Int. J. Anal. Appl. (2025), 23:122

International Journal of Analysis and Applications

Analysis of Weighted Sobolev Space Solutions for Nonlinear Capillarity Problem With Dirichlet Boundary Conditions

Lhoucine Hmidouch[1,*], **Manal Badgaish**[2,*]

[1]*Mathematics Department, Faculty of Science, Chouaib Doukkali University, BP. 20, El Jadida 24000, Morocco*

[2]*Mathematics Department, Faculty of Science, Umm Al-Qura University, Makkah, KSA*

[*]*Corresponding authors:* hmidouchlhoucine@gmail.com, mobadgaish@uqu.edu.sa

Abstract. This work establishes the existence of weak solutions for p-Laplacian-like equations in weighted Sobolev spaces under Dirichlet boundary conditions, assuming the data is in the weighted Lebesgue space.

1. Introduction

In this paper we study the existence and uniqness of solutions the following nonlinear problem arising from capillarity phenomena

$$\begin{cases} \Delta[\omega_2|\Delta u|^{p-2}\Delta u] - \operatorname{div}\omega_1\left(|\nabla u|^{p-2} + \dfrac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\right)\nabla u + \omega_1|u|^{p-2}u = f \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, $(N \geq 2)$, $T > 0$, $p > 1$, ∇u is the gradient of u, ω_1 and ω_2 are weight functions (i.e., a locally integrable functions on $\mathbb{R}^N$, such that $0 < \omega_1(x) < \infty, 0 < \omega_2(x) < \infty$, *a.e.* $x \in \mathbb{R}^N$), satisfied suitable assumptions (see Section 2 for more details).

The rise in capillaries in a thin vertical tube is a significant physical phenomenon seen in several everyday contexts. A prevalent natural illustration of capillary action is the movement of water in soil or vegetation. To succinctly describe capillarity, consider the effects of two opposing forces: adhesion, which pulls or repels molecules from the liquid and the container, and cohesion, which is the attractive force between molecules in the liquid. The study of capillary phenomena has recently received more attention. Capillary phenomena are gaining popularity because of a variety of factors, including a fascination with natural phenomena such as drop, bubble, and wave

Received: Mar. 19, 2025.

2020 *Mathematics Subject Classification.* 35A15, 35J60, 35J65, 74G30.

Key words and phrases. p-Laplacian-like problem; weak solution; weighted Sobolev space.

https://doi.org/10.28924/2291-8639-23-2025-122
ISSN: 2291-8639

motion, as well as their importance in applied fields ranging from industrial, biomedical, and pharmaceutical to microfluidic systems.

Given the applicability of capillary flow in industrial contexts, the development of suitable mathematical models to describe its behavior is crucial. The known equation that models the kinetics of capillary rise was proposed by Washburn in 1921. The Washburn equation is a valuable simplification of liquid ascent in a capillary tube. Alongside theoretical modeling, numerous experiments have revealed that for specific fluids, their free boundary oscillates around the equilibrium rather than approaching it consistently. In 1985, Ni and Serrin launched their investigation of fundamental states for equations of the specified form [7, 8]

$$-div\frac{\nabla u}{1+|\nabla u|^2} = f(u) \qquad \text{in } \mathbb{R}^{\mathbb{N}}.$$

After that, an extensive research has been conducted to investigate capillary action. In [9], Rodrigues employed the Mountain Pass Lemma and the Fountain Theorem to examine the existence of nontrivial solutions to the subsequent problem.

$$\begin{cases} -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u + \dfrac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\right) = \lambda f(x,u) \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^{\mathbb{N}} (N \geq 2)$ is a bounded regular domain, λ is a positive parameter, and f is a Carathéodory function.

Moreover, Shokoo, Afrouzi and Heidarkhani addressed the existence of three weak solutions to the subsequent problem in [10]

$$\begin{cases} -\operatorname{div}\left(1 + \dfrac{|\nabla u|^{2p(x)-2}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)\nabla u + a(x)|u|^{p(x)-2}u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega \\ \dfrac{u}{\partial \nu} = 0\Big|_{\partial}\Omega, \end{cases} \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ with $N \geq 2$ be a bounded domain, a C^1 -class boundary,ν be the outer unit normal to $\partial\Omega)$, $\lambda > 0$, $\mu \geq 0$, and $a \in L^{\infty}(\Omega)$, $a \geq 0$ a.e. in Ω.

Note that the functions $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, and $p \in C(\overline{\Omega})$ satisfies the condition

$$N < p_- := \inf_{x\in\overline{\Omega}} p(x) \leq p_+ := \sup_{x\in\overline{\Omega}} p(x) < +\infty.$$

This paper demonstrates the existence and uniqueness of weak solutions to problems (1.1), utilizing the theory of weighted Sobolev spaces and the theory of monotone type operators in reflexive Banach spaces. The subsequent sections of this work are organized as follows. Section 2 presents definitions and essential properties of weighted Sobolev spaces. Furthermore, we reference several established lemmas to be utilized in the proof of the principal results. In Section 3, we demonstrate the existence and uniqueness of weak solutions for (1.1).

2. Preliminaries and Notations

This section provides notations, definitions, and outcomes that will be utilized in this work. Let Ω denote a smooth bounded domain in $\mathbb{R}^N$. A weight refers to a locally integrable function ω on $\mathbb{R}^N$ satisfying $0 < \omega < \infty$ for almost every $x \in \mathbb{R}^N$. We will designate by $L^p(\Omega, \omega)$ the collection of all measurable functions u on Ω with a finite norm.

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega \omega(x)\,|u|^p dx\right)^{\frac{1}{p}},\ 1 \leq p < \infty.$$

The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as the set of all functions $u \in L^p(\Omega)$.Having the derivatives $\nabla u \in L^p(\Omega, \omega)$ with a finite norm.

$$\|u\|_{W^{1,p}(\Omega,\omega)} := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega,\omega)^N}.$$

The space of all functions with continuous derivatives of arbitrary order and compact support in Ω is denoted by $C_0^\infty(\Omega)$, and the space $W_0^{1,p}(\Omega, \omega)$ denotes the closure $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega, \omega)$. In this study, we presuppose that ω meets the following criteria:

- (H_1) $\omega \in L^1_{loc}(\Omega)$, $\omega^{\frac{-1}{p-1}} \in L^1_{loc}(\Omega)$,
- (H_2) $\omega^{-s} \in L^1(\Omega)$ where $s \in \left(\frac{N}{p}, \infty\right) \cap \left[\frac{1}{p-1}, \infty\right)$.

For additional information regarding weighted Sobolev spaces, we direct the reader to [1,5,11].

Proposition 2.1. *[3] Assume that the hypothesis (H) holds, then for $s + 1 \leqslant ps < N(s + 1)$ we have the continuous embedding*

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow W_0^{1,p_1}(\Omega) \hookrightarrow L^q(\Omega), \tag{2.1}$$

where $p_1 = \frac{ps}{1+s}$, $1 \leq q = \frac{Np_1}{N-p_1} = \frac{Nps}{N(s+1)-ps}$, and for $ps \geqslant N(s+1)$ the embedding (2.1) holds with arbitrary $1 \leq q < \infty$. Moreover, the compact embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^r(\Omega)$$

holds provided $1 \leq r < q$.

Proposition 2.2. *[3] (**Hardy-type inequality**) There exist a weight function ω on Ω and a parameter $q, 1 < q < \infty$ such that the inequality*

$$\left(\int_\Omega \omega |u(x)|^q dx\right)^{\frac{1}{q}} \leqslant C\left(\int_\Omega \omega |\nabla u|^p dx\right)^{\frac{1}{p}} \tag{2.2}$$

holds for every $u \in W_0^{1,p}(\Omega, \omega)$ with a constant $C > 0$ independent of u and, moreover the embedding

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow L^q(\Omega, \omega)$$

determined by the inequality (2.2) is compact.

Theorem 2.1. *[6] Let Y be a reflexive real Banach space and $A : Y \to Y'$ be a bounded operator, hemicontinuous, coercive and monotone on space Y. Then the equation $Au = v$ has at least one solution $u \in Y$ for each $v \in Y'$.*

Lemma 2.1. *[2] Let $1 < p < \infty$. There exist two positive constants α_p and β_p such that for every $(\xi, \eta \in \mathbb{R}^N)$ with $N \geq 1$,*

$$\alpha_p(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \leq \langle |\xi|^{p-2}\xi - |\eta|^{p-2}\eta, \xi - \eta\rangle \leq \beta_p(|\xi| + |\eta|)^{p-2}|\xi - \eta|.$$

Lemma 2.2. *For $\xi, \eta \in \mathbb{R}^N$ and $1 < p < \infty$, we have*

$$\left(\frac{|\xi|^{p-2}}{\sqrt{1+|\xi|^p}}\xi - \frac{|\eta|^{p-2}}{\sqrt{1+|\eta|^p}}\eta\right)\cdot(\xi - \eta) \geq 0.$$

Proof. Consider the function $G : \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ defined by

$$G(\xi,\eta) = \left(\frac{|\xi|^{2p-2}}{\sqrt{1+|\xi|^{2p}}}\xi - \frac{|\eta|^{2p-2}}{\sqrt{1+|\eta|^{2p}}}\eta\right)\cdot(\xi - \eta).$$

By Cauchy-Schwarz inequality, we can write

$$\begin{aligned}
G(\xi,\eta) &= \frac{|\xi|^{2p}}{\sqrt{1+|\xi|^{2p}}} + \frac{|\eta|^{2p}}{\sqrt{1+|\eta|^{2p}}} - \Big(\frac{|\xi|^{2p-2}}{\sqrt{1+|\xi|^{2p}}} + \frac{|\eta|^{2p-2}}{\sqrt{1+|\eta|^{2p}}}\Big)\xi\cdot\eta \\
&\geq \frac{|\xi|^{2p}}{\sqrt{1+|\xi|^{2p}}} + \frac{|\eta|^{2p}}{\sqrt{1+|\eta|^{2p}}} - \Big(\frac{|\xi|^{2p-2}}{\sqrt{1+|\xi|^{2p}}} + \frac{|\eta|^{2p-2}}{\sqrt{1+|\eta|^{2p}}}\Big)|\xi||\eta| \\
&\geq \frac{|\xi|^{2p}}{\sqrt{1+|\xi|^{2p}}} + \frac{|\eta|^{2p}}{\sqrt{1+|\eta|^{2p}}} - \Big(\frac{|\xi|^{2p-1}}{\sqrt{1+|\xi|^{2p}}}|\eta| + \frac{|\eta|^{2p-1}}{\sqrt{1+|\eta|^{2p}}}|\xi|\Big) \\
&\geq \left(\frac{|\xi|^{2p-1}}{\sqrt{1+|\xi|^{2p}}} - \frac{|\eta|^{2p-1}}{\sqrt{1+|\eta|^{2p}}}\right)(|\xi| - |\eta|).
\end{aligned}$$

Since the function $\lambda \mapsto \frac{\lambda^{2p-1}}{\sqrt{1+\lambda^{2p}}}$ is nondecreasing for all λ in $\mathbb{R}^+$, then

$$\left(\frac{|\xi|^{2p-1}}{\sqrt{1+|\xi|^{2p}}} - \frac{|\eta|^{2p-1}}{\sqrt{1+|\eta|^{2p}}}\right)(|\xi| - |\eta|) \geq 0,$$

which implies that

$$G(\xi,\eta) = \left(\frac{|\xi|^{2p-2}}{\sqrt{1+|\xi|^{2p}}}\xi - \frac{|\eta|^{2p-2}}{\sqrt{1+|\eta|^{2p}}}\eta\right)\cdot(\xi - \eta) \geq 0.$$

□

3. Existence Result

This section investigates the existence and uniqueness of the weak solution to (1.1). The concept of a weak solution to problem (1.1) can now be introduced.

Definition 3.1. *A measurable function $u : \Omega \to \mathbb{R}$ is called a weak solution ofthe Problem (1.1) if $u \in W_0^{1,p}(\Omega, \omega_1) \cap W_0^{2,p}(\Omega, \omega_2)$ and*

$$\begin{aligned}
\int_\Omega \omega_2|\Delta u|^{p-2}\Delta u\Delta\phi dx + \int_\Omega \omega_1\left(|\nabla u|^{p-2} + \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}}\right)\nabla u\cdot\nabla\phi dx + \int_\Omega \omega_1|u|^{p-2}u\phi dx \\
= \int_\Omega f\phi dx,
\end{aligned}$$

for all $\phi \in W_0^{1,p}(\Omega, \omega_1) \cap W_0^{2,p}(\Omega, \omega_2)$.

The part that follows presents the principal result of this work along with its proof.

Theorem 3.1. *Let* $f/\omega_1 \in L^{p'}(\Omega, \omega_1)$, *and assume that hypothesis* (H_1) *and* (H_2) *holds. Then problem (1.1) has a unique weak solution.*

Proof. **Existence of the weak solutions**
Let

$$E := W_0^{1,p}(\Omega, \omega_1) \cap W_0^{2,p}(\Omega, \omega_2),$$

which is a Banach space equipped with the following norm

$$||u||_E := \left(||u||^p_{W_0^{1,p}(\Omega,\omega_1)} + ||\Delta u||^p_{L^p(\Omega,\omega_2)} \right)^{1/p}.$$

We consider the operator $A : E \to E'$ such that

$$A = A^1 + A^2 + A^3 + A^4 - A^5,$$

where for $\forall u, v \in E$

$$< A^1 u, v > = \int_\Omega \omega_1 |\nabla u|^{p-2} \nabla u . \nabla v dx,$$
$$< A^2 u, v > = \int_\Omega \omega_1 \frac{|\nabla u|^{2p-2}}{\sqrt{1 + |\nabla u|^{2p}}} \nabla u \cdot \nabla v dx,$$
$$< A^3 u, v > = \int_\Omega \omega_1 |u|^{p-2} u v dx,$$
$$< A^4 u, v > = \int_\Omega \omega_2 |\Delta u|^{p-2} \Delta u \Delta v dx,$$
$$< A^5 u, v > = \int_\Omega f v dx.$$

- **Assertion 1**. The operator A is monotone.

We have by Lemma (2.1) and (2.2)

$$\langle A^1 u - A^1 v, u - v \rangle \geq 0, \tag{3.1}$$

$$\langle A^2 u - A^2 v, u - v \rangle \geq 0. \tag{3.2}$$

On the other hand, as the monotony of $\lambda \to |\lambda|^{p-2}\lambda$, we obtain

$$\langle A^3 u - A^3 v, u - v \rangle = \int_\Omega \omega_1 \left(|u|^{p-2} u - |v|^{p-2} v \right) (u - v) dx \geq 0, \tag{3.3}$$

and

$$\langle A^4 u - A^4 v, u - v \rangle = \int_\Omega \omega_2 \left(|\Delta u|^{p-2} \Delta u - |\Delta v|^{p-2} \Delta v \right) (\Delta u - \Delta v) dx \geq 0. \tag{3.4}$$

Then, we deduce from (3.1), (3.2), (3.3) and (3.4) that

$$\langle Au - Av, u - v\rangle \geq 0.$$

Hence, the operator A is monotone.

- **Assertion 2**. The operator A is coercive.

Using Hölder's inequality, and Proposition 2.1, we obtain

$$\begin{aligned} < A^5u, u >= \int_\Omega fu &= \int_\Omega \omega_1^{1/p'} \frac{f}{\omega_1} \omega_1^{1/p} u dx \\ &\leq \left(\int_\Omega \omega_1 \left|\frac{f}{\omega_1}\right|^{p'}\right)^{1/p'} \left(\int_\Omega \omega_1 |u|^p dx\right)^{1/p} \\ &\leq C \left\|\frac{f}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)} \|u\|_{W_0^{1,p}(\Omega,\omega_1)} \\ &\leq C \left\|\frac{f}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)} \|u\|_E . \end{aligned} \tag{3.5}$$

On the other hand, it is obvious that

$$< A^3u, u > + < A^4u, u >= \int_\Omega \omega_1 \frac{|\nabla u|^{2p}}{\sqrt{1 + |\nabla u|^{2p}}} dx + \int_\Omega \omega_1 |u|^p dx \geq 0,$$

which implies

$$\begin{aligned} < Au, u > &=< A^1u, u > + < A^2u, u > + < A^3u, u > + < A^4u, u > - < A^5u, u > \\ &\geq \|\Delta u\|^p_{L^p(\Omega,\omega_2)} + \|u\|^p_{W_0^{1,p}(\Omega,\omega_1)} - C \left\|\frac{f}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)} \|u\|_E \\ &\geq \|u\|^p_E - C \left\|\frac{f}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)} \|u\|_E . \end{aligned}$$

Therefore

$$\frac{< Au, u >}{\|u\|_E} \to +\infty \qquad \text{as} \qquad \|u\|_E \to +\infty. \tag{3.6}$$

Thus, the operator A is coercive.

Assertion 3. The operator A is bounded

Applying Hölder's inequality once more, we find

$$\begin{aligned} | < A^1u, v > | = | \int_\Omega \omega_1 |\nabla u|^{p-2} \nabla u . \nabla v dx| &\leq \int_\Omega \omega_1^{1/p'} |\nabla u|^{p-1} \omega_1^{1/p} |\nabla v| dx \\ &\leq \left(\int_\Omega \omega_1 |\nabla u|^p dx\right)^{1/p'} \left(\int_\Omega \omega_1 |\nabla v|^p dx\right)^{1/p} \\ &\leq \|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_{W_0^{1,p}(\Omega,\omega_1)} \\ &\leq \|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_E, \end{aligned} \tag{3.7}$$

$$
\begin{aligned}
| < A^2u, v > | =& | \int_\Omega \omega_1 \frac{|\nabla u|^{2p-2}}{\sqrt{1+|\nabla u|^{2p}}} \nabla u . \nabla v dx | \\
\leq& \int_\Omega \omega_1 \frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}} |\nabla u|^{p-2} |\nabla u . \nabla v| dx \\
\leq& \int_\Omega \omega_1 \frac{|\nabla u|^{p}}{\sqrt{1+|\nabla u|^{2p}}} |\nabla u|^{p-1} |\nabla v| dx \\
\leq& \int_\Omega \omega_1 |\nabla u|^{p-1} |\nabla v| dx \\
\leq& \|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_{W^{1,p}_0(\Omega,\omega_1)} \\
\leq& \|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_E,
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
| < A^4u, v > | =& | \int_\Omega \omega_2 |\Delta u|^{p-2} \Delta u \Delta v dx | \\
\leq& \int_\Omega \omega_2 |\Delta u|^{p-1} |\Delta v| dx \\
\leq& \|\Delta u\|^{p/p'}_{L^p(\Omega,\omega_2)} \|\Delta v\|_{L^p(\Omega,\omega_2)} \\
\leq& \|\Delta u\|^{p/p'}_{L^p(\Omega,\omega_2)} \|v\|_E.
\end{aligned} \tag{3.9}
$$

Similarly, by applying Hölder's inequality once again, together with proposition 2.1, we have

$$
\begin{aligned}
&\left| < A^3u, v > - < A^5u, v > \right| \\
&= \left| \int_\Omega \omega_1 |u|^{p-2} u v dx - \int_\Omega f v dx \right| \\
&\leq \int_\Omega \omega_1^{1/p'} |u|^{p-1} \omega_1^{1/p} |v| dx + \int_\Omega \omega_1^{1/p'} |\frac{f}{\omega_1}| \omega_1^{1/p} |v| dx \\
&\leq \left(\int_\Omega \omega_1 |u|^p dx \right)^{1/p'} \left(\int_\Omega \omega_1 |v|^p dx \right)^{1/p} + \left(\int_\Omega \omega_1 |\frac{f}{\omega_1}|^{p'} dx \right)^{1/p'} \left(\int_\Omega \omega_1 |v|^p dx \right)^{1/p} \\
&\leq C \|u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_{W^{1,p}_0(\Omega,\omega_1)} + C \left\| \frac{f}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \|v\|_{W^{1,p}_0(\Omega,\omega_1)} \\
&\leq C \|u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_E + C \left\| \frac{f}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \|v\|_E \, .
\end{aligned} \tag{3.10}
$$

Combining of (3.7), (3.8), (3.9) and (3.10), we derive

$$
\begin{aligned}
| < Au, v > | \leq& | < A^1u, v > | + | < A^2u, v > | + | < A^4u, v > | + | < A^3u, v > - < A^5u, v > | \\
\leq& 2 \|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_E + \|\Delta u\|^{p/p'}_{L^p(\Omega,\omega_2)} \|v\|_E \\
&+ C \|u\|^{p/p'}_{L^p(\Omega,\omega_1)} \|v\|_E + C \left\| \frac{f}{\omega_1} \right\|_{L^{p'}(\Omega,\omega_1)} \|v\|_E \\
\leq& C_1 \|v\|_E \, ,
\end{aligned} \tag{3.11}
$$

where $C_1 := 2\|\nabla u\|^{p/p'}_{L^p(\Omega,\omega_1)} + \|\Delta u\|^{p/p'}_{L^p(\Omega,\omega_2)} + C\|u\|^{p/p'}_{L^p(\Omega,\omega_1)} + C\left\|\frac{f}{\omega_1}\right\|_{L^{p'}(\Omega,\omega_1)}$.
Therefore, A is bounded.
Furthermore, the operator A exhibits hemi-continuity. By Theorem 2.1, we conclude that the problem (1.1) has weak solutions.

Uniqueness of a weak solutions

We shall now demonstrate that the problem (1.1) possesses a unique weak solution. We assume the existence of two weak solutions, u_1 and u_2, for problem (1.1). Consequently, we possess

$$\begin{aligned}
&\int_\Omega \omega_2 \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)\Delta\phi dx \\
&+ \int_\Omega \omega_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla\phi dx \\
&+ \int_\Omega \omega_1 \left(\frac{|\nabla u_1|^{2p-2}}{\sqrt{1+|\nabla u_1|^{2p}}}\nabla u_1 - \frac{|\nabla u_2|^{2p-2}}{\sqrt{1+|\nabla u_2|^{2p}}}\nabla u_2\right)\cdot\nabla\phi dx \\
&+ \int_\Omega \omega_1 (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)\phi dx \\
&\qquad = 0.
\end{aligned}$$

By substituting $\phi = u_1 - u_2$ into (1.1), we obtain

$$\begin{aligned}
&\int_\Omega \omega_2 \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)\Delta(u_1-u_2) dx \\
&+ \int_\Omega \omega_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla(u_1-u_2) dx \\
&+ \int_\Omega \omega_1 \left(\frac{|\nabla u_1|^{2p-2}}{\sqrt{1+|\nabla u_1|^{2p}}}\nabla u_1 - \frac{|\nabla u_2|^{2p-2}}{\sqrt{1+|\nabla u_2|^{2p}}}\nabla u_2\right)\cdot\nabla(u_1-u_2) dx \\
&+ \int_\Omega \omega_1 (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1-u_2) dx \\
&= 0.
\end{aligned}$$

Following that, by utilizing Lemmas 2.1 and 2.2, we derive

$$\begin{aligned}
&\int_\Omega \omega_1 (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1-u_2) dx \\
&= -\int_\Omega \omega_1 \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right)\cdot\nabla(u_1-u_2) dx \\
&- \int_\Omega \omega_1 \left(\frac{|\nabla u_1|^{2p-2}}{\sqrt{1+|\nabla u_1|^{2p}}}\nabla u_1 - \frac{|\nabla u_2|^{2p-2}}{\sqrt{1+|\nabla u_2|^{2p}}}\nabla u_2\right)\cdot\nabla(u_1-u_2) dx \\
&- \int_\Omega \omega_2 \left(|\Delta u_1|^{p-2}\Delta u_1 - |\Delta u_2|^{p-2}\Delta u_2\right)\Delta(u_1-u_2) dx \\
&\leq 0.
\end{aligned}$$

Employing Lemma 2.1, we infer the existence of $\alpha_p > 0$ such that

$$\alpha_p \int_\Omega \omega_1(|u_1| + |u_2)^{p-2}|u_1 - u_2|^2 dx \leq \int_\Omega \omega_1(|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)(u_1 - u_2)dx \leq 0.$$

Therefore, $|u_1 - u_2|^2 = 0$,
i.e $u_1 = u_2$. a.e in Ω. □

4. Example

Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, and define the functions $\omega_1(x,y)$, $\omega_2(x,y)$ and $f(x,y)$ as follows:

$\omega_1(x,y) = (x^2+y^2)^{-1/2}$, $\omega_2(x,y) = (x^2+y^2)^{-1/4}$, $p = 4$, $f(x,y) = \frac{\cos(xy)}{(x^2+y^2)^{1/6}}$.

Then by applying Theorem 3.1, we can deduce that that the problem

$$\begin{cases} \Delta[\omega_2|\Delta u|^2\Delta u] - \operatorname{div} \omega_1 \left(|\nabla u|^2 \nabla u + \dfrac{|\nabla u|^6 \nabla u}{\sqrt{1+|\nabla u|^4}} \right) + \omega_1 |u|^2 u = f \quad \text{in } \Omega \\ u|_{\partial\Omega} = 0, \end{cases} \tag{4.1}$$

has a unique solution in $W_0^{1,4}(\Omega,\omega_1) \cap W_0^{2,4}(\Omega,\omega_2)$.

5. Conclusion

Capillary action, a physical phenomenon that involves the movement of water in a thin vertical tube, is gaining popularity because of its relevance in various fields such as industrial, biomedical, pharmaceutical, and microfluidic systems. The study of capillary phenomena has gained attention due to its fascination with natural phenomena and its importance in applied fields. Many researchers have proposed well-known equations to model the kinetics of capillary rise, which is crucial for understanding its application in industrial contexts, and several investigations have been carried out to identify a weak solution to these problems and to establish the uniqueness and existence of such solutions. This paper explores the existence and uniqueness of weak solutions to problems (1.1) using weighted Sobolev spaces and monotone-type operators in reflexive Banach spaces. It provides definitions, properties,lemmas, and their proof.

Acknowledgments: The authors extend their appreciation to Umm Al-Qura University, Saudi Arabia for funding this research work through grant number 25UQU4340156GSSR01.

Funding: This research work was funded by Umm Al-Qura University, Saudi Arabia under grant number 25UQU4340156GSSR01.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] I. Aydin, Weighted Variable Sobolev Spaces and Capacity, J. Funct. Spaces Appl. 2012 (2012), 132690. https://doi.org/10.1155/2012/132690.

[2] M. Chipot, Elliptic Equations: An Introductory Course, Birkhäuser, Basel, 2009. https://doi.org/10.1007/978-3-7643-9982-5.

[3] P. Drábek, A. Kufner, F. Nicolosi, Nonlinear Elliptic Equations, Singular and Degenerate Cases, Preprint, University of West Bohemia, 1996.

[4] L. Hmidouch, A. Jamea, M. Laghdir, Existence of Entropy Solution for a Nonlinear Parabolic Problem in Weighted Sobolev Space via Optimization Method, Math. Model. Anal. 28 (2023), 393–414. https://doi.org/10.3846/mma.2023.17010.

[5] J. Heinonen, T. Kilpeläinen, O. Martio, Nonlinear Potential Theory of Degenerate Elliptic Equations, Dover Publications, Mineola, 2006.

[6] J.L. Lions, Quelques Methodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[7] W.M. Ni, J. Serrin, Non-Existence Theorems for Quasilinear Partial Differential Equations, Rend. Circ. Mat. Palermo 2 Suppl. 8 (1985), 171–185.

[8] W.M. Ni, J. Serrin, Existence and Nonexistence Theorems for Ground States for Quasilinear Partial Differential Equations, Accad. Naz. dei Lincei 77 (1986), 231–257.

[9] M.M. Rodrigues, Multiplicity of Solutions on a Nonlinear Eigenvalue Problem for p(x)-Laplacian-like Operators, Mediterr. J. Math. 9 (2012), 211–223. https://doi.org/10.1007/s00009-011-0115-y.

[10] S. Shokooh, G.A. Afrouzi, S. Heidarkhani, Multiple Solutions for $P(X)$-Laplacian-Like Problems With Neumann Condition, Acta Univ. Apulensis 49 (2017), 111–128.

[11] B.O. Turesson, Nonlinear Potential Theory and Weighted Sobolev Spaces, Springer, Berlin, 2000.