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CHARACTERIZATION OF BIORTHOGONAL MULTIWAVELET PACKETS WITH ARBITRARY DILATION MATRIX

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Abstract. In this paper, we investigate the characterization of biorthogonal multiwavelet packets associated with arbitrary matrix dilations and particularly of orthonormal multiwavelet packets by means of basic equations in Fourier domain.

1. INTRODUCTION

It is well-known that the classical orthonormal wavelet bases have poor frequency localization. For example, if the wavelet ψ is band limited, then the measure of the supp of $\hat{\psi}_{j,k}$ is 2^j-times that of supp $\hat{\psi}$. To overcome this disadvantage, Coifman *et al.* [8] constructed univariate orthogonal wavelet packets. The fundamental idea of wavelet packet analysis is to construct a library of orthonormal bases for $L^2(\mathbb{R})$, which can be searched in real time for the best expansion with respect to a given application. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Chui and Li [6] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be can be employed to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [7]. They have also shown that all the wavelet packets, constructed in this way, are not led to Riesz bases for $L^2(\mathbb{R})$. Shen [18] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the wavelet packets related to the Walsh polynomials on $\mathbb{R}^+$ [13,14,16], higher dimensional wavelet packets with arbitrary dilation matrix [9], the orthogonal version of vector-valued wavelet packets [5] and the M-band framelet packets [17].

On the other hand, multiwavelets are natural extension and generalization of traditional wavelets. They have received considerable attention from the wavelet research communities both in the theory as well as in applications. They can be seen as vector valued-wavelets that satisfy conditions in which matrices are involved rather than scalars as in the wavelet case. Multiwavelets can own symmetry, orthogonality, short support and high order vanishing moments, however traditional wavelets can not possess all these properties at the same time (see [10]). As far as the characterization of multiwavelets is concerned, Calogero studied the characterization of all multiwavelets associated with general expanding maps of $\mathbb{R}^n$ in [4].

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The Calogero's work was further extended by Bownik [2], taking into consideration the dilation matrices which preserves the standard lattice $\mathbb{Z}^n$ in terms of affine systems. In the same year, another characterization of orthonormal multiwavelets was given by Rzeszotnik [11] for expanding dilations that preserves the lattice $\mathbb{Z}^n$. However, Bownik [3] has presented a new approach to characterize all orthonormal multiwavelets by means of basic equations in the Fourier domain.

Recently, Yang and Cheng [20] have generalized the concept of wavelet packets to the case of multiwavelet packets associated with a dilation factor a which were more flexible in applications. Subsequently, Behera [1] extended the results of Yang and Cheng to the multivariate multiwavelet packets associated with a dilation matrix A. He proved lemmas on the so-called splitting trick and several theorems concerning the Fourier transform of the multiwavelet packets and the construction of multiwavelet packets to show that their translates form an orthonormal basis of $L^2(\mathbb{R}^d)$. Later on, Sun and Li [19] have given the construction and properties of generalized orthogonal multiwavelet packets based on the results discussed in [20].

Orthogonal wavelet packets have many desired properties such as compact support, good frequency localization and vanishing moments. However, there is no continuous symmetry which is a much desired property in imaging the compression and signal processing. To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. The characterization of multiwavelet packets associated with the dilation matrix A on general lattices has been studied by the author in [12, 15]. In this paper, we further investigate the characterization of biorthogonal multiwavelet packets associated with arbitrary matrix dilations and particularly of orthonormal multiwavelet packets by means of basic equations in Fourier domain.

We have structured the article as follows. In Section 2, we state some basic preliminaries, notations and definitions including the definition of multiresoltion analysis associated with arbitrary dilation matrix A and the corresponding multiwavelet packets. In Section 3, we establish our main results concerning with the characterization of biorthogonal multiwavelet packets on $\mathbb{R}^d$.

2. NOTATIONS AND PRELIMINARIES

Throughout, this paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be all real and complex numbers, respectively. $\mathbb{Z}$ and $\mathbb{Z}^+$ denote all integers and all non-negative integers, respectively. $\mathbb{Z}^d$ and $\mathbb{R}^d$ denote the set of all d-tuples integers and d-tuples of reals, respectively. Assume that we have an expansive dilation matrix A, i.e., all eigenvalues λ of A satisfy $|\lambda| > 1$ and preserves the lattice Γ. Let $a = |detA|, A^* =$ transpose of A and B be a $d \times d$ non-singular matrix. Also, if A is expanding so is A^*. Considering $\mathbb{Z}^d$ as an additive group, we see that $A\mathbb{Z}^d$ is a normal subgroup of $\mathbb{Z}^d$ so we can form the cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$. It is well known fact that the number of distinct cosets of $A\mathbb{Z}^d$ in $\mathbb{Z}^d$ is equal to $a = |detA|$ (see[21]). With A and B defined as above, we consider

$$\Lambda(A,B)=\Big\{\alpha\in\mathbb{R}^d:\exists\,(j,m)\in\mathbb{Z}\times B^{*-1}(\mathbb{Z}^d):\alpha=A^{*-j}m\Big\},\tag{2.1}$$

and

$$I_{A,B}(\alpha)=\Big\{(j,m)\in\mathbb{Z}\times B^{*-1}(\mathbb{Z}^d):\alpha=A^{*-j}m\Big\}.\tag{2.2}$$

The set $\Lambda(A,B)$ is thought of as the set of all A-adic vectors relative to the lattice $B^{*-1}(\mathbb{Z}^d)$, i. e., the set of representatives of the equivalence classes of $\mathbb{Z}\times B^{*-1}(\mathbb{Z}^d)$ with respect to the equivalence relation defined by $(j,m)\sim(j',m')$ if and only if $\alpha=A^{*-j}m=A^{*-j'}$. Further, the set $I_{A,B}(\alpha)$ is the set of points of $\mathbb{Z}\times B^{*-1}(\mathbb{Z}^d)$ in the equivalence class of $\alpha\in\Lambda(A,B)$.

Since it is a well known fact that for every dilation matrix A, there exists a Hermitian norm $\|\cdot\|_*$ in $\mathbb{R}^d$, and constants $\lambda_{\max}\geq\lambda_{\min}>1$, such that if $\mathcal{B}$ denotes the unit ball in the new norm, centered at the origin, then

$$\mathcal{B}\subset\lambda_{\min}\mathcal{B}\subset A^*(\mathcal{B})\subset\lambda_{\max}\mathcal{B}.$$

For each $k\in\mathbb{Z}$, we define H_k as

$$H_k=A^{*k}(\mathcal{B}),\quad 2H_0\subset H_\eta,\ |\mathcal{B}|=1.$$

where η be the smallest integer. Then, the quasi-distance ρ on $\mathbb{R}^d$ induced by the dilation A^* is given by

$$\rho(\xi,\zeta)=\begin{cases}|detA|^j & \text{if } \xi-\zeta\in H_{j+1}\setminus H_j\\ 0 & \text{if } \xi=\zeta.\end{cases}$$

Furthermore, it is easy to verify that the *Hardy-Littlewood maximal operator*

$$M_{HL}f(\zeta)=\sup_{k\in\mathbb{Z}}\frac{1}{|H_k|}\int_{\zeta+H_k}\big|f(\xi)\big|\,d\xi$$

is bounded from L^1 to L^1-weak norm and

$$\lim_{k\to-\infty}\frac{1}{|H_k|}\int_{\xi+H_k}f(\xi)\,d\xi=f(\xi),\quad\text{for } a.e.\ \xi\in\mathbb{R}^d.\tag{2.3}$$

Definition 2.1. A countable family $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ of elements in a separable Hilbert space $\mathbb{H}$ is a *frame* if there exist constants $A,B,\ 0<A\leq B<\infty$ satisfying

$$A\big\|f\big\|_2^2\leq\sum_{\alpha\in\mathcal{A}}\big|\langle f,f_\alpha\rangle\big|^2\leq B\big\|f\big\|_2^2\tag{2.4}$$

for all $f\in\mathbb{H}$. The constants A and B independent of f for which (2.4) holds are called *frame bounds*. A frame is a tight frame if A and B can be chosen so that $A=B$ and is a *normalized tight frame* if $A=B=1$. If only the right hand side inequality holds in (2.4), we say that $\{f_\alpha\}_{\alpha\in\mathcal{A}}$ is a *Bessel squence* with constant B.

Lemma 2.2 [3]. Two families $\{f_\alpha : \alpha \in \mathcal{A}\}$ and $\left\{\widetilde{f}_\alpha : \alpha \in \mathcal{A}\right\}$ constitute a biorthogonal pair if and only if they are Bessel sequences and satisfy

$$P(f,g) = \sum_{\alpha \in \mathcal{A}} \langle f, f_\alpha \rangle \langle \widetilde{f}_\alpha, g \rangle = \langle f, g \rangle$$

for all f, g in a dense subset $\mathcal{D}$ of $\mathbb{H}$, where $P(f,g)$ is a bi-linear functional on $\mathbb{H} \times \mathbb{H}$.

Using polarization identity along with the Definition 2.1 implies that

$$P(f,f) = \|f\|^2, \quad \text{for all } f \in L^2(\mathbb{R}^d), \tag{2.5}$$

which is equivalent to

$$P(f,g) = \langle f, g \rangle, \quad \text{for } f \in \mathcal{D}.$$

We recall the notion of higher dimensional multiresolution analysis associated with multiplicity L and orthogonal multiwavelets of $L^2(\mathbb{R}^d)$ (see [1]).

Definition 2.3. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ is called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^d)$ of multiplicity L associated with the dilation matrix A if the following conditions are satisfied:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(ii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R}^d)$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$;

(iii) $f \in V_j$ if and only if $f(A\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(iv) there exist L-functions $\Phi = \{\varphi_1, \varphi_2, \ldots, \varphi_L\} \in V_0$, such that the system of functions $\{\varphi_\ell(x-k)\}_{\ell=1,k\in\mathbb{Z}^d}^{L}$, forms an orthonormal basis for subspace V_0.

The L-functions whose existence is asserted in (iv) are called *scaling functions* of the given MRA. Given a multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$, we define another sequence $\{W_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R}^d)$ by $W_j = V_{j+1} \ominus V_j$, $j \in \mathbb{Z}$. These subspaces inherit the scaling property of $\{V_j\}$, namely

$$f \in W_j \quad \text{if and only if} \quad f(A\cdot) \in W_{j+1}. \tag{2.6}$$

Further, they are mutually orthogonal, and we have the following orthogonal decompositions:

$$L^2(\mathbb{R}^d) = \bigoplus_{j\in\mathbb{Z}} W_j = V_0 \oplus \Big(\bigoplus_{j\geq 0} W_j\Big). \tag{2.7}$$

A set of functions $\{\psi_\ell^r : 1 \leq \ell \leq L, 1 \leq r \leq a-1\}$ in $L^2(\mathbb{R}^d)$ is said to be a set of *basic multiwavelets* associated with the MRA of multiplicity L if the collection

$$\left\{\psi_\ell^r(.-k) : 1 \leq r \leq a-1, 1 \leq \ell \leq L,\ k \in \mathbb{Z}^d\right\}$$

forms an orthonormal basis for W_0. Now, in view of (2.6) and (2.7), it is clear that if $\{\psi_\ell^r : 1 \le \ell \le L, 1 \le r \le a-1\}$ is a basic set of multiwavelets, then

$$\Big\{a^{j/2}\psi_\ell^r(A^j.-k) : j \in \mathbb{Z},\, k \in \mathbb{Z}^d, 1 \le \ell \le L, 1 \le r \le a-1\Big\}$$

forms an orthonormal basis for $L^2\big(\mathbb{R}^d\big)$ (see [1]).

For any $n \in \mathbb{Z}^+$, we define the *basic multiwavelet packets* ω_ℓ^n; $1 \le \ell \le L$ recursively as follows. We denote $\omega_\ell^0 = \varphi_\ell$, $1 \le \ell \le L$, the scaling functions and $\omega_\ell^r = \psi_\ell^r, r \in \mathbb{Z}^+, 1 \le \ell \le L$ as the possible candidates for basic multiwavelets. Then, define

$$\omega_\ell^{s+ar}(x) = \sum_{j=1}^{L}\sum_{k\in\mathbb{Z}^d} h_{\ell jk}^s\, a^{1/2}\, \omega_\ell^r(Ax-k), \quad 0 \le s \le a-1, 1 \le \ell \le L \tag{2.8}$$

where $\Big(h_{\ell jk}^s\Big)$ is a unitary matrix (see [1]).

Taking Fourier transform on both sides of (2.8), we obtain

$$\big(\omega_\ell^{s+ar}\big)^\wedge(\xi) = \sum_{j=1}^{L} h_{\ell j}^s(B^{-1}\xi)\,\big(\omega_\ell^r\big)^\wedge(B^{-1}\xi). \tag{2.9}$$

Note that (2.8) defines ω_ℓ^n for every non-negative integer n and every ℓ such that $1 \le \ell \le L$. The set of functions $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ as defined above are called the *basic multiwavelet packets* corresponding to the MRA $\{V_j\}_{j\in\mathbb{Z}}$ of $L^2(\mathbb{R}^d)$ of multiplicity L associated with matrix A.

Definition 2.4. Let $\{\omega_\ell^n : n \in \mathbb{Z}^+, 1 \le \ell \le L\}$ be the basic multiwavelet packets. The collection

$$\mathcal{P} = \Big\{|detA|^{j/2}\omega_\ell^n(A.-\,k) : 1 \le \ell \le L, j \in \mathbb{Z}, k \in \mathbb{Z}^d\Big\}$$

is called the *general multiwavelet packets* associated with MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2\big(\mathbb{R}^d\big)$ of multiplicity L over matrix dilation A.

Corresponding to some orthonormal scaling vector $\Phi = \omega_\ell^0$, the family of multiwavelet packets ω_ℓ^n defines a family of subspaces of $L^2(\mathbb{R}^d)$ as follows:

$$U_j^n = \overline{\operatorname{span}}\Big\{a^{j/2}\omega_\ell^n(A^jx-k) : k \in \mathbb{Z}^d, 1 \le \ell \le L\Big\}; \quad j \in \mathbb{Z}, n \in \mathbb{Z}^+. \tag{2.10}$$

Observe that

$$U_j^0 = V_j, \quad U_j^1 = W_j = \bigoplus_{r=1}^{a-1} U_j^r, \quad j \in \mathbb{Z}$$

so that the orthogonal decomposition $V_{j+1} = V_j \oplus W_j$, can be written as

$$U^0_{j+1} = \bigoplus_{r=0}^{a-1} U^r_j. \tag{2.11}$$

A generalization of this result for other values of $n = 1, 2, \ldots$ can be written as

$$U^n_{j+1} = \bigoplus_{r=0}^{a-1} U^{an+r}_j, \quad j \in \mathbb{Z}. \tag{2.12}$$

The following proposition is proved in [1].

Proposition 2.5. If $j \geq 0$, then

$$W_j = \bigoplus_{r=0}^{a-1} U^r_j = \bigoplus_{r=a}^{a^2-1} U^r_{j-1} = \cdots = \bigoplus_{r=a^t}^{a^{t+1}-1} U^r_{j-t} = \bigoplus_{r=a^j}^{a^{j+1}-1} U^r_0$$

where U^n_j is defined in (2.10). Using this decomposition, we get the multiwavelet packets decomposition of subspaces W_j, $j \geq 0$.

Similar to the orthogonal multiwavelet packets, the *biorthogonal multiwavelet packets* associated with the biorthogonal scaling vector $\tilde{\Phi}$ are given by

$$\tilde{\omega}^{s+ar}_\ell(x) = \sum_{j=1}^{L} \sum_{k \in \mathbb{Z}^d} \tilde{h}^s_{\ell jk}\, a^{1/2}\, \tilde{\omega}^r_\ell(Ax - k), \quad 0 \leq s \leq a-1, 1 \leq \ell \leq L. \tag{2.13}$$

Implementation of Fourier transform of (2.13) yields

$$\left(\tilde{\omega}^{s+ar}_\ell\right)^\wedge(\xi) = \sum_{j=1}^{L} \tilde{h}^s_{\ell j}(B^{-1}\xi)\left(\tilde{\omega}^r_\ell\right)^\wedge(B^{-1}\xi). \tag{2.14}$$

Let ω^n_ℓ be general multiwavelet packets associated with the dilation matrix A. Then, we consider the system

$$\mathcal{F}(A, B) = \left\{\omega^n_{\ell,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, \ldots, L, a^j \leq n < a^{j+1}\right\} \tag{2.15}$$

where $\omega^n_{\ell,j,k}(x) = |detA|^{j/2}\, \omega^n_\ell\big(A^j x - Bk\big)$.

Similarly, for the biorthogonal multiwavelet packets, we have

$$\tilde{\mathcal{F}}(A, B) = \left\{\tilde{\omega}^n_{\ell,j,k} : j \in \mathbb{Z}, k \in \mathbb{Z}^d, \ell = 1, \ldots, L, a^j \leq n < a^{j+1}\right\} \tag{2.16}$$

where $\tilde{\omega}^n_{\ell,j,k}(x) = |detA|^{j/2}\, \tilde{\omega}^n_\ell\big(A^j x - Bk\big)$.

The bi-linear functional $P(f,g)$ associated to the multiwavelet packets systems $\mathcal{F}(A,B)$ and $\tilde{\mathcal{F}}(A,B)$ is given by

$$P(f,g)=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^d}\left\langle f,\omega^n_{\ell,j,k}\right\rangle\left\langle\tilde{\omega}^n_{\ell,j,k},g\right\rangle. \tag{2.16}$$

We will also consider the set $\mathcal{D}$ as a dense subset of $L^2\left(\mathbb{R}^d\right)$ defined by

$$\mathcal{D}=\left\{f\in L^2\big(\mathbb{R}^d\big):\hat{f}\in L^\infty(\mathbb{R}^d),\ \hat{f}\text{ has compact support in }\mathbb{R}^d\setminus\{0\}\right\}.$$

3. CHARACTERIZATION OF BIORTHOGONAL MULTIWAVELET PACKETS

In this section, we prove our main results concerning the characterization of biorthogonal multiwavelet packets associated with arbitrary matrix dilations by means of the Fourier transform.

Theorem 3.1. *Suppose* $\{\omega^n_\ell:n\in\mathbb{Z}^+,\ell=1,\ldots,L\}$ *and* $\{\tilde{\omega}^n_\ell:n\in\mathbb{Z}^+,\ell=1,\ldots,L\}$ *are the basic multiwavelet packets associated with a pair of biorthogonal scaling functions* Φ *and* $\tilde{\Phi}$ *such that the following functions are locally integrable:*

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\omega}^n_\ell\big(A^{*j}\xi\big)\right|^2,\quad \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\left|\hat{\tilde{\omega}}^n_\ell\big(A^{*j}\xi\big)\right|^2. \tag{3.1}$$

Then, the bi-linear functional $P(f,g)$ *converges absolutely for all* $f,g\in\mathcal{D}$. *Moreover, the multiwavelet packets* ω^n_ℓ *and* $\tilde{\omega}^n_\ell$ *satisfy:*

$$\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\tilde{\omega}}^n_\ell\big(A^{*j}\xi\big)\,\overline{\hat{\omega}^n_\ell\big(A^{*j}(\xi+A^{*-j}m)\big)}=\delta_{\alpha,0}, \tag{3.2}$$

for a.e. $\xi\in\mathbb{R}^d$ *and for all* $\alpha\in\Lambda(A,B)$, *if and only if* $P(f,g)=\langle f,g\rangle$, *for all* $f,g\in\mathcal{D}$.

Proof. First of all we prove that $P(f,g)$ is absolutely convergent. For this, fix $j\in\mathbb{Z}$ and let

$$G_j=\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d}\langle f,\omega^n_{\ell,j,k}\rangle\,\langle\tilde{\omega}^n_{\ell,j,k},f\rangle. \tag{3.3}$$

Implementation of Parseval's identity gives

$$\left\langle f,\omega^n_{\ell,j,k}\right\rangle=|detA|^{j/2}\int_{\mathbb{R}^d}\hat{f}\big(A^{*j}\zeta\big)\,\overline{\hat{\omega}^n_\ell(\zeta)}\,e^{2\pi iB(k)\cdot\zeta}\,d\zeta,$$

and

$$\langle \tilde{\omega}^n_{\ell,j,k}, f\rangle = |detA|^{j/2} \int_{\mathbb{R}^d} \overline{\hat{f}\big(A^{*j}\zeta\big)}\, \hat{\tilde{\omega}}^n_\ell(\xi)\, e^{-2\pi i B(k)\cdot\xi}\, d\xi.$$

Let

$$F^n_{\ell,j}(\xi) = \sum_{s\in\mathbb{Z}^d} \hat{f}\big(A^{*j}(\xi + B^{*-1}s)\big)\, \overline{\hat{\omega}^n_\ell\big(\xi + B^{*-1}s\big)}.$$

Then, by virtue of Fourier inversion formula for the function $F^n_{\ell,j} \circ B^{*-1}$, we obtain

$$\begin{aligned} F^n_{\ell,j}(\xi) &= \sum_{k\in\mathbb{Z}^d} \left\{ |detB| \int_{B^{*-1}([0,1]^d)} F^n_{\ell,j}(\zeta)\, e^{2\pi i Bk\cdot\zeta}\, d\zeta \right\} e^{-2\pi i Bk\cdot\xi} \\ &= |detB| \sum_{k\in\mathbb{Z}^d} \left\{ \int_{\mathbb{R}^d} \hat{f}\big(A^{*j}\zeta\big)\, \overline{\hat{\omega}^n_\ell(\zeta)}\, e^{2\pi i Bk\cdot\zeta}\, d\zeta \right\} e^{-2\pi i Bk\cdot\xi}. \end{aligned}$$

Thus, G_j as defined in (3.3) can be written as

$$\begin{aligned} G_j &= \frac{|detA|^j}{|detB|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \overline{\hat{f}\big(A^{*j}\xi\big)}\, \hat{\tilde{\omega}}^n_\ell(\xi)\, F^n_{\ell,j}(\xi)\, d\xi \\ &= \frac{1}{|detB|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\, \hat{\tilde{\omega}}^n_\ell\big(A^{*-j}\xi\big)\, F^n_{\ell,j}\big(A^{*-j}\xi\big)\, d\xi \\ &= \frac{1}{|detB|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\, \hat{\tilde{\omega}}^n_\ell\big(A^{*-j}\xi\big) \\ &\qquad \left\{ \textstyle\sum_{s\in\mathbb{Z}^d} \hat{f}\big(\xi + A^{*j}B^{*-1}s\big)\, \overline{\hat{\omega}^n_\ell\big(A^{*-j}\xi + B^{*-1}s\big)} \right\} d\xi. \end{aligned}$$

Now, in order to show that the convergence of $\sum_{j\in\mathbb{Z}} G_j$ is absolute and unconditional, it is sufficient to prove that the following two series are absolutely convergent:

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\, \hat{\tilde{\omega}}^n_\ell\big(A^{*-j}\xi\big)\, \hat{f}(\xi)\, \overline{\hat{\omega}^n_\ell\big(A^{*-j}\xi\big)}\, d\xi,$$

and

$$\sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}^d} \overline{\hat{f}(\xi)}\, \hat{\tilde{\omega}}^n_\ell\big(A^{*-j}\xi\big) \left\{ \sum_{s\in\mathbb{Z}^d\setminus\{0\}} \hat{f}\big(\xi + A^{*j}B^{*-1}s\big)\, \overline{\hat{\omega}^n_\ell\big(A^{*-j}\xi + B^{*-1}s\big)} \right\} d\xi.$$

From our assumptions on the basic multiwavelet packets ω_ℓ^n and $\tilde{\omega}_\ell^n$, it is clear that the first of these series converges absolutely. Moreover, we have

$$2\left|\hat{\tilde{\omega}}_\ell^n\left(A^{*-j}\xi\right)\hat{\omega}_\ell^n\left(A^{*-j}\xi+B^{*-1}s\right)\right| \le \left|\hat{\tilde{\omega}}_\ell^n\left(A^{*-j}\xi\right)\right|^2+\left|\hat{\omega}_\ell^n\left(A^{*-j}\xi+B^{*-1}s\right)\right|^2.$$

Further, it is easy to verify that the convergence of the second series follows from the convergence of:

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^d\setminus\{0\}}\int_{\mathbb{R}^d}\left|\hat{f}(\xi)\right|\left|\hat{f}\left(\xi+A^{*j}B^{*-1}s\right)\right|\left|\hat{\tilde{\omega}}_\ell^n\left(A^{*-j}\xi\right)\right|^2 d\xi$$

$$=\int_{\mathbb{R}^d}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\left\{\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^d\setminus\{0\}}|detA|^j\left|\hat{f}\left(A^{*j}\xi\right)\right|\left|\hat{f}\left(A^{*j}\xi+A^{*j}B^{*-1}s\right)\right|\right\}\left|\hat{\tilde{\omega}}_\ell^n(\xi)\right|^2 d\xi,$$

and from the convergence of a similar series, with $\tilde{\omega}_\ell^n$ replaced by ω_ℓ^n. But as $s\neq 0$, therefore there exists $J\in\mathbb{Z}$ such that

$$\hat{f}\left(A^{*j}\xi\right)\hat{f}\left(A^{*j}\xi+A^{*j}B^{*-1}s\right)=0,\quad \text{for all } j\ge J.$$

On the other hand, for each fixed $j\in\mathbb{Z}$, and $\xi\in\mathbb{R}^d$, the number of $s\in\mathbb{Z}^d$, for which the above product is nonzero, is less than or equal to $C|detA|^{-j}$ for some constant C. Thus, we have

$$\sum_{j\in\mathbb{Z}}\sum_{s\in\mathbb{Z}^d\setminus\{0\}}|detA|^j\left|\hat{f}\left(A^{*j}\xi\right)\right|\left|\hat{f}\left(A^{*j}\xi+A^{*j}B^{*-1}s\right)\right|\le C\sum_{j\le J}\left\|\hat{f}\right\|_\infty^2\chi_F\left(A^{*j}\xi\right),$$

where F is compact in $\mathbb{R}^d\setminus\{0\}$. Observe that if $b'<|A^{*j}\xi|<b$, there exists $K>0$, which does not depend on ξ, such that the number of j for which this is nonzero is less than K for every ξ. Hence, the above sum can be estimated from above by $CK\|\hat{f}\|_\infty^2$ and it proves the convergence of second sum.

Hence, we can rearrange the series for $P(f,g)$ to obtain

$$\begin{aligned}P(f,f) &= \sum_{\alpha\in\Lambda(A,B)}\int_{\mathbb{R}^d}\overline{\hat{f}(\xi)}\hat{f}(\xi+\alpha)\\ &\quad\times\left\{\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)}\right\}d\xi.\end{aligned}$$

Therefore, it is enough to show that if $P(f,g)=\langle f,g\rangle$ for all $f,g\in\mathcal{D}$, then the second condition follows. For this, we write

$$P(f,g)=M(f,g)+R(f,g),$$

with

$$M(f,g)=\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\hat{f}(\xi)\left\{\sum_{j\in\mathbb{Z}}\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\overline{\hat{\omega}_\ell^n(A^{*j}\xi)}\right\}d\xi,$$

and

$R(f,g)=$

$$\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{\alpha\in\Lambda(A,B)\setminus\{0\}}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\hat{f}(\xi+\alpha)\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)}\,d\xi.$$

Now, let us fix, $\xi_0\in\mathbb{R}^d\setminus\{0\}, k\in\mathbb{Z}$, and consider $f=g=f_1$, where f_1 is defined by

$$\hat{f}_1(\xi)=\frac{1}{|H_k|^{1/2}}\chi_{H_k}(\xi).$$

Then,

$$M(f_1,f_1)=\frac{1}{|detB||H_k|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{H_k}\sum_{j\in\mathbb{Z}}\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\overline{\hat{\omega}_\ell^n\left(A^{*j}\xi\right)}d\xi,$$

and

$$\begin{aligned}\left|R(f_1,f_1)\right| &\le \frac{1}{|detB||H_k|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{\substack{\alpha\in\Lambda(A,B)\\ \alpha\neq0}}\sum_{(j,m)\in I_{A,B}(\alpha)}\\ &\qquad\times\int_{H_k\cap(\alpha+H_k)}\left|\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)\right|d\xi\\ &\le \frac{1}{|detB||H_k|}\left\{\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{\substack{\alpha\in\Lambda(A,B)\\ \alpha\neq0}}\sum_{(j,m)\in I_{A,B}(\alpha)}\int_{H_k\cap(\alpha+H_k)}\left|\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2d\xi\right\}^{1/2}\\ &\qquad\times\left\{\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{\substack{\alpha\in\Lambda(A,B)\\ \alpha\neq0}}\sum_{(j,m)\in I_{A,B}(\alpha)}\int_{H_k\cap(\alpha+H_k)}\left|\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)\right|^2d\xi\right\}^{1/2}.\end{aligned}$$

To estimate $R(f_1,f_1)$, we observe that if $\alpha\notin H_{k+\eta}$, then $H_k\cap(\alpha+H_k)=\emptyset$. Therefore, we may assume that $\alpha\in H_{k+\eta}$ and $\alpha\neq0$. Also, if $(j,m)\in I_{A,B}(\alpha)$, then

$$m\in A^{*j}(H_{k+\eta})\cap B^{*-1}(\mathbb{Z}^d)\quad\text{and}\quad j\ge -k+c_1,$$

where c_1 is the largest integer such that $H_{k+\eta} \cap B^{*-1}(\mathbb{Z}^d) = \{0\}$. Therefore, under these observations, we have

$$|R(f_1, f_1)| \leq \frac{1}{|detB||H_k|} \left\{ \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \geq -k+c_1} \sum_{\substack{m \neq 0 \\ m \in A^{*j}(H_{k+\eta}) \cap B^{*-1}(\mathbb{Z}^d)}} \int_{\xi_0+H_k} \left| \hat{\tilde{\omega}}_\ell^n \left(A^{*j}\xi\right)\right|^2 d\xi \right\}^{1/2}$$

$$\times \left\{ \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \geq -k+c_1} \sum_{\substack{m \neq 0 \\ m \in A^{*j}(H_{k+\eta}) \cap B^{*-1}(\mathbb{Z}^d)}} \int_{\xi_0+H_k} \left| \hat{\omega}_\ell^n \left(A^{*j}\xi\right)\right|^2 d\xi \right\}^{1/2} .$$

Now, in order to estimate the first factor in the above product, we observe that

$$\begin{aligned} & \frac{1}{|H_k|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \geq -k+c_1} \sum_{\substack{m \neq 0 \\ m \in A^{*j}(H_{k+\eta}) \cap B^{*-1}(\mathbb{Z}^d)}} \int_{\xi_0+H_k} \left| \hat{\tilde{\omega}}_\ell^n \left(A^{*j}\xi\right)\right|^2 d\xi \\ \leq \; & |detA|^{-k} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \geq -k+c_1} C|detA|^{j+k}|detA|^{-j} \int_{A^{*j}(\xi_0+H_k)} \left| \hat{\tilde{\omega}}_\ell^n (\xi)\right|^2 d\xi \\ \leq \; & C \sum_{j \geq -k+c_1} \int_{A^{*j}(\xi_0+H_k)} \left| \hat{\tilde{\omega}}_\ell^n (\xi)\right|^2 d\xi. \end{aligned}$$

Here, we have used the fact that the number of points of the lattice $B^{*-1}(\mathbb{Z}^d)$, different from the origin and contained in the set $A^{*j}(H_{k+\eta}) = H_{j+k+\eta}$, is smaller than a constant multiple of the volume of this set.

Similar estimate holds for the second factor. Since the sets $A^{*j}(\xi_0+H_k)$, $j \in \mathbb{Z}$, are pairwise disjoint for sufficiently large $|k|$, so we may conclude that $R(f_1, f_1) \to 0$, as $k \to -\infty$ by the Lebesgue Dominated Convergence Theorem. Therefore, we have

$$\begin{aligned} 1 \; &= \lim_{k \to -\infty} \frac{1}{|detB|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \frac{1}{|H_k|} \int_{\xi_0+H_k} \sum_{j \in \mathbb{Z}} \hat{\tilde{\omega}}_\ell^n \left(A^{*j}\xi\right) \overline{\hat{\omega}_\ell^n \left(A^{*j}\xi\right)} \, d\xi \\ &= \frac{1}{|detB|} \sum_{n=a^j}^{a^{j+1}-1} \sum_{\ell=1}^{L} \sum_{j \in \mathbb{Z}} \hat{\tilde{\omega}}_\ell^n \left(A^{*j}\xi_0\right) \overline{\hat{\omega}_\ell^n (A^{*j}\xi_0)}, \end{aligned}$$

which proves our claim for $\alpha = 0$. This also shows that $M(f, g) = \langle f, g \rangle$, and thus $R(f, g) = 0$, for $f, g \in \mathcal{D}$.

Now, we choose $\alpha_0 \in \Lambda(A, B) \setminus \{0\}$, and write

$$R(f, g) = R_1(f, g) + R_2(f, g),$$

where

$$R_1(f,g)=\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\hat{f}(\xi+\alpha_0)\sum_{(j,m)\in I_{A,B}(\alpha_0)}\hat{\tilde{\omega}}_\ell^n(A^{*j}\xi)\,\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha_0)\right)}d\xi,$$

and

$$R_2(f,g)=\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{\substack{\alpha\in\Lambda(A,B)\\ \alpha\neq 0,\alpha_0}}\int_{\mathbb{R}^d}\overline{\hat{g}(\xi)}\hat{f}(\xi+\alpha)\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\tilde{\omega}}_\ell^n(A^{*j}\xi)\,\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)}d\xi.$$

Let $\xi_0\in\mathbb{R}^d\setminus\{0\}$ be a Lebesgue point of differentiability for the functions

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=J}^{\infty}\left|\hat{\omega}_\ell^n(A^{*j}\xi)\right|^2\quad\text{and}\quad\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=J}^{\infty}\left|\hat{\tilde{\omega}}_\ell^n(A^{*j}\xi)\right|^2,\quad J\in\mathbb{Z}.$$

Then, for given $k\in\mathbb{Z}$, we define f_2 and g_2 as follows:

$$\hat{f}_2(\xi+\alpha_0)=\frac{1}{|H_k|^{1/2}}\chi_{\xi_0+H_k}(\xi),\quad \hat{g}_2(\xi+\alpha_0)=\frac{1}{|H_k|^{1/2}}\chi_{\xi_0+H_k}(\xi).$$

Using equation (2.3), we obtain

$$\lim_{k\to-\infty}R_1(f_2,g_2)=\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha_0)}\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi_0\right)\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi_0+\alpha_0)\right)}.$$

To estimate $R_2(f_2,g_2)$, we observe that $\hat{g}_2(\xi)\hat{f}_2(\xi+\alpha)\not\equiv 0$ is only possible when $\alpha\in\alpha_0+H_{k+\eta}$. Since $\alpha=(A^*)^{-j}m\in\Lambda(A,B)\setminus\{0,\alpha_0\}$, there exists $J_0\in\mathbb{Z}$ such that $(A^*)^{-j}m\notin\alpha_0+H_\eta$ for any $m\in B^{*-1}(\mathbb{Z}^d)\setminus\{0\}$ and $j\leq J_0$. Thus, $R_2(f_2,g_2)$ can be re-written as

$$\begin{aligned}
&R_2(f_2,g_2)\\
&=\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=J_1}^{\infty}\sum_{\substack{m\neq 0,\\ A^{*-j}m-\alpha_0\in H_{k+\eta}}}\int_{\mathbb{R}^d}\overline{\hat{g}_2(\xi)}\hat{f}_2(\xi+\alpha)\,\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\,\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)}d\xi\\
&\quad+\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j=J_0}^{J_1}\sum_{\substack{m\neq 0,\\ A^{*-j}m-\alpha_0\in H_{k+\eta}}}\int_{\mathbb{R}^d}\overline{\hat{g}_2(\xi)}\hat{f}_2(\xi+\alpha)\,\hat{\tilde{\omega}}_\ell^n\left(A^{*j}\xi\right)\overline{\hat{\omega}_\ell^n\left(A^{*j}(\xi+\alpha)\right)}d\xi\\
&=R_{2,1}(f_2,g_2)+R_{2,2}(f_2,g_2),
\end{aligned}$$

where $J_1\in\mathbb{Z}$. Since $R_{2,2}(f_2,g_2)$ is now a finite sum, and the number of m's satisfying the condition $A^{*-j}m-\alpha_0\in H_{k+\eta}\subset H_\eta$ may now be estimated independently of $k\leq 0$, we have $\lim_{k\to-\infty}R_{2,2}(f_2,g_2)=0$ by Lebesgue Dominated Convergence Theorem. To estimate $R_{2,1}(f_2,g_2)$, we will show that for every $\varepsilon>0$, there exists

$J_1 \in \mathbb{Z}$ such that $|R_{2,1}(f_2, g_2)| \leq \varepsilon$ for sufficiently large $|k|$. In fact, as in the case of $R(f_1, g_1)$, we have

$$\begin{aligned} R_{2,1}(f_2,g_2) \quad \leq \quad & \frac{1}{|detB||H_k|}\left\{\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\sum_{\substack{m\neq 0,\\ A^{*-j}m-\alpha_0\in H_{k+\eta}}}\int_{\xi_0+H_k}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi\right\}^{1/2} \\ & \times\left\{\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\sum_{\substack{m\neq 0,\\ A^{*-j}m-\alpha_0\in H_{k+\eta}}}\int_{\xi_0+H_k}\left|\hat{\omega}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi\right\}^{1/2}. \end{aligned}$$

Therefore, it is enough to estimate just one of these factors, namely:

$$\begin{aligned} & \frac{1}{|H_k|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\sum_{\substack{m\neq 0,\\ A^{*-j}m-\alpha_0\in H_{k+\eta}}}\int_{\xi_0+H_k}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi \\ \leq \quad & \frac{1}{|H_k|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\left(1+C|detA|^{k+j+\eta}\right)\int_{\xi_0+H_k}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi \\ = \quad & \sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\frac{1}{|H_k|}\int_{\xi_0+H_k}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi+\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\int_{(A^*)^j(\xi_0+H_k)}\left|\hat{\hat{\omega}}_\ell^n(\xi)\right|^2 d\xi. \end{aligned}$$

Here, we have used the fact that the number of points of the lattice $B^{*-1}(\mathbb{Z}^d)$ that are contained in the set $A^{*j}(\alpha_0 + H_{k+\eta}) = (A^*)^j\alpha_0 + H_{j+k+\eta}$, is smaller than one plus a constant multiple of the volume of this set.

Let $J_1 \in \mathbb{Z}$ be such that

$$\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi_0\right)\right|^2 < \varepsilon/2.$$

Then, by our choice of ξ_0 and equation (2.3), we have

$$\lim_{k\to-\infty}\sup\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\frac{1}{|H_k|}\int_{\xi_0+H_k}\left|\hat{\hat{\omega}}_\ell^n\left(A^{*j}\xi\right)\right|^2 d\xi < \varepsilon/2.$$

Therefore, by virtue of Lebesgue Dominated Convergence Theorem, we get

$$\lim_{k\to-\infty}\sup\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{j\geq J_1}\int_{A^{*j}(\xi_0+H_k)}\left|\hat{\hat{\omega}}_\ell^n(\xi)\right|^2 d\xi = 0.$$

Since the sets $A^{*j}(\alpha_0 + H_{k+\eta}),\ j \in \mathbb{Z}$, are pairwise disjoint for sufficiently large $|k|$, therefore, for every $\varepsilon > 0$, there exist J_1 such that

$$\lim_{k\to-\infty}\sup\left|R_{2,1}(f_2, g_2)\right| \leq \varepsilon.$$

Combining these observations with the fact that ε is arbitrary, we obtain

$$\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha_0)}\hat{\tilde{\omega}}_\ell^n\big(A^{*j}\xi\big)\,\overline{\hat{\omega}_\ell^n\big(A^{*j}(\xi+\alpha_0)\big)}=0,\text{ for all }\alpha_0\in\Lambda(A,B)\setminus\{0\}.$$

An immediate consequence of the above theorem is the following:

Corollary 3.2. *Let $\{\omega_\ell^n : n\in\mathbb{Z}^+, \ell=1,\dots,L\}$ be the basic multiwavelet packets associated with the scaling vector Φ. Then*

$$\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\omega}_\ell^n\big(A^{*j}\xi\big)\,\overline{\hat{\omega}_\ell^n\big(A^{*j}(\xi+A^{*-j}m)\big)}=\delta_{\alpha,0}, \tag{3.4}$$

for a.e. $\xi\in\mathbb{R}^d$ and for all $\alpha\in\Lambda(A,B)$, if and only if the system $\mathcal{F}(A,B)$ given by (2.15) is a normalized tight frame for $L^2\big(\mathbb{R}^d\big)$.

Theorem 3.3. *If $\{\omega_\ell^n : n\in\mathbb{Z}^+, \ell=1,\dots,L\}$ and $\{\tilde{\omega}_\ell^n : n\in\mathbb{Z}^+, \ell=1,\dots,L\}$ are Bessel families and have the property that the functions in (3.1) are locally integrable. Then, they are biorthogonal if and only if*

$$\frac{1}{|detB|}\sum_{n=a^j}^{a^{j+1}-1}\sum_{\ell=1}^{L}\sum_{(j,m)\in I_{A,B}(\alpha)}\hat{\tilde{\omega}}_\ell^n\big(A^{*j}\xi\big)\,\overline{\hat{\omega}_\ell^n\big(A^{*j}(\xi+A^{*-j}m)\big)}=\delta_{\alpha,0}, \tag{3.5}$$

for a.e. $\xi\in\mathbb{R}^d$ and for all $\alpha\in\Lambda(A,B)$. Moreover, if $\omega_\ell^n=\tilde{\omega}_\ell^n$ and $\|\omega_\ell^n\|_2=\|\tilde{\omega}_\ell^n\|_2=1$ for $n\in\mathbb{Z}^+, \ell=1,\dots,L$. Then, the system $\mathcal{F}(A,B)$ forms an orthonormal basis for $L^2\big(\mathbb{R}^d\big)$.

Proof. The proof of this theorem follows from (2.2) and Theorem 3.1. □

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