^7} \right| \mathrm{d}\xi = 1152\gamma.$$

Therefore

$$\|\Pi\lambda\| = \max\Big\{ |(\Pi\lambda)|_0,\ \tfrac{7}{24}|(\Pi\lambda)'|_0,\ \tfrac{1}{8}|(\Pi\lambda)''|_0,\ \tfrac{5}{48}|(\Pi\lambda)'''|_0,\ \tfrac{1}{48}|(\Pi\lambda)^{(4)}|_0,$$
$$\tfrac{1}{192}|(\Pi\lambda)^{(5)}|_0,\ \tfrac{1}{576}|(\Pi\lambda)^{(6)}|_0,\ \tfrac{1}{1152}|(\Pi\lambda)^{(7)}|_0 \Big\} \leq \gamma,$$

implies $\Pi\lambda \in \mathrm{Y}_\gamma$. Thus, according to the Leray-Schauder fixed point Theorem, Y exists a fixed point $\lambda^* \in \mathrm{Y}_\gamma$, which is a solution of the problem (1.1). We conclude $f(t,0,0,0,0,0,0,0,0) \neq 0$. Then, trivial solution is not achieved for the problem (1.1), therefore, $|\lambda^*|_0 > 0$. By (3.15) we know that $\lambda^*(\varpi)$ is nondecreasing and convex on $\bar{\mathrm{I}}$, thus $\lambda^*(\varpi) \geq \varpi|\lambda^*|_0 > 0$, for $\varpi \in \bar{\mathrm{I}}$. So, $\lambda^*(\varpi)$ is a non-negative solution of the problem (1.1). Here the proof is finished. □

3.1. **Application.** To show the desired outcomes, we solve an example.

Example 3.1. *Consider the following problem*

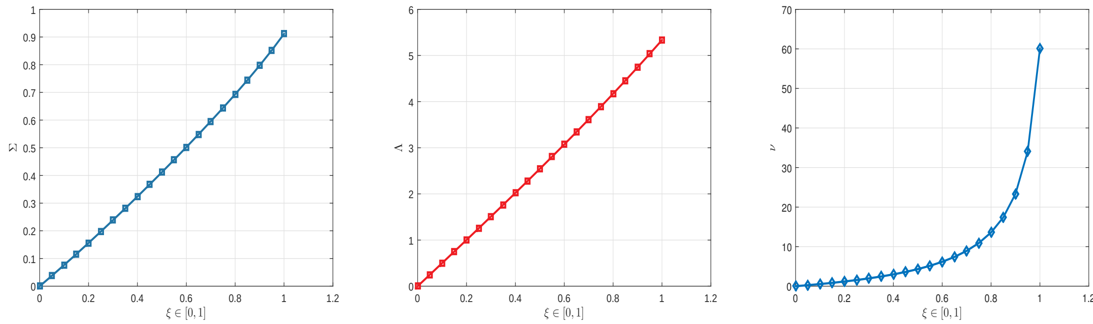
$$\begin{cases} \chi^{(8)}(\varpi) = \frac{\sqrt[3]{\varpi}}{6}\chi(\varpi) + \frac{\varpi^{17}}{3}\chi'(\varpi) + \frac{\varpi^{13}}{5}\chi''(\varpi) + \frac{\sqrt[5]{\varpi}}{47}\chi'''(\varpi) + \frac{\varpi^7}{11}\chi^{(4)}(\varpi) \\ \qquad -\frac{3\sqrt[3]{\varpi}}{5}\chi^{(5)}(\varpi) + \frac{e^{\varpi}}{4}\chi^{(6)}(\varpi) - \frac{1}{2}\cos\varpi\,\chi^{(7)}(\varpi) + \varpi^2 + 3, \\ \chi(0) = \chi'(0) = \chi''(0) = \chi'''(0) = 0 \\ \chi^{(4)}(1) = \chi^{(5)}(1) = \chi^{(6)}(1) = \chi^{(7)}(1) = 0 \end{cases} \tag{3.18}$$

We have

$$\psi(\varpi,\chi_0,\chi_1,\chi_2,\chi_3,\chi_4,\chi_5,\chi_6,\chi_7) = \tfrac{\sqrt[3]{\varpi}}{6}\chi(\varpi) + \tfrac{\varpi^{17}}{3}\chi'(\varpi) + \tfrac{\varpi^{13}}{5}\chi''(\varpi) + \tfrac{\sqrt[5]{\varpi}}{47}\chi'''(\varpi)$$
$$+ \tfrac{\varpi^7}{11}\chi^{(4)}(\varpi) - \tfrac{3\sqrt[3]{\varpi}}{5}\chi^{(5)}(\varpi) + \tfrac{e^{\varpi}}{4}\chi^{(6)}(\varpi) - \tfrac{1}{2}\cos\varpi\,\chi^{(7)}(\varpi) + \varpi^2 + 3,$$

then $\psi(\varpi,0,0,0,0,0,0,0,0) = \varpi^2 + 3 \neq 0$, *for each* $\varpi \in \bar{\mathrm{I}}$. *Clearly* $\zeta_0(\varpi) = \frac{\varpi}{6}$ *and* $\zeta_1(\varpi) = \varpi^{17}$, $\zeta_2(\varpi) = \frac{\varpi^{13}}{2}$, $\zeta_3(\varpi) = \frac{\varpi}{10}$, $\zeta_4(\varpi) = \varpi^7$, $\zeta_5(\varpi) = \frac{\varpi}{7}$, $\zeta_6(\varpi) = \frac{e^{\varpi}}{4}$, $\zeta_7(\varpi) = \frac{1}{2}\cos\varpi$, $\zeta_8(\varpi) = \varpi^2 + 5$ *are non-negatives functions and* $\zeta_\jmath \in L^1(\bar{\mathrm{I}})$, $\jmath = 0,1,2,3,4,5,6,7,8$. *Therefore, we find*

$$\Sigma = \underbrace{\tfrac{1}{252}\int_0^1 \zeta_0(\xi)\,\mathrm{d}\xi}_{I_0} + \underbrace{\tfrac{1}{72}\int_0^1 \zeta_1(\xi)\,\mathrm{d}\xi}_{I_1} + \underbrace{\tfrac{1}{30}\int_0^1 \zeta_2(\xi)\,\mathrm{d}\xi}_{I_2} + \underbrace{\tfrac{1}{24}\int_0^1 \zeta_3(\xi)\,\mathrm{d}\xi}_{I_3}$$
$$+ \underbrace{\tfrac{1}{6}\int_0^1 \zeta_4(\xi)\,\mathrm{d}\xi}_{I_4} + \underbrace{\tfrac{1}{2}\int_0^1 \zeta_5(\xi)\,\mathrm{d}\xi}_{I_5} + \underbrace{\int_0^1 \zeta_6(\xi)\,\mathrm{d}\xi}_{I_6} + \underbrace{\int_0^1 \zeta_7(\xi)\,\mathrm{d}\xi}_{I_7} \approx 0.911 < 1.$$



(A) Numerical results of Σ (B) Numerical results of Λ (C) Numerical results of ν

FIGURE 7. Obtained results of Σ, Λ and ν for $\mathbb{BVP}$ (3.18) whenever $\varpi \in \bar{\mathrm{I}}$.

Figures 7a, 7b and 7c show the plot of Σ, Λ *and* ν *for* $\mathbb{BVP}$ (3.18). *One can check these results in Tables 1. And for every*

$$(\varpi, \chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7) \in \bar{\mathrm{I}} \times \left[0, \tfrac{1}{252}\nu\right] \times \left[0, \tfrac{1}{72}\nu\right] \times \left[0, \tfrac{1}{30}\nu\right]$$
$$\times \left[0, \tfrac{1}{24}\nu\right] \times \left[0, \tfrac{1}{6}\nu\right] \times \left[-\tfrac{1}{2}\nu, 0\right] \times [0, \nu] \times [-\nu, 0],$$

we have ψ *satisfies*

$$\psi(\varpi, \chi_0, \chi_1, \chi_2, \chi_3, \chi_4, \chi_5, \chi_6, \chi_7) \leq \zeta_0(\varpi)\chi_0 + \zeta_1(\varpi)\chi_1 + \zeta_2(\varpi)\chi_2$$
$$+ \zeta_3(\varpi)\chi_3 + \zeta_4(\varpi)\chi_4 - \zeta_5(\varpi)\chi_5 + \zeta_6(\varpi)\chi_6 - \zeta_7(\varpi)\chi_7 + \zeta_8(\varpi),$$

where $\Lambda = \int_0^1 \zeta_8(\xi)\,\mathrm{d}\xi = 5.333$, $\nu = \Lambda(1-\Sigma)^{-1} = 60.080$.

TABLE 1. Obtained results of $\Sigma < 1$, Λ and ν for $\varpi \in \bar{\mathrm{I}}$.

ϖ	I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7	Σ	Λ	ν
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.05	0.000	0.000	0.000	0.000	0.000	0.000	0.013	0.025	0.038	0.250	0.260
0.10	0.001	0.000	0.000	0.001	0.000	0.001	0.026	0.050	0.077	0.500	0.542
0.15	0.002	0.000	0.000	0.001	0.000	0.002	0.040	0.075	0.116	0.751	0.850
0.20	0.003	0.000	0.000	0.002	0.000	0.003	0.055	0.099	0.156	1.003	1.188
0.25	0.005	0.000	0.000	0.003	0.000	0.004	0.071	0.124	0.197	1.255	1.563
0.30	0.008	0.000	0.000	0.005	0.000	0.006	0.087	0.148	0.239	1.509	1.982
0.35	0.010	0.000	0.000	0.006	0.000	0.009	0.105	0.171	0.281	1.764	2.453
0.40	0.013	0.000	0.000	0.008	0.000	0.011	0.123	0.195	0.324	2.021	2.989
0.45	0.017	0.000	0.000	0.010	0.000	0.014	0.142	0.217	0.367	2.280	3.604
0.50	0.021	0.000	0.000	0.013	0.000	0.018	0.162	0.240	0.412	2.542	4.319
0.55	0.025	0.000	0.000	0.015	0.001	0.022	0.183	0.261	0.456	2.805	5.161
0.60	0.030	0.000	0.000	0.018	0.002	0.026	0.206	0.282	0.502	3.072	6.168
0.65	0.035	0.000	0.000	0.021	0.004	0.030	0.229	0.303	0.548	3.342	7.397
0.70	0.041	0.000	0.000	0.025	0.007	0.035	0.253	0.322	0.595	3.614	8.934
0.75	0.047	0.000	0.001	0.028	0.013	0.040	0.279	0.341	0.644	3.891	10.917
0.80	0.053	0.001	0.002	0.032	0.021	0.046	0.306	0.359	0.693	4.171	13.586
0.85	0.060	0.003	0.004	0.036	0.034	0.052	0.335	0.376	0.744	4.455	17.397
0.90	0.068	0.008	0.008	0.041	0.054	0.058	0.365	0.392	0.797	4.743	23.342
0.95	0.075	0.022	0.017	0.045	0.083	0.064	0.396	0.407	0.852	5.036	34.084
1.00	0.083	0.056	0.036	0.050	0.125	0.071	0.430	0.421	0.911	5.333	60.080

Thereby, according to Theorem 3.1, the problem (3.18) *admits at least one non-negative solution* $\chi^* \in C^8(\bar{\mathrm{I}})$ *such that*

Table 2. Numerical results of all variables in inequalities (3.19) for $\varpi \in \bar{\mathrm{I}}$.

ϖ	$C_i\ (i=0,1,\ldots,7)$								ν
	C_0	C_1	C_2	C_3	C_4	C_5	C_6	C_7	
0.00	0.000	0.000	0.000	0.000	0.000	0.000	0.000	8.258	0.000
0.05	0.000	0.000	0.000	0.000	0.001	−0.021	0.413	8.258	0.260
0.10	0.000	0.000	0.000	0.001	0.008	−0.083	0.826	8.258	0.542
0.15	0.000	0.000	0.000	0.004	0.028	−0.186	1.239	8.258	0.850
0.20	0.000	0.000	0.001	0.013	0.066	−0.330	1.652	8.258	1.188
0.25	0.000	0.000	0.002	0.032	0.129	−0.516	2.064	8.258	1.563
0.30	0.000	0.001	0.005	0.067	0.223	−0.743	2.477	8.258	1.982
0.35	0.000	0.002	0.011	0.124	0.354	−1.012	2.890	8.258	2.453
0.40	0.001	0.003	0.021	0.211	0.528	−1.321	3.303	8.258	2.989
0.45	0.002	0.007	0.038	0.339	0.752	−1.672	3.716	8.258	3.604
0.50	0.003	0.013	0.065	0.516	1.032	−2.064	4.129	8.258	4.319
0.55	0.006	0.023	0.104	0.756	1.374	−2.498	4.542	8.258	5.161
0.60	0.012	0.039	0.161	1.070	1.784	−2.973	4.955	8.258	6.168
0.65	0.020	0.062	0.240	1.474	2.268	−3.489	5.367	8.258	7.397
0.70	0.034	0.097	0.347	1.983	2.832	−4.046	5.780	8.258	8.934
0.75	0.055	0.147	0.490	2.613	3.484	−4.645	6.193	8.258	10.917
0.80	0.087	0.216	0.676	3.382	4.228	−5.285	6.606	8.258	13.586
0.85	0.132	0.311	0.916	4.310	5.071	−5.966	7.019	8.258	17.397
0.90	0.197	0.439	1.219	5.418	6.020	−6.689	7.432	8.258	23.342
0.95	0.288	0.607	1.597	6.726	7.080	−7.452	7.845	8.258	34.084
1.00	0.413	0.826	2.064	8.258	8.258	−8.258	8.258	8.258	60.080

$$
\begin{aligned}
&\underbrace{252\max_{\varpi\in\bar{\mathrm{I}}}\chi^*(\varpi)}_{C_0} \le \underbrace{72\max_{\varpi\in\bar{\mathrm{I}}}(\chi^*)'(\varpi)}_{C_1} \le \underbrace{30\max_{\varpi\in\bar{\mathrm{I}}}(\chi^*)''(\varpi)}_{C_2} \le \underbrace{24\max_{\varpi\in\bar{\mathrm{I}}}(\chi^*)'''(\varpi)}_{C_3}\\
&\le \underbrace{6\max_{\varpi\in\bar{\mathrm{I}}}(\chi^*)^{(4)}(\varpi)}_{C_4} \le \underbrace{2\max_{\varpi\in\bar{\mathrm{I}}}\Big(-(\chi^*)^{(5)}(\varpi)\Big)}_{C_5}\\
&\le \underbrace{\max_{\varpi\in\bar{\mathrm{I}}}(\chi^*)^{(6)}(\varpi)}_{C_6} \le \underbrace{\max_{\varpi\in\bar{\mathrm{I}}}\Big(-(\chi^*)^{(7)}(\varpi)\Big)}_{C_7} \le \nu.
\end{aligned}
\tag{3.19}
$$

We compare the numerical results of C_i *in* (3.19) *with* ν *when* $\chi^*(\varpi) = \left(\frac{2\varpi}{5}\right)^7$ *in Table 2. Thesrs results show that the inequalities* (3.19) *hold for* $\varpi \in \bar{\mathrm{I}}$*. One can see the solution curve* ξ^* *in Fig 8. Also, we compare these numerical results in 2D plot 9.*

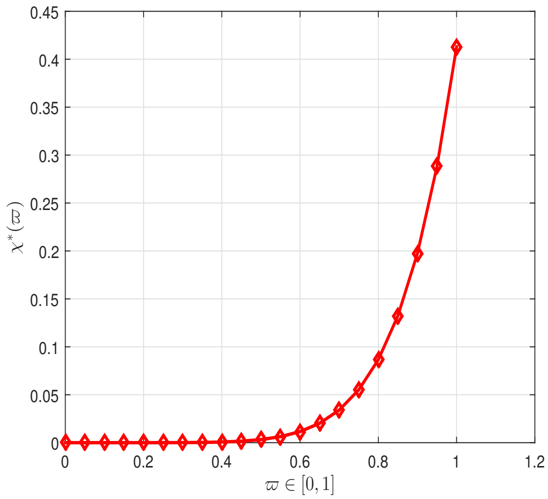


FIGURE 8. Solution curve, $\chi^*(\varpi)$. (The solution is non-decreasing and convex).

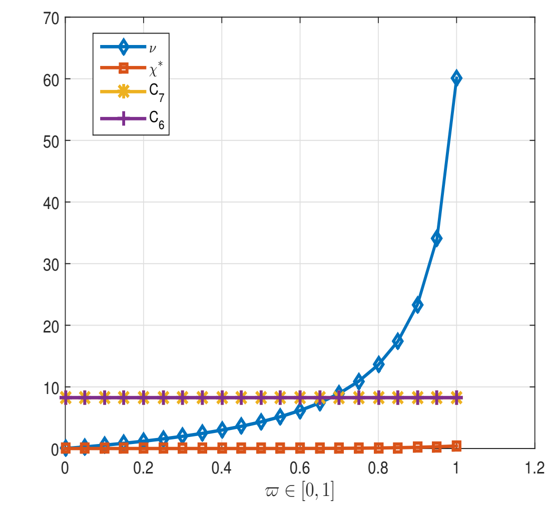


FIGURE 9. 2D plot of the solution curve vs ν and C_7, C_6.

4. CONCLUSION

In this article, we have investigated the existence of non-negative solutions for the $\mathbb{BVP}$ of eighth-order. We have used the classical theorem of the fixed point of Leray-Schauder with all the necessary conditions. The considered Green functions are briefly explained and plotted for clear visualization. We have implemented our fundamental results by considering an example and justified that the proposed results are correct and fully novel. In the future, the proposed results can be used to further analyze the eighth-order $\mathbb{BVP}$s.

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