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Third Order Hankel Determinant for Inverse Functions of a Classes of Univalent Functions with Bounded Turning

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Abstract. The main goal of this paper is to determine an upper bound for the third Hankel determinant for the inverse functions of f, belonging to the two classes of univalent functions with bounded turning.

1. Introduction

Let $\mathcal{A}$ is the class of functions f which are analytic in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and are normalized such that $f(0) = 0 = f'(0) - 1$, i.e.,

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \tag{1.1}$$

The general Hankel determinant $H_q(n)(f)$ of a given function f, for $q \geq 1$ and $n \geq 1$ is defined with

$$H_q(n)(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

The third Hankel determinant is

$$H_3(1)(f) = \begin{vmatrix} 1 & a_2 & a_3 \\ a_2 & a_3 & a_4 \\ a_3 & a_4 & a_5 \end{vmatrix} = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

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This research is focused on the class of $\mathcal{R} \subset \mathcal{A}$ of univalent functions satisfying

$$\operatorname{Re} f'(z) > 0 \ (z \in \mathbb{D}), \tag{1.2}$$

and the class $\mathcal{R}_1 \subset \mathcal{A}$ satisfying

$$\operatorname{Re} \{f'(z) + zf''(z)\} > 0, (z \in \mathbb{D}).$$

The functions from the class $\mathcal{R}$ are said to be of bounded turning since $\operatorname{Re} f'(z) > 0$ is equivalent to $|argf'(z)| < \pi/2$, and $argf'(z)$ is the angle of rotation of the image of a line segment starting from z under the mapping f. They are of special interest since they are not part of class of starlike functions which is very wide subclass of univalent functions. For the class $\mathcal{R}$ in [10] the authors showed that

$$|H_2(1)| \leq \frac{4}{9} = 0.444\ldots,$$

and in [11] (with $\alpha = 1$ in Corollary 2.8),

$$|H_3(1)| \leq \frac{1}{540}\left(\frac{877}{3} + 25\sqrt{5}\right) = 0.64488\ldots.$$

While the first estimate is sharp, the second one is not and it is improved in [12] where is given an upper bounds of the third Hankel determinant for class of univalent functions with bounded turning and class $\mathcal{R}_1$. To prove the main result we will use the method based on the estimates of the coefficients of Schwarz function due to Prokhorov and Szynal ([13], Lemma 2). For the proofs needed the result:

Lemma 1.1. *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then, for any real numbers μ and ν such that $(\mu, \nu) \in D_1 \cup D_2$, where*

$$D_1 = \left\{(\mu, \nu) : |\mu| \leq \frac{1}{2}, -1 \leq \nu \leq 1\right\}$$

and

$$D_2 = \left\{(\mu, \nu) : \frac{1}{2} \leq |\mu| \leq 2, \frac{4}{27}(|\mu| + 1)^3 - (|\mu| + 1) \leq \nu \leq 1\right\},$$

the following sharp estimate holds

$$|c_3 + \mu c_1 c_2 + \nu c_1^3| \leq 1.$$

We will also use the following, almost forgotten result of Carleson ([14]) that can also be found in [[15], Problem 16, p.78].

Lemma 1.2. *Let $\omega(z) = c_1 z + c_2 z^2 + \cdots$ be a Schwarz function. Then*

$$|c_2| \leq 1 - |c_1|^2$$

and

$$|c_4| \leq 1 - |c_1|^2 - |c_2|^2 - |c_3|^2.$$

Some of the more significant results for the Hankel determinant of second order for the inverse functions of convex and starlike function can be found in Obradović et al. [5] and for the second Hankel determinant for starlike and convex functions of order alpha in Sim et al. [9]. Sharp bound of third Hankel determinant for inverse coefficients of convex functions can be found in Raza et al. [7] and the sharp bound of the third Hankel determinant $|H_3(1)(f)|$ in Lecko et al., Ahamed et al., and Rath at al., respectively [3], [1] and [6]. Karamazova Gelova and Tuneski in [16] determine an upper bound for the third Hankel determinant for the inverse functions of f, belonging to the class of starlike functions of order $\alpha (0 \leq \alpha \leq \frac{1}{2})$.

Krishna and Ramreddy in Krishna et al. [2] obtain an upper bound of the second Hankel determinant $|a_2a_4 - a_3^2|$ for starlike and convex functions of order α. The bounds of some initial coefficients, the Fekete-Szegö -type inequality and estimation of Hankel determinants of second and third order were discussed in Shi et al. [8]. Upper bound of the Hankel determinant of third order for inverse functions of functions from some classes of univalent functions can find in Obradović et al. [4].

For every univalent function in $\mathbb{D}$, exists inverse at least on the disk with radius 1/4. If the inverse has an expansion

$$f^{-1}(\omega) = \omega + A_2\omega^2 + A_3\omega^3 + \cdots, \tag{1.3}$$

then, by using the identity $f(f^{-1}(\omega)) = \omega$, from (1.1) and (1.3) we receive

$$\begin{aligned} A_2 &= -a_2 \\ A_3 &= -a_3 + 2a_2^2 \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3 \\ A_5 &= -a_5 + 6a_2a_4 - 21a_2^2a_3 + 3a_3^2 + 14a_2^4. \end{aligned} \tag{1.4}$$

By using the definition of $H_3(1)(f)$ and the relations (1.4), after some calculations, we receive:

$$\begin{aligned} H_3(1)(f^{-1}) &= A_3(A_2A_4 - A_3^2) - A_4(A_4 - A_2A_3) + A_5(A_3 - A_2^2) \\ &= a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2) - (a_3 - a_2^2)^3 \\ &= H_3(1)(f) - (a_3 - a_2^2)^3, \end{aligned} \tag{1.5}$$

i.e.,

$$H_3(1)(f^{-1}) = H_3(1)(f) - (a_3 - a_2^2)^3 \tag{1.6}$$

2. Main results

Theorem 2.1. *If $f(z) \in \mathcal{R}$ is of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, then*

$$\left|H_3(1)(f^{-1})\right| \leq \frac{1249}{3840} = 0.32526\ldots.$$

Proof. First let note that (1.2) is equivalent with

$$f'(z) = \frac{1 + \omega(z)}{1 - \omega(z)}$$

or

$$f'(z)[1-\omega(z)] = 1+\omega(z), \tag{2.1}$$

where ω is analytic in $\mathbb{D}$, $\omega(0)=0$ and $|\omega(z)|<1$ for all z in $\mathbb{D}$. If $\omega(z)=c_1z+c_2z^2+\cdots$, by (1.1) and equating the coefficients in (2.1), we have

$$\begin{aligned} a_2 &= c_1, \\ a_3 &= \frac{2}{3}(c_1^2+c_2), \\ a_4 &= \frac{1}{2}(c_3+2c_1c_2+c_1^3), \\ a_5 &= \frac{2}{5}(c_4+2c_1c_3+3c_1^2c_2+c_1^4+c_2^2). \end{aligned} \tag{2.2}$$

Using (1.5) i.e. (1.6) and (2.2) after some calculation we get:

$$\begin{aligned} H_3(1)(f^{-1}) &= \frac{1}{540}\Big(-7c_1^6-12c_1^4c_2-16c_2^3-54c_1^3c_3+108c_1c_2c_3 \\ &\quad -135c_3^2+60c_1^2c_2^2-72c_1^2c_4+144c_2c_4\Big)-\left(\frac{2}{3}c_2-\frac{1}{3}c_1^2\right)^3 \\ &= \frac{1}{540}\Big(-7c_1^6-12c_1^4c_2-16c_2^3-54c_1^3c_3+108c_1c_2c_3-135c_3^2 \\ &\quad +60c_1^2c_2^2-72c_1^2c_4+144c_2c_4-160c_2^3+20c_1^6+240c_2^2c_1^2-120c_2c_1^4\Big). \end{aligned}$$

From here again we do some calculation and obtain finally:

$$\begin{aligned} H_3(1)(f^{-1}) = \frac{1}{540}\Bigg[&13c_1^6-135c_3\left(c_3-\frac{4}{5}c_1c_2+\frac{2}{5}c_1^3\right) \\ &-132c_1^4c_2-176c_2^3+300c_1^2c_2^2+72\left(2c_2-c_1^2\right)c_4\Bigg]. \end{aligned}$$

Now,

$$\begin{aligned} |H_3(1)(f^{-1})| \le \frac{1}{540}\Bigg[&13|c_1|^6+135|c_3|\left|c_3-\frac{4}{5}c_1c_2+\frac{2}{5}c_1^3\right|+132|c_1|^4|c_2| \\ &+176|c_2|^3+300|c_1|^2|c_2|^2+72\left(2|c_2|+|c_1|^2\right)|c_4|\Bigg]. \end{aligned} \tag{2.3}$$

From Lemma 1.1 for $\mu=-\frac{4}{5}, \nu=\frac{2}{5}$ and $(\mu,\nu)\in D_2$ and also the inequalities for the function ω given in Lemma 1.2 from (2.3) we have

$$\begin{aligned} |H_3(1)(f^{-1})| \le \frac{1}{540}\Big[&13|c_1|^6+135|c_3|+132|c_1|^4|c_2|+176|c_2|^2(1-|c_1|^2) \\ &+300|c_1|^2|c_2|^2+72(2-|c_1|^2)(1-|c_1|^2-|c_2|^2-|c_3|^2)\Big]. \end{aligned}$$

Now, after some calculation we get

$$|H_3(1)(f^{-1})| \leq \frac{1}{540}\Big[144+135|c_3|-144|c_3|^2-72|c_1|^2(1-|c_3|^2)-196|c_2|^2(1-|c_1|^2)$$
$$+228|c_2|^2-|c_1|^2(144-132|c_1|^2|c_2|-72|c_1|^2-13|c_1|^4)\Big],$$

i.e.,

$$|H_3(1)(f^{-1})| \leq \frac{1}{540}\left(144+135|c_3|-144|c_3|^2\right),$$

because all other terms are less or equal to zero. Since, $0 \leq |c_3| \leq 1$, we have

$$|H_3(1)(f^{-1})| \leq \frac{1249}{3840} = 0.32526\ldots,$$

where maximum is attained for $|c_3| = \frac{15}{32}$. □

The result of Theorem 2.1 are probably not sharp.

Next, upper estimate we will give for the modulus of the third Hankel determinant for the class $\mathcal{R}_1$.

Theorem 2.2. *If $f(z) \in \mathcal{R}_1$ is of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$, then*

$$\left|H_3(1)(f^{-1})\right| \leq \frac{4}{225}\left[1+\frac{1}{4}\left(\frac{15}{16}\right)^4\right] = 0.021211\ldots.$$

Proof. As in the proof of the previous theorem, for each function f from $\mathcal{R}_1$, there exists a function $\omega(z) = c_1z + c_2z^2 + \cdots$, analytic in $\mathbb{D}$, such that $|\omega(z)| < 1$ for all z in $\mathbb{D}$, and

$$f'(z) + zf''(z) = \frac{1+\omega(z)}{1-\omega(z)},$$

or

$$[f'(z) + zf''(z)][1-\omega(z)] = 1 + \omega(z). \tag{2.4}$$

Now, with equating the coefficients in the previous expression (2.4) we get

$$\begin{aligned} a_2 &= \frac{c_1}{2}, \\ a_3 &= \frac{2}{9}(c_1^2 + c_2), \\ a_4 &= \frac{1}{8}(c_3 + 2c_1c_2 + c_1^3), \\ a_5 &= \frac{2}{25}(c_4 + 2c_1c_3 + 3c_1^2c_2 + c_1^4 + c_2^2). \end{aligned} \tag{2.5}$$

Using (1.5) i.e. (1.6) and (2.5) we obtain:

$$\begin{aligned} H_3(1)(f^{-1}) &= \frac{1}{1166400}\Big[-1217c_1^6 - 1140c_1^4c_2 + 1311c_1^2c_2^2 + 7936c_2^3 \\ &\quad -9234c_1^3c_3 + 972c_1c_2c_3 - 18225c_3^2 + 2592\left(8c_2 - c_1^2\right)c_4 + 25c_1^6 \\ &\quad -12800c_2^3 - 600c_1^4c_2 + 4800c_1^2c_2^2\Big] \\ &= \frac{1}{1166400}\Big[-1192c_1^6 - 1740c_1^4c_2 + 6111c_1^2c_2^2 - 4864c_2^3 \\ &\quad -9234c_1^3c_3 + 972c_1c_2c_3 - 18225c_3^2 + 2592\left(8c_2 - c_1^2\right)c_4\Big] \\ &= \frac{1}{1166400}\Bigg[-18225c_3\left(c_3 - \frac{4}{75}c_1c_2 + \frac{38}{75}c_1^3\right) - 1740c_1^4c_2 \\ &\quad + 6111c_1^2c_2^2 - 4864c_2^3 - 1192c_1^6 + 2592\left(8c_2 - c_1^2\right)c_4\Bigg] \end{aligned}$$

Then,

$$\begin{aligned} |H_3(1)(f^{-1})| \leq \frac{1}{1166400}\Bigg[&18225|c_3|\left|c_3 - \frac{4}{75}c_1c_2 + \frac{38}{75}c_1^3\right| \\ &+ 1740|c_1|^4|c_2| + 6111|c_1|^2|c_2|^2 + 4864|c_2|^3 \\ &+1192|c_1|^6 + 2592\left(8|c_2| + |c_1|^2\right)|c_4|\Bigg]. \end{aligned} \tag{2.6}$$

From Lemma 1.1 for $\mu = -\frac{4}{75}, \nu = \frac{38}{75}$ and $(\mu, \nu) \in D_2$ and also the inequalities for the function ω given in Lemma 1.2 from (2.6) we have

$$\begin{aligned} |H_3(1)(f^{-1})| &\leq \frac{1}{1166400}\Big[18225|c_3| + 1740|c_1|^4|c_2| + 4864|c_2|^2(1 - |c_1|^2) \\ &\quad +6111|c_1|^2|c_2|^2 + 1192|c_1|^6 + 2592\left(8 - 7|c_1|^2\right)(1 - |c_1|^2 - |c_2|^2 - |c_3|^2)\Big] \\ &= \frac{1}{1166400}\Big[20736 + 18225|c_3| - 20736|c_3|^2 - 18144|c_1|^2(1 - |c_3|^2) \\ &\quad -15872|c_2|^2\left(1 - |c_1|^2\right) + M\Big], \end{aligned}$$

where

$$\begin{aligned} M &= -20736|c_1|^2 + 1740|c_1|^4|c_2| + 18144|c_1|^4 + 1192|c_1|^6 + 3519|c_1|^2|c_2|^2 \\ &\leq -20736|c_1|^2 + 1740|c_1|^4(1 - |c_1|^2) + 18144|c_1|^4 + 1192|c_1|^6 \\ &\quad + 3519|c_1|^2(1 - 2|c_1|^2 + |c_1|^4) \\ &= -17217|c_1|^2 + 12846|c_1|^4 + 2971|c_1|^6 \\ &= -|c_1|^2[17217(1 - |c_1|^4) + |c_1|^4(12846 - 14246|c_1|^2)] \\ &\leq 0. \end{aligned}$$

From here, for $|c_3| \le 1$ we have

$$
\begin{aligned}
|H_3(1)(f^{-1})| &\le \frac{1}{1166400}\left(20736 + 18225|c_3| - 20736|c_3|^2\right)\\
&\le \frac{4}{225}\left[1 + \left(\frac{15}{16}\right)^2 |c_3| - |c_3|^2\right]\\
&\le \frac{4}{225}\left[1 + \frac{1}{4}\left(\frac{15}{16}\right)^4\right],
\end{aligned}
$$

where maximum is attained for $|c_3| = \frac{225}{512}$. □

The result of Theorem 2.2 are probably not sharp.

Conjecture 2.1. *Let $f \in \mathcal{A}$ and is of the form $f(z) = z + a_2z^2 + a_3z^3 + \cdots$.*

- *If $f \in R$, then $|H_3(1)(f^{-1})| \le 1/4$;*
- *If $f \in R_1$, then $|H_3(1)(f^{-1})| \le 1/64$.*

Both estimates are sharp. An extremal function is received for $\omega(z) = z^3$ in (2.1) and (2.4), respectively.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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