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A State-Dependent Dynamic Nonlinear Problems on Time Scales

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Abstract. Considering the phenomena that depend on their past state or past history, it was noted that they were given more importance. The mathematical models of these phenomena can be explained by state-dependent differential equations or a self-referred type. This article is dedicated to studying the solvability of state-dependent or self-referred dynamic nonlinear problems on time scales. Here, we got the existence of at least one solution to state-dependent dynamic nonlinear problems on time scales and a unique solution it has. Further more, we obtained results on the dependency of solutions for state-dependent dynamic nonlinear problems on time scales with respect to initial values.

1. Introduction

The state-dependent dynamic nonlinear problems on time scales are one of the recent kinds of functional dynamic nonlinear problems on time scales. Most of the dynamic nonlinear problems on time scales with deviating arguments that appear in much literature, the deviation of the argument usually involves only the time itself. However, another case in which the deviating arguments depend on both the state variable x and the time t is of importance in theory and practice. Several papers have appeared recently that are kinds of state-dependent differential equations [10, 11, 13] and references therein.

The theory of time scales, which has recently received a lot of attention, was initiated by Stefan Hilger in his PhD thesis in order to unify discrete and continuous analysis [14]. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so-called time scale $\mathbb{T}$, which may be an arbitrary closed subset of the real numbers $\mathbb{R}$ see [3, 6]. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see [16]), i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{Z}$ and

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$\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The book on the subject of time scales by Bohner and Peterson [5] summarizes and organizes much of time scales calculus. All time scales notations are defined in the next section.

Dynamic equations on time scales play a significant role in the mathematical modelling of numerous real-world phenomena involving continuous and discrete data simultaneously. The sphere of study of dynamic equations covers various aspects like qualitative and quantitative properties of solutions, stability of solutions, controllability of solutions, and applications in various areas of applied science and engineering [1,4,9,12,15,18,20,21,23].

In [24] Christopher et al. considered first-order dynamic equations of the type:

$$x^{\Delta}(t) = f(t, x(t)) \quad t \in [a, b]_{\mathbb{T}}, \tag{1.1}$$

$$x^{\Delta}(t) = f(t, x^{\sigma}(t)) \quad t \in [a, b]_{\mathbb{T}}, \tag{1.2}$$

subject to the initial condition

$$x(a) = A, \tag{1.3}$$

where $f : [a, b]_{\mathbb{T}} \times \mathbb{R}^n \times \mathbb{R}^n \to \times \mathbb{R}^n$, may be a nonlinear function, $n \geq 1$ and $A \in \mathbb{R}$ is given.

In [17] the authors considered a generalised form of (1.1) and (1.2) with supposing the function f is re-continuous, as follows:

$$x^{\Delta}(t) = f(t, x(t), x^{\sigma}(t)) \quad t \in [a, b]_{\mathbb{T}}. \tag{1.4}$$

without supposing f is rd-continuous, the authors in [22] obtained the existence of solutions to Eqs. (1.4) and (1.3) with respect to initial values.

As we know, the obtained results of dynamic equations with nonlocal conditions are better than those with local conditions, see for instance [2,8,23]. For example, by Banach's fixed-point theorem, [7] Bohner et al. studied the existence of solutions of the first-order dynamic equation of the type:

$$x^{\Delta}(t) + \psi_1(t)x^{\sigma}(t) = f(t, x(t)), \quad t \in [a, b]_{\mathbb{T}^{\kappa}}; \tag{1.5}$$

subject to the nonlocal condition

$$x(S) = x_0.$$

Here, in this study we proved the existence and uniqueness of the state-dependent dynamic nonlinear problem of the type:

$$\chi^{\Delta}(t) = \psi_1\Big(t, \chi\big(\int_S^t \psi_2(s, \chi(s))\Delta s\big)\Big), \quad t \in I = \mathbb{T}^k, \tag{1.6}$$

subject to the nonlocal condition

$$\alpha_1\chi(S) + \alpha_2\chi(\sigma(T)) = \chi_0. \tag{1.7}$$

Where $\mathbb{T} = [S, T]_{\mathbb{T}},\ S,\ T \in \mathbb{T},\ S < T,\ \chi_0 \in \mathbb{R}$ *and* $\psi_{1,2} \in C_{rd}(\mathbb{T} \times \mathbb{R}, \mathbb{R}),\ \alpha_1,\ \alpha_2 \in \mathbb{R}$.

Also, we study dependency of solutions to initial value problems with respect to initial values.

The paper is organized as follows. In Section 2, we briefly recall necessary results and notions; the original results being then given and proved in Section 3.

2. Preliminaries on Time Scales

First, we recall some time scales essentials and some universal symbols used in the present article. From now on, $\mathbb{R}$ and $\mathbb{Z}$ denote the set of all real numbers and the set of all integers, respectively.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. We suppose throughout the article that $\mathbb{T}$ has the topology that it inherits from the standard topology on $\mathbb{R}$. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined for any $\tau \in \mathbb{T}$ by

$$\sigma(\tau) := \inf\{s \in \mathbb{T} : s > \tau\},$$

and the backward jump operator $\rho : \mathbb{T} :\to \mathbb{T}$ is defined for any $\tau \in \mathbb{T}$ by

$$\rho(\tau) := \sup\{s \in \mathbb{T} : s < \tau\}.$$

A point $\tau \in \mathbb{T}$ with $\inf \mathbb{T} < \tau < \sup \mathbb{T}$ is said to be right-scattered if $\sigma(\tau) > \tau$, right-dense if $\sigma(\tau) = \tau$, left-scattered if $\rho(\tau) < \tau$, and left-dense if $\rho(\tau) = \tau$. Points that are simultaneously right-dense and left-dense are called dense points. Whereas points that are simultaneously right-scattered and left-scattered are called isolated points.

We define the forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ for any $\tau \in \mathbb{T}$ by $\mu(\tau) := \sigma(\tau) - \tau$, and the backward graininess function $\nu : \mathbb{T} \to [0, \infty)$ is defined for any $\tau \in \mathbb{T}$ by $\nu(\tau) := \tau - \rho(\tau)$.

Let $\omega : \mathbb{T} \to \mathbb{R}$ be a function. Then the function $\omega^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $\omega^\sigma(\tau) = \omega(\sigma(\tau)), \forall \tau \in \mathbb{T}$, that is $\omega^\sigma = \omega \circ \sigma$. In a similar manner, the function $\omega^\rho : \mathbb{T} \to \mathbb{R}$ is defined by $\omega^\rho(\tau) = \omega(\rho(\tau)), \forall \tau \in \mathbb{T}$, that is $\omega^\rho = \omega \circ \rho$.

The sets $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $\mathbb{T}^\kappa_\kappa$ are defined as follows: If $\mathbb{T}$ has a left–scattered maximum τ_1, then $\mathbb{T}^\kappa = \mathbb{T} - \{\tau_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If $\mathbb{T}$ has a right–scattered minimum τ_2, then $\mathbb{T}_\kappa = \mathbb{T} - \{\tau_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, we have $\mathbb{T}^\kappa_\kappa = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

The interval $[a, b]$ in $\mathbb{T}$ is defined by

$$[a, b]_\mathbb{T} = \{\tau \in \mathbb{T} : a \leq \tau \leq b\}.$$

Open intervals and half-closed interval are defined similarly.

Suppose $\omega : \mathbb{T} \to \mathbb{R}$ is a function and $\tau \in \mathbb{T}^\kappa$. Then we say that $\omega^\Delta(\tau) \in \mathbb{R}$ is the delta derivative of ω at τ if for any $\varepsilon > 0$ there exists a neighborhood U of τ such that for all $\varsigma \in U$

$$\left|[\omega(\sigma(\tau)) - \omega(\varsigma)] - \omega^\Delta(\tau)[\sigma(\tau) - \varsigma]\right| \leq \varepsilon|\sigma(\tau) - \varsigma|.$$

Furthermore, ω is said to be delta differentiable on $\mathbb{T}^\kappa$ if it is delta differentiable at each $\tau \in \mathbb{T}^\kappa$.

A function $\omega : \mathbb{T} \to \mathbb{R}$ is called right-dense continuous (rd-continuous) if ω is continuous at all the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$. On the other hand, ω is called left-dense continuous (ld-continuous) if ω is continuous at all the left-dense points in $\mathbb{T}$ and its right-sided limits exist at all right-dense points in $\mathbb{T}$.

A function $F : \mathbb{T} \to \mathbb{R}$, is said to be a delta antiderivative of $\omega : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta} = \omega(\tau)$ for all $\tau \in \mathbb{T}^{\kappa}$. In this case, the definite integral of ω is defined by

$$\int_a^b \omega(\tau)\Delta\tau = F(b) - F(a) \qquad for\ all \quad a, b \in \mathbb{T}.$$

We will need the following important relations between calculus on time scales $\mathbb{T}$ and continuous calculus on $\mathbb{R}$, discrete calculus on $\mathbb{Z}$. Note that:

(*i*): If $\mathbb{T} = \mathbb{R}$, then

$$\begin{gathered} \sigma(\tau) = \tau, \quad \mu(\tau) = 0, \quad u^{\Delta}(\tau) = u'(\tau), \\ \int_a^b u(\tau)\Delta\tau = \int_a^b u(\tau)d\tau. \end{gathered} \tag{2.1}$$

(*ii*): If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{gathered} \sigma(\tau) = \tau + 1, \quad \mu(\tau) = \nu(\tau) = 1, \\ u^{\Delta}(\tau) = \Delta u(\tau), \\ \int_a^b u(\tau)\Delta\tau = \sum_{\tau=a}^{b-1} u(\tau), \end{gathered} \tag{2.2}$$

Lemma 2.1 (See [5]). *If $p \in \Re$ and fix $t \in \mathbb{T}$, then the exponential function $e_p(t, t_0)$ is the unique solution of the following initial value problem:*

$$\begin{cases} y^{\Delta}(t) = & p(t)y(t), \\ y(t_0) = & 1. \end{cases} \tag{2.3}$$

Definition 2.1 (See [19]). *A mapping between normed linear spaces is called compact provided bounded sets are mapped into relatively compact sets.*

Definition 2.2 (See [7]). *A set M is called relatively compact provided its closure is compact.*

By using the following theorem, Schauder fixed point, we obtain existence of at least one solution to Eqs. (1.6) and (1.7).

Theorem 2.1 (See [19]). *Let U be a convex subset of a Banach space F, and $T : U \to U$ is compact, continuous map. Then T has at least one fixed point in U.*

Theorem 2.2 (See [25]). *A subset of $C(I, \mathbb{R})$ which is both equicontinuous and bounded is relatively compact.*

Now, we are ready to state and prove our main results.

3. Main Results

First, we prove the following lemma which establishes the equivalence to Eqs. (1.6)–(1.7) and a delta integral equation.

Lemma 3.1. *Let $S \in \mathbb{T}$, $\chi_0 \in \mathbb{R}$, $\psi_{1,2} \in C_{rd}(\mathbb{T}, \mathbb{R})$ and $\alpha_2 + \alpha_1 \neq 0$. Then, x solution to Eqs.* (1.6) *and* (1.7) *iff*

$$\begin{aligned} \chi(t) &= (\alpha_2+\alpha_1)^{-1}[\chi_0 - \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s] \\ &\quad + \int_S^t \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s. \end{aligned} \tag{3.1}$$

Proof. First, assume $\chi : I \to \mathbb{R}$ solution to Eqs. (1.6) and (1.7). Now integrating from S to $t \in I$, we obtain

$$\chi(t) = \chi(S) + \int_S^t \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s, \tag{3.2}$$

and

$$\chi(\sigma(T)) = \chi(S) + \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s.$$

Using the condition (1.7), we obtain

$$\begin{aligned} \alpha_2\chi(\sigma(T)) &= \alpha_2\chi(S) + \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s, \\ \chi_0 - \alpha_1\chi(S) &= \alpha_2\chi(S) + \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s, \end{aligned} \tag{3.3}$$

and rearranging (3.3) yields

$$\chi(S) = (\alpha_2+\alpha_1)^{-1}[\chi_0 - \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s].$$

Hence, from (3.2) and (3.3), we conclude that x satisfies equation (3.1).
Conversely, suppose that χ satisfies equation (3.1).
Taking the $\Delta-$derivatives on both sides of equation (3.1), we obtain the equation (1.6). Now, we calculate $\chi(\sigma(T))$ and $\chi(S)$ from equation (3.1)

$$\begin{aligned} \alpha_2\chi(\sigma(T)) &= \alpha_2(\alpha_2+\alpha_1)^{-1}[\chi_0 - \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s] \\ &\quad +\alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s, \\ \alpha_1\chi(S) &= \alpha_1(\alpha_2+\alpha_1)^{-1}[\chi_0 - \alpha_2 \int_S^{\sigma(T)} \psi_1\Big(s, \chi(\int_S^s \psi_2(\theta, \chi(\theta))\Delta\theta)\Big)\Delta s]. \end{aligned}$$

Therefor

$$\alpha_2\chi(\sigma(T)) + \alpha_1\chi(S) = \chi_0.$$

Hence, χ satisfies Eqs. (1.6) and (1.7). □

For the main theorems presented below, we employ the notation

$$\mathbb{E} := C(I,\mathbb{R}),\ d(\chi,y) = \sup_{t\in I}|\chi(t)-y(t)| \text{ for } \chi,y\in\mathbb{E},\ \|\chi\| := d(x,0) \text{ for } \chi\in\mathbb{E},$$

and we introduce $\Phi : \mathbb{E}\to\mathbb{E}$ by

$$\begin{aligned}\Phi(\chi)(t) &= (\alpha_2+\alpha_1)^{-1}[\chi_0 - \alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s]\\ &\quad + \int_S^t\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s,\ t\in I,\ x\in\mathbb{E}.\end{aligned}$$

In the following theorem, we obtain the existence of solutions to Eqs. (1.6) and (1.7) applying Theorem 2.1.

Theorem 3.1. *Let $\psi_{1,2}\in C_{rd}(\mathbb{T}\times\mathbb{R},\mathbb{R})$. If there exists $N_1>0$, with*

$$|\psi_1(t,u,v)| \le N_1(1+|u|),\ |\psi_2(t,w)|\le 1 \text{ and } (|\alpha_2|+1)\sigma(T)N_1 < 1, \tag{3.4}$$

for all $u,\ v,\ w\in\mathbb{R},\ t\in I$, then Eqs. (1.6) *and* (1.7) *has at least one solution.*

Proof. In a first step, we demonstrate that $\Phi:\mathbb{E}\to\mathbb{E}$ is continuous.
Let $\{x_n : n\in\mathbb{N}\}\subset\mathbb{E}$ be such that $\chi_n\to\chi\in\mathbb{E}$ as $n\to\infty$. Then, for $t\in I$, we find

$$\begin{aligned}|\Phi(\chi_n)(t)-\Phi(\chi)(t)| &= \Big|(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)\Delta s]\\ &\quad + \int_S^t\psi_1\Big(s,\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)\\ &\quad -(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s]\\ &\quad -\int_S^t\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s\Big|\\ &\le \Big|(\alpha_2+\alpha_1)^{-1}\alpha_2\Big|\int_S^{\sigma(T)}\Big|\psi_1\Big(s,\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)\\ &\quad -\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Big|\Delta s] + \int_S^t\Big|\psi_1\Big(s,\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)\\ &\quad -\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Big|\Delta s,\end{aligned}$$

hence

$$\begin{aligned}&|\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)-\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)|\\ &\qquad\le|\chi_n(\int_S^s\psi_2(\theta,\chi_n(\theta))\Delta\theta)-\chi_n(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)|\end{aligned}$$

$$
\begin{aligned}
&+|\chi_n(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)| \\
&\le L\int_S^s |\psi_2(\theta,\chi_n(\theta))-\psi_2(\theta,\chi(\theta))|\Delta\theta+\frac{\epsilon}{2} \\
&\le \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
d(\Phi(\chi_n)(t),\Phi(\chi)(t)) \;\le\; & \left|(\alpha_2+\alpha_1)^{-1}\alpha_2\right|\int_S^{\sigma(T)}\left|\psi_1\Big(s,\chi_n(\int_S^s \psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)\right. \\
& \left.-\psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\right|\Delta s] \\
& +\int_S^t \left|\psi_1\Big(s,\chi_n(\int_S^s \psi_2(\theta,\chi_n(\theta))\Delta\theta)\Big)-\psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\right|\Delta s.
\end{aligned}
$$

Due to the imposed conditions on ψ_1 and ψ_2, we have $\Phi(\chi_n)\to\Phi(\chi)$ as $n\to\infty$. Hence, $\Phi:\mathbb{E}\to\mathbb{E}$ is indeed continuous. In a second step, we show that the class of functions $\{\Omega\chi\}$ is uniformly bounded and equi-continuous in Ω, where the set Ω is defined by

$$
\Omega=\{\chi\in\mathbb{E}:|\chi(t_1)-\chi(t_2)|\le L(t_1-t_2)\ \text{ for some } L>0\},
$$

where

$$
L=\frac{N_1+(|\alpha_2+\alpha_1|)^{-1}[N_1|\chi_0|]}{1-(|\alpha_2|+1)\sigma(T)N_1}.
$$

Now, let $x\in\Omega$. Then, for $t\in I$, we obtain

$$
\begin{aligned}
|\Phi(\chi)(t)| \;=\; & \left|(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s]\right. \\
& \left.+\int_S^t \psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s\right| \\
\le\; & |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}\left|\psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\right|\Delta s] \\
& +\int_S^t\left|\psi_1\Big(s,\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\right|\Delta s \\
\le\; & |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}\Big|N_1\Big(1+|\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(S)|+|\chi(S)|\Big)\Delta s] \\
& +\int_S^t N_1\Big(1+|\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(S)|+|\chi(S)|\Big)\Delta s \\
\le\; & |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}\Big|N_1\Big(1+L\int_S^s |\psi_2(\theta,\chi(\theta))\Delta\theta|+|\chi(S)|\Big)\Delta s] \\
& +\int_S^t N_1\Big(1+L\int_S^s |\psi_2(\theta,\chi(\theta))|\Delta\theta+|\chi(S)|\Big)\Delta s
\end{aligned}
$$

$$\leq \quad |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}\Big|N_1\Big(1+LT+|\chi(S)|\Big)\Delta s]$$
$$+\int_S^t N_1\Big(1+LT+|\chi(S)|\Big)\Delta s. \tag{3.5}$$

But

$$\begin{aligned}
|\chi(S)| &\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)|\Delta s]\\
&\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}N_1\Big(1+|\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(S)|+|\chi(S)|\Big)|\Delta s]\\
&\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}N_1\Big(1+L\int_S^s|\psi_2(\theta,\chi(\theta))\Delta\theta|+|\chi(S)|\Big)\Delta s]\\
&\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\int_S^{\sigma(T)}N_1\Big(1+LT+|\chi(S)|\Big)\Delta s]\\
&\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|N_1\sigma(T)\Big(1+LT+|\chi(S)|\Big)]\\
&\leq |(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\sigma(T)L].
\end{aligned}$$

Hence

$$|\chi(S)|\leq\frac{|(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|\sigma(T)N_1(1+LT)]}{1-|\alpha_2|\sigma(T)N_1}.$$

using (3.5) and (3.6), we obtain

$$|\Phi(\chi)(t)|\leq|(\alpha_2+\alpha_1)^{-1}|[|\chi_0|+|\alpha_2|L\sigma(T)]+LT.$$

Let $t_1,\ t_2\in I$ with $t_1\leq t_2$, we obtain

$$\begin{aligned}
&|\Phi(\chi)(t_2)-\Phi(\chi)(t_1)| = \Big|(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s]\\
&+\int_S^{t_2}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s\\
&-(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s]\\
&-\int_S^{t_1}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Delta s\Big|\\
&\leq\int_{t_1}^{t_2}\Big|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\Big|\Delta s\\
&\leq\int_{t_1}^{t_2}N_1\Big(1+\Big|\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(S)\Big|+|\chi(S)|\Big)\Delta s\\
&\leq\int_{t_1}^{t_2}N_1\Big(1+L\int_S^s\Big|\psi_2(\theta,\chi(\theta))\Big|\Delta\theta+|\chi(S)|\Big)\Delta s
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{t_1}^{t_2} N_1\left(1+LT+\frac{(\alpha_2+\alpha_1)^{-1}[|\chi_0|+|\alpha_2|\sigma(T)N_1(1+LT)]}{1-|\alpha_2|\sigma(T)N_1}\right)\Delta s\\
&\leq (t_1-t_2)\frac{N_1+N_1LT-|\alpha_2|\sigma(T)N_1^2-|\alpha_2|N_1^2L\sigma(T)T+(\alpha_2+\alpha_1)^{-1}[N_1|\chi_0|+|\alpha_2|\sigma(T)N_1^2(1+LT)]}{1-|\alpha_2|\sigma(T)N_1}\\
&=(t_1-t_2)L.
\end{aligned} \tag{3.6}$$

If $t_1 \leq t_2$, then a similar calculation leads to the same result. Altogether, for any $t_1, t_2 \in I$, we have

$$|\Phi(\chi)(t_2)-\Phi(\chi)(t_1)| \leq (t_2-t_1)L.$$

As $t_2-t_1 \to 0$, the right-hand side of this inequality tends to zero. Thus, this proves that $\Phi : \Omega \to \Omega$; the class of functions $\{\Phi\chi\}$ is uniformly bounded and equi-continuous in Ω, by Theorem 2.1, Φ has a fixed point. □

In the next theorem, we establish the existence exactly one solution to Eqs. (1.6) and (1.7).

Theorem 3.2. *Let $\psi_{1,2} \in C_{rd}(\mathbb{T}\times\mathbb{R},\mathbb{R})$. If there exists $L_1, L_2 > 0$ with*

$$\begin{aligned}
&|\psi_{1,2}(t,u_1)-\psi_{1,2}(t,u_2)| \leq L_{1,2}|u_1-u_2|, \\
&\textit{for all } u_1,\ u_2,\ v_1,\ v_2 \in \mathbb{R},\ t \in I,
\end{aligned} \tag{3.7}$$

then Eqs. (1.6) *and* (1.7) *has a unique solution.*

Proof. Let the assumptions of Theorem 3.1 are satisfied. Then the solution of Eqs. (1.6) and (1.7) exists. Now, let χ, y be two the solutions of (1.6) and (1.7). Then

$$\begin{aligned}
|\chi(t)-y(t)| \ &= \ \Bigg|(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta\Big)\Delta s]\\
&\quad+\int_S^t\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta\Big)\Delta s\\
&\quad-(\alpha_2+\alpha_1)^{-1}[\chi_0-\alpha_2\int_S^{\sigma(T)}\psi_1\Big(s,y(\int_S^s\psi_2(\theta,y(\theta))\Delta\theta\Big)\Delta s]\\
&\quad-\int_S^t\psi_1\Big(s,y(\int_S^s\psi_2(\theta,y(\theta))\Delta\theta\Big)\Delta s\Bigg|\\
&\leq \ (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)}|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta\Big)\\
&\quad-\psi_1\Big(s,y(\int_S^s\psi_2(\theta,y(\theta))\Delta\theta\Big)|\Delta s\\
&\quad+\int_S^t\Bigg|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta\Big)-\psi_1\Big(s,y(\int_S^s\psi_2(\theta,y(\theta))\Delta\theta\Big)\Bigg|\Delta s\\
&\leq \ (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)}L_1|\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)-y(\int_S^s\psi_2(\theta,y(\theta))\Delta\theta)|\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+\int_S^t L_1|\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-y(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)|\Delta s\\
\leq\ & (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)} L_1[|\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)|\\
&+|\chi(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)-y(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)|]\Delta s\\
&+\int_S^t L_1[|\chi(\int_S^s \psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)|\\
&+|\chi(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)-y(\int_S^s \psi_2(\theta,y(\theta))\Delta\theta)|]\Delta s\\
\leq\ & (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)} L_1[L\int_S^s |\psi_2(\theta,\chi(\theta))-\psi_2(\theta,y(\theta))|\Delta\theta+d(\chi,y)]\Delta s\\
&+\int_S^t L_1[L\int_S^s |\psi_2(\theta,\chi(\theta))-\psi_2(\theta,y(\theta))|\Delta\theta+d(\chi,y)]\Delta s\\
\leq\ & (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)} L_1[LL_2\int_S^s |\chi(\theta)-y(\theta)|\Delta\theta+d(\chi,y)]\Delta s\\
&+\int_S^t L_1[LL_2\int_S^s |\chi(\theta)-y(\theta)|\Delta\theta+d(\chi,y)]\Delta s\\
\leq\ & (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|(L_1LL_2T+L_1)d(\chi,y)(\sigma(T)-S)\\
&+(L_1LL_2T+L_1)d(\chi,y)(t-S)\\
\leq\ & (|\alpha_2+\alpha_1|)^{-1}|\alpha_2|Bd(\chi,y)(\sigma(T)-S)+B\,d(\chi,y)(t-S).
\end{aligned}
$$

Therefor

$$d(\chi,y)\leq[(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|B\sigma(T)+BT]d(\chi,y),$$

where $B=L_1LL_2T+L_1$. Since $[(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|B\sigma(T)+BT]<1$, which implies $\chi(t)=y(t)$ and the solution of Eqs. (1.6) and (1.7) is unique. □

Below we state and prove the result on the dependency of solutions to Eqs. (1.6) and (1.7) with respect to χ_0.

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied and the solution χ to Eqs.* (1.6) *and* (1.7), *depends continuously on χ_0, if*

$$\forall\epsilon>0,\quad \exists\quad \delta(\epsilon)\quad s.t\quad |\chi_0-\chi_0^*|<\delta\ \Rightarrow\ d(\chi,\chi^*)<\epsilon,$$

where χ^* is the solution of the dynamic boundary value problem

$$\chi^{*\Delta}=\psi_1\Big(t,\chi^*(\int_S^t \psi_2(s,\chi^*(s))\Delta s)\Big),\quad t\in I^k, \tag{3.8}$$

with condition

$$\alpha_2\chi^*(S)+\alpha_1\chi^*(\sigma(T))=\chi_0^*. \tag{3.9}$$

Proof. Let $\chi,\ \chi^*$ be two solutions of to Eqs. (1.6), (1.7), (3.8) and (3.9) respectively. Then

$$\begin{aligned}
&|\chi(t)-\chi^*(t)|\\
\leq\ & \frac{|\chi_0-\chi_0^*|}{\alpha_2+\alpha_1}+(\alpha_2+\alpha_1)^{-1}\alpha_2\int_S^{\sigma(T)}|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)\\
&-\psi_1\Big(s,x^*(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)\Big)|\Delta s\\
&+\int_S^t\Big|\psi_1\Big(s,\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)\Big)-\psi_1\Big(s,\chi^*(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)\Big)\Big|\Delta s\\
\leq\ & \frac{|\chi_0-\chi_0^*|}{\alpha_2+\alpha_1}+(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)}L_1[|\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)|\\
&+|\chi(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)-y(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)|]\Delta s\\
&+\int_S^t L_1[|\chi(\int_S^s\psi_2(\theta,\chi(\theta))\Delta\theta)-\chi(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)|\\
&+|\chi(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)-\chi^*(\int_S^s\psi_2(\theta,\chi^*(\theta))\Delta\theta)|]\Delta s\\
\leq\ & \frac{\delta}{\alpha_2+\alpha_1}+(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)}L_1[L\int_S^s|\psi_2(\theta,\chi(\theta))-\psi_2(\theta,\chi^*(\theta))|\Delta\theta+d(x,\chi^*)]\Delta s\\
&+\int_S^t L_1[L\int_S^s|\psi_2(\theta,\chi(\theta))-\psi_2(\theta,\chi^*(\theta))|\Delta\theta+d(\chi,\chi^*)]\Delta s\\
\leq\ & \frac{\delta}{\alpha_2+\alpha_1}+(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|\int_S^{\sigma(T)}L_1[LL_2\int_S^s|\chi(\theta)-\chi^*(\theta)|\Delta\theta+d(\chi,\chi^*)]\Delta s\\
&+\int_S^t L_1[LL_2\int_S^s|\chi(\theta)-\chi^*(\theta)|\Delta\theta+d(\chi,\chi^*)]\Delta s\\
\leq\ & \frac{\delta}{\alpha_2+\alpha_1}+(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|(L_1LL_2T+L_1)d(\chi,\chi^*)(\sigma(T)-S)\\
&+(L_1LL_2T+L_1)d(\chi,\chi^*)(t-S)\\
\leq\ & \frac{\delta}{\alpha_2+\alpha_1}+(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|Bd(\chi,\chi^*)(\sigma(T)-S)+Bd(\chi,\chi^*)(t-S),
\end{aligned}$$

therefor

$$d(\chi,\chi^*)\leq\frac{(\alpha_2+\alpha_1)^{-1}\,\delta}{1-[(|\alpha_2+\alpha_1|)^{-1}|\alpha_2|B\sigma(T)+BT]}.$$

This mean that the solution to Eqs. (1.6) and (1.7). depends continuously on χ_0. The proof is completed. □

Below, we offer one example to illustrate our results.

Example 3.1. *Let $\mathbb{T}:=[0,1]\cup[2,3]$. Consider the dynamic boundary value problem*

$$x^{\Delta}=\frac{1}{4}t^3+\frac{1}{t^2+4}\left(\chi\left(\int_0^t\frac{|\chi(s)|}{1+|\chi(s)|}\Delta s\right)\right),\quad t\in\mathbb{T}^k, \tag{3.10}$$

with the condition

$$0.75\chi(S) + 0.25\chi(\sigma(T)) = \chi_0. \tag{3.11}$$

A delta integral equation for (3.10) *and* (3.11)

$$\begin{aligned}\chi(t) &= \chi_0 - 0.25\int_S^{\sigma(T)}\Big[\frac{1}{4}\theta^4 + \frac{1}{\theta^2+3}\Big(\chi\Big(\int_S^{\theta}\frac{|\chi(s)|}{1+|\chi(s)|}\Delta s\Big)\Big)\Big]\Delta\theta \\ &\quad + \int_S^{t}\Big[\frac{1}{4}\theta^3 + \frac{1}{\theta^2+4}\Big(\chi\Big(\int_S^{\theta}\frac{|\chi(s)|}{1+|\chi(s)|}\Delta s\Big)\Big)\Big]\Delta\theta. \end{aligned}\tag{3.12}$$

Set

$$\psi_1\Big(t, x\Big(\int_S^t \psi_2(s,\chi(s))\mathrm{d}s\Big)\Big) = \frac{1}{4}t^3 + \frac{1}{t^2+4}\Big(\chi\Big(\int_S^t \frac{|\chi(s)|}{1+|\chi(s)|}\Delta s\Big)\Big).$$

Then

$$\psi_1\Big(t, x\Big(\int_S^t \psi_2(s,\chi(s))\mathrm{d}s\Big)\Big) \le \frac{1}{3}(1+(|x|)),$$

and also

$$|\psi_2(s,\chi(s))| \le 1.$$

It is clear that the assumptions of Theorem 3.1 are satisfied with $N_1 = \frac{1}{4}$, $(|\alpha_2|+1)N_1\sigma(T) = (0.25+1)\times\frac{1}{4}\times 3 = \frac{15}{16} < 1$. *Therefore, by applying to Theorem 3.1, the given dynamic boundary value problem* (3.10)–(3.11) *has a solution given by the delta integral equation* (3.12).

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