Int. J. Anal. Appl. (2025), 23:87

International Journal of Analysis and Applications

Some New Versions of Various Inequalities over Trapezoidal Fuzzy Codomain

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ABSTRACT. Considerable attention has been given to Hölder's inequality, its extensions, and its reverse within the realms of differential equations and mathematical analysis. This study uses a new approach to find the novel version of Hölder's inequality by employing a fundamental analytical approach rooted in algebra and calculus known as trapezoidal fuzzy Hölder's inequality. With the help of Hölder's inequality, trapezoidal fuzzy Minkowski's inequality and trapezoidal fuzzy Beckenbach's inequality are also obtained. As specific examples of the inequalities mentioned earlier, our results illustrate various outcomes related to trapezoidal fuzzy Hölder's inequality. These outcomes show that the behavior of these inequalities is better than the classical results. For the validation of the results, some examples are also provided.

1. Introduction

A key idea in mathematical analysis, Hölder's inequality [1] extends the Cauchy-Schwarz inequality to numerous sequences and variable exponents. Differential equations, probability theory, and functional analysis are just a few of the mathematical disciplines that use it, see [2]. This inequity has undergone numerous revisions and reversals over time, see [3]. For example,

Received Feb. 7, 2025

2020 Mathematics Subject Classification. 26A33, 26A51, 26D10.

Key words and phrases. Hölder's inequality over trapezoidal fuzzy $\mathbb{L}_P$ space; Minkowski's inequality; Beckenbach's inequality.

https://doi.org/10.28924/2291-8639-23-2025-87

ISSN: 2291-8639

the square (or higher-power) roots of expressions in inequalities are handled by the reverse Hölder inequality, which eliminates them by repeated multiplication, see [4]. In many branches of mathematics, both the Cauchy-Schwarz inequality and Hölder's inequality are important tools for carrying out analysis.

Many researchers have explored and developed ways to generalize, refine, sharpen, and apply this inequality in the literature. Cauchy [5] introduced a class of inequalities involving inner products of vectors and functions. Subsequently, these generalizations and extensions led to the Schwarz, Minkowski, and Hölder inequalities. Mathematicians and scientists employ inequalities as crucial tools to describe relationships, establish boundaries, and solve various problems. Inequalities are prevalent in numerous branches of mathematics. They are frequently encountered in algebra, geometry, and analysis, see [6], and [7].

Hölder's inequality has significant implications in functional analysis, probability theory, and partial differential equations. It is particularly useful for establishing connections between different function spaces, estimating the norms of integral operators, and proving the convergence properties of function sequences. Hölder's inequality, a crucial concept in mathematics, relates norms, integrals, and the inner products of functions and vectors. By adjusting the exponents or incorporating additional terms, Hölder's inequality can be refined to provide more precise upper bounds in specific situations, see [8]. These enhancements are particularly useful when dealing with specific types of functions or when more information about the functions is available. By customizing the inequality, these enhancements yield more accurate estimates and highlight complex interactions between functions.

On the other hand, Hölder's inequality reversals seek to provide lower bounds for the supplied phrase (see [9],[10]). A reverse inequality gives information on the lowest possible value, whereas the original inequality gives an upper bound. Reversals aid in illuminating Hölder's inequality's optimality and clarifying the accuracy of the constraints it establishes. They are especially useful in defining situations in which there are particular interdependencies across functions, see [11].

Hölder's inequality is integral to various contemporary mathematical fields, such as fuzzy measure theory, real and complex analysis, probability and statistics, qualitative theory of differential equations and their applications, and numerical analysis. This classical inequality has been extensively studied by many authors, forming the basis for numerous research papers that offer different proofs, generalizations, variations, and applications (see, for example, [12], [13], [14] and [15] and references therein). Notable sources on this inequality include works by Agahi et al. [16], [17] and [18], Flores-Franulić and Román-Flores [19], Hu [20], Khrennikov [21], Mesiar and Mesiarová [22], Mesiar and Ouyang [23], Ouyang and Mesiar [24], [25], Ouyang et al. [26], Román-Flores et al. [27], [28], and Wu [29]. Additionally, various authors have explored the applications of Hölder's inequality in the information sciences. For instance, Li and Wu [30] introduced a distinct class of Cohen-Grossberg neural networks with delays, employing inverse Hölder neuron activation functions. Özkan et al. [31] applied Hölder's inequality to temporal scales. Singh et al. [32] utilized Hölder's inequality to establish two coding theorems for the R-norm information measure.

Khan et al. have developed new forms of fuzzy integral inequalities using fuzzy fractional integrals, establishing a connection between inclusion relations and UD fuzzy relations [33], [34]. Furthermore, several compelling examples are provided to support the validity of these findings. For more information on this topic, refer to the cited study. For additional details on fuzzy theory, see [35], [36], [37], [38], [39] and the referenced works. Khastan and Rodríguez-López [40] recently introduced real-valued functions across the fuzzy domain and explored specific properties of such functions over fuzziness using Lebesgue measures. After that Khan and Guirao [41] extended this version of integral to fractional integrals that are known as Riemann-Liouville fractional-like integrals over fuzzy domain. Moreover, the properties of convex-like functions over fuzzy domain are discussed. For more information, see [42], [43], [44], [45], [46], [47], [48], [49], [50], [51], [52], [53], [54], [55], respectively, and the references therein.

Encouraged and driven by ongoing research, this study is organized as follows. Section 2 defines preliminary concepts such as young inequality, Hölder's inequality, fuzzy number ($F \cdot N$), and explains integral over trapezoidal fuzzy number ($T \cdot F \cdot N$) of a real valued mapping. In Section 3, the main results are discussed such that a new version of the Hölder's integral inequality over $T \cdot F \cdot Ns$ is introduced, where integrable functions are real valued functions over $T \cdot F \cdot N$s. Moreover, some classical and new exceptional cases are also acquired. Section 4 discusses the trapezoidal fuzzy Minkowski's and, Beckenbach's-type integral inequalities in the context of real valued functions over $T \cdot F \cdot N$s. Furthermore, a nontrivial case is given to explain the applicability of trapezoidal fuzzy logic and the validity of the primary results. It also proves Hölder's integral inequality. Finally, Section 5 provides concluding remarks.

2. Preliminaries

For two scalar quantities, the -weighted arithmetic–geometric mean inequality corresponds to the widely recognized Young's inequality. When both $ʐ$ and y are positive, and $ɤ$ lies within the interval $[0, 1]$, the relationship $ʐ^{ɤ} y^{1-ɤ} \leq ɤʐ + (1-ɤ)y$

$$ʐ^{ɤ} y^{1-ɤ} \leq ɤʐ + (1-ɤ)y, \tag{2.1}$$

holds true, (2.1) with equality occurring exclusively when $ʐ = y$, as indicated by this inequality. Let $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Inequality (2.1) can be written as

$$ʐy \leq \frac{ʐ^p}{p} + \frac{y^q}{q}, \tag{2.2}$$

where the both real numbers $\mathfrak{z}$ and y are positive.
In this formulation, inequality (2.2) was used to derive the famous Hölder's inequality, a basic inequality in mathematical analysis. It has many uses in both theoretical and practical mathematics and is a vital instrument for solving many issues in the social, cultural, and natural sciences.

Theorem 2.1. (Hölder's inequality [1]) Suppose $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{Q}$ and $\mathcal{G}$ are two real functions stated on $[\mathfrak{a}, \mathfrak{b}]$ such that $|\mathcal{Q}|^p$ and $|\mathcal{G}|^q$ are integrable functions on $[\mathfrak{a}, \mathfrak{b}]$, then,

$$\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{Q}(\mathfrak{z})\mathcal{G}(\mathfrak{z})| \, d\mathfrak{z} \leq \left[\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{Q}(\mathfrak{z})|^p \, d\mathfrak{z}\right]^{\frac{1}{p}} \left[\int_{\mathfrak{a}}^{\mathfrak{b}} |\mathcal{G}(\mathfrak{z})|^q \, d\mathfrak{z}\right]^{\frac{1}{q}}, \tag{2.3}$$

with equality if and only if $\mathcal{Q}$ and $\mathcal{G}$ are proportional.

Fuzzy theory

Definition 2.1. [38] A mapping $\widetilde{\mathcal{D}}: \mathbb{R} \to [0,1]$, also referred to as the membership mapping of T, is what defines a fuzzy subset T of $\mathbb{R}$. Therefore, we use this nomenclature going forward for our research. To denote the collection of all fuzzy subsets of $\mathbb{R}$, we call it $\mathbb{E}$. Fuzzy numbers' central concept was first presented by Goetschel and Voxman in [37] and goes as follows:

Assume $\widetilde{\mathcal{D}}$ to be in $\mathbb{E}$. Then, if $\widetilde{\mathcal{D}}$ meets the following criteria, it is identified as a $F{\cdot}N$ or fuzzy interval:

(1) $\widetilde{\mathcal{D}}$ should be normal if there exists $\mathfrak{z} \in \mathbb{R}$ and $\widetilde{\mathcal{D}}(\mathfrak{z}) = 1$;

(2) $\widetilde{\mathcal{D}}$ should be upper semi-continuous on $\mathbb{R}$ if for given $\mathfrak{z} \in \mathbb{R}$, and $\varepsilon > 0$ there exist $\delta > 0$ such that $\widetilde{\mathcal{D}}(\mathfrak{z}) - \widetilde{\mathcal{D}}(y) < \varepsilon$ for all $y \in \mathbb{R}$ with $|\mathfrak{z} - y| < \delta$;

(3)$\widetilde{\mathcal{D}}$ should be fuzzy convex, meaning $\widetilde{\mathcal{D}}\big((1-\upsilon)\mathfrak{z} + \upsilon y\big) \geq min\left(\widetilde{\mathcal{D}}(\mathfrak{z}), \widetilde{\mathcal{D}}(y)\right)$, for all $\mathfrak{z}, y \in \mathbb{R}$, and $\upsilon \in [0, 1]$;

(4) $\widetilde{\mathcal{D}}$ should be compactly supported, i.e., $cl\{\mathfrak{z} \in \mathbb{R} |\ \widetilde{\mathcal{D}}(\mathfrak{z}) > 0\}$ is compact.

We designate $\mathbb{E}_C$ to represent the set of all $F{\cdot}N$s of $\mathbb{R}$.

Definition 2.2. [38] Given $\widetilde{\mathcal{D}} \in \mathbb{E}_C$, the level sets or cut sets are stated as $\left[\widetilde{\mathcal{D}}\right]^{\lambda} = \{\mathfrak{z} \in \mathbb{R} |\ \widetilde{\mathcal{D}}(\mathfrak{z}) > \lambda\}$ for all $\lambda \in [0, 1]$.

From these definitions, we have

$$\left[\widetilde{\mathcal{D}}\right]^{\lambda} = [\Lambda(\lambda), \upsilon(\lambda)], \tag{2.4}$$

Where,

$$\Lambda(\lambda) = inf\{\mathfrak{z} \in \mathbb{R} |\ \widetilde{\mathcal{D}}(\mathfrak{z}) \geq \lambda\},$$
$$\upsilon(\lambda) = sup\{\mathfrak{z} \in \mathbb{R} |\ \widetilde{\mathcal{D}}(\mathfrak{z}) \geq \lambda\}.$$

Remark 2.1. [39] Let $\mathcal{X}_C$ be the set of intervals. Then, for each interval $[\mathfrak{a}, \mathfrak{b}] \in \mathcal{X}_C$, the characteristic function $\widetilde{[\mathfrak{a}, \mathfrak{b}]}: \mathbb{R} \to [0,1]$ is stated by

$$\widetilde{[\mathfrak{a}, \mathfrak{b}]}(\mathfrak{z}) = \begin{cases} 1, & \mathfrak{z} \in [\mathfrak{a}, \mathfrak{b}] \\ 0, & \text{otherwise.} \end{cases} \tag{2.5}$$

Degraded intervals, being interpretable as real numbers, allow us to consider $F{\cdot}N$s as a generalization of the set of closed intervals of real numbers, i.e., $\mathcal{X}_C \subseteq \mathbb{E}_C$ and, consequently, $\mathbb{R} \subseteq \mathbb{E}_C$. Instead of denoting $[\text{б}, \text{б}]$, we simply use $\tilde{\text{б}}$. As stated in [39], a $F{\cdot}N$ $\tilde{\text{б}}$ is also referred to as a fuzzy singleton or a crisp number.

Keeping in mind the ideas frequently seen in the literature, if $\widetilde{\mathcal{D}}, \widetilde{\text{Л}} \in \mathbb{E}_C$, then the arithmetic operations are expressed as follows for each $\curlywedge \in [0,1]$:

$$\left[\widetilde{\mathcal{D}} \oplus \widetilde{\text{Л}}\right]^{\curlywedge} = \left[\widetilde{\mathcal{D}}\right]^{\curlywedge} + \left[\widetilde{\text{Л}}\right]^{\curlywedge}, \tag{2.6}$$

$$\left[\widetilde{\mathcal{D}} \otimes \widetilde{\text{Л}}\right]^{\curlywedge} = \left[\widetilde{\mathcal{D}}\right]^{\curlywedge} \times \left[\widetilde{\text{Л}}\right]^{\curlywedge}, \tag{2.7}$$

$$\left[\curlyvee \odot \widetilde{\mathcal{D}}\right]^{\curlywedge} = \curlyvee . \left[\widetilde{\mathcal{D}}\right]^{\curlywedge}. \tag{2.8}$$

Theorem 2.2. [38] The space $\mathbb{E}_C$ dealing with a supremum metric, i.e., for $\widetilde{\mathcal{D}}, \widetilde{\text{Л}} \in \mathbb{E}_C$, the expression

$$d_\infty\left(\widetilde{\mathcal{D}}, \widetilde{\text{Л}}\right) = \sup_{0 \le \curlywedge \le 1} d_H\left(\left[\widetilde{\mathcal{D}}\right]^{\curlywedge}, \left[\widetilde{\text{Л}}\right]^{\curlywedge}\right), \tag{2.9}$$

defines a complete metric space, where H refers to the widely recognized Hausdorff metric used in the context of interval spaces.

Let us revisit some of the core concepts of integration over a fuzzy domain, where real-valued functions defined on fuzzy domains are considered integrable functions.

Definition 2.3. [40] If $\widetilde{\mathcal{D}} \in \mathbb{E}_C$, and $\mathcal{Q}: \widetilde{\mathcal{D}} \subseteq \mathbb{R}^n \to \mathbb{R}$ is measurable on $\left[\widetilde{\mathcal{D}}\right]^0$ (and consequently every $\left[\widetilde{\mathcal{D}}\right]^{\curlywedge}$, for all $\curlywedge \in [0,1]$), the definition is established as follows

$$\left(\int_{\widetilde{\mathcal{D}}} \mathcal{Q}\right)(\curlywedge) = \int_{\left[\widetilde{\mathcal{D}}\right]^{\curlywedge}} \mathcal{Q}(\text{z})\, d\text{z}, \tag{2.10}$$

when employing Lebesgue integration to evaluate the integral on the right-hand side, if the integral $\int_{\left[\widetilde{\mathcal{D}}\right]^0} \mathcal{Q}(\text{z})\, d\text{z}$ is finite, it signifies that $\mathcal{Q}$ is integrable over the fuzzy domain. Under these circumstances, the mapping can be represented as follows:

$$\int_{\widetilde{\mathcal{D}}} \mathcal{Q} : [0,1] \to \mathbb{R}$$

$$\curlywedge \to \left(\int_{\widetilde{\mathcal{D}}} \mathcal{Q}\right)(\curlywedge) = \int_{\left[\widetilde{\mathcal{D}}\right]^{\curlywedge}} \mathcal{Q}(\text{z})\, d\text{z}. \tag{2.11}$$

Remark 2.2. Using Remark 2.1, we obtain the classical definition of the integral, which applies to integrable real-valued functions.

3. Trapezoidal fuzzy $\mathbb{L}_p$ space and Hölder's Inequality

In this study, we obtain new versions of integral forms of Hölder's inequality using a simple proof procedure.

Conversely, taking into account that the $T{\cdot}F{\cdot}N$s $\widetilde{\mathcal{D}} = (\text{ŧ}, \text{ɳ};\ \text{л}, \mu)$, where $\text{ŧ}, \text{ɳ} \in \mathbb{R}$, and $\text{л}, \mu \in \mathbb{R}$, thus

$$\widetilde{\mathcal{D}}(\text{z}) = \begin{cases} 1, & \upsilon \in [\text{ŧ}, \text{ɳ}] \\ \frac{\upsilon - \text{ŧ} + \text{л}}{\text{л}}, & \upsilon \in [\text{ŧ} - \text{л}, \text{ŧ}] \\ \frac{\text{ɳ} + \mu - \upsilon}{\mu}, & \upsilon \in [\text{ɳ}, \text{ɳ} + \mu] \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

The accompanying figure illustrates the geometric depiction of $T \cdot F \cdot N$s:

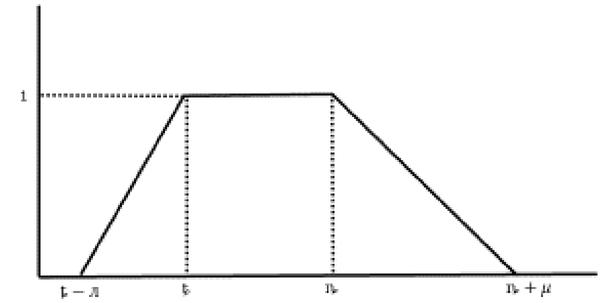

Figure 1: Trapezoidal fuzzy number

with a parametrized form of

$$\left[\widetilde{\mathcal{D}}\right]^{⋋} = [ŧ - л(1-⋋), ɳ + \mu(1-⋋)], \text{ for all } ⋋ \in [0,1]. \quad (3.2)$$

Then

$$\mathbb{L}_P\left[\widetilde{\mathcal{D}}\right] = \Big\{\mathcal{Q} \mid \mathcal{Q}: [ŧ - л(1-⋋), ɳ + \mu(1-⋋)] \to \mathbb{R} \text{ is measurable on}\big[[ŧ - л(1-⋋), ɳ + \mu(1-⋋)]\big] \text{ and } \int_{[[ŧ-л(1-⋋), ɳ+\mu(1-⋋)]]} \mathcal{Q}(ȥ)\, dȥ < \infty, \text{for all } ⋋ \in [0,1]\Big\}. \quad (3.3)$$

Remark 3.1. Utilizing Remark 2.1 and Remark 2.2, we derive the classical $\mathbb{L}_P\widetilde{[\mathfrak{a},\mathfrak{b}]}$ space.

The following result discusses the new version of Hölder's inequality.

Theorem 3.1. (Trapezoidal fuzzy Hölder's inequality) Suppose $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If $\mathcal{Q}$ and $\mathcal{G}$ are two real functions stated on $\left[\widetilde{\mathcal{D}}\right]^{⋋} = [ŧ - л(1-⋋), ɳ + \mu(1-⋋)]$ such that $|\mathcal{Q}|^p$ and $|\mathcal{G}|^q$ are integrable functions on $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, then, for each $⋋ \in [0,1]$

$$\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(ȥ)\mathcal{G}(ȥ)|\, dȥ \leq \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(ȥ)|^p\, dȥ\right]^{\frac{1}{p}} \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{G}(ȥ)|^q\, dȥ\right]^{\frac{1}{q}}, \quad (3.4)$$

with equality if and only if $\mathcal{Q}$ and $\mathcal{G}$ are proportional.

Proof. Since $\left[\widetilde{\mathcal{D}}\right]^0 = [ŧ - л, ɳ + \mu]$ (and consequently every $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, for all $⋋ \in [0,1]$)

$\int_{[\widetilde{\mathcal{D}}]^0} |\mathcal{Q}(ȥ)\mathcal{G}(ȥ)|\, dȥ = \int_{ŧ-л}^{ɳ+\mu} |\mathcal{Q}(ȥ)\mathcal{G}(ȥ)|\, dȥ$, on every $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, for all $⋋ \in [0,1]$.

Case I. If $\eta = \left[\int_{ŧ-л}^{ɳ+\mu} |\mathcal{Q}(ȥ)|^p\, dȥ\right]^{\frac{1}{q}} = 0$, and $\xi = \left[\int_{ŧ-л}^{ɳ+\mu} |\mathcal{G}(ȥ)|^q\, dȥ\right]^{\frac{1}{q}} = 0$, it is obvious that equality will hold because functions $\mathcal{Q}$ and $\mathcal{G}$ are measurable on $\left[\widetilde{\mathcal{D}}\right]^{⋋} = [ŧ - л(1-⋋), ɳ + \mu(1-⋋)]$.

Case II. Consider $\eta = \left[\int_{ŧ-л}^{ɳ+\mu} |\mathcal{Q}(ȥ)|^p\, dȥ\right]^{\frac{1}{q}} \neq 0$, and $\xi = \left[\int_{ŧ-л}^{ɳ+\mu} |\mathcal{G}(ȥ)|^q\, dȥ\right]^{\frac{1}{q}} \neq 0$ (and consequently every $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, for all $⋋ \in [0,1]$). Let $\mathcal{u} = \frac{|\mathcal{Q}(ȥ)|}{\eta}, \mathcal{v} = \frac{|\mathcal{G}(ȥ)|}{\xi}$, then by using inequality (2.2) we have

$$\frac{|\mathcal{Q}(ȥ)||\mathcal{G}(ȥ)|}{\eta\xi} \leq \frac{|\mathcal{Q}(ȥ)|^p}{p\eta^p} + \frac{|\mathcal{G}|^q}{q\xi^q}.$$

Considering integration over $\left[\widetilde{\mathcal{D}}\right]^0 = [ŧ - л, ɳ + \mu]$ (and consequently every $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, for all $⋋ \in [0,1]$) with respect to $ȥ$, we have

$$\frac{1}{\eta\xi}\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{Q}(\text{ȥ})\mathcal{G}(\text{ȥ})|\,d\text{ȥ} \le \frac{1}{p\eta^{p}}\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{Q}(\text{ȥ})|^{p}\,d\text{ȥ} + \frac{1}{q\xi^{q}}\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{G}|^{q}\,d\text{ȥ},$$

(and consequently every $[\widetilde{\mathcal{D}}]^{⋋}$, for all $⋋\in[0,1]$), which implies that

$$\frac{1}{\eta\xi}\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{Q}(\text{ȥ})\mathcal{G}(\text{ȥ})|\,d\text{ȥ} \le \frac{1}{p\eta^{p}}(\eta^{p}) + \frac{1}{q\xi^{q}}(\xi^{q}),$$

$= \frac{1}{p}+\frac{1}{q} = 1$, (and consequently every $[\widetilde{\mathcal{D}}]^{⋋}$, for all $⋋\in[0,1]$).

Then,

$$\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{Q}(\text{ȥ})\mathcal{G}(\text{ȥ})|\,d\text{ȥ} \le \left[\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{Q}(\text{ȥ})|^{p}\,d\text{ȥ}\right]^{\frac{1}{p}}\left[\int_{\text{ƫ}-\text{л}}^{\text{ɳ}+\mu}|\mathcal{G}(\text{ȥ})|^{q}\,d\text{ȥ}\right]^{\frac{1}{q}},$$

(and consequently every $[\widetilde{\mathcal{D}}]^{⋋}$, for all $⋋\in[0,1]$), which implies that

$$\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ɳ}+\mu(1-⋋)}|\mathcal{Q}(\text{ȥ})\mathcal{G}(\text{ȥ})|\,d\text{ȥ} \le \left[\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ɳ}+\mu(1-⋋)}|\mathcal{Q}(\text{ȥ})|^{p}\,d\text{ȥ}\right]^{\frac{1}{p}}\left[\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ɳ}+\mu(1-⋋)}|\mathcal{G}(\text{ȥ})|^{q}\,d\text{ȥ}\right]^{\frac{1}{q}},$$

for each $⋋\in[0,1]$.

Remark 3.2. If $\text{ɳ} = \text{ƫ}$, then triangular $F\cdot N$s ($T\cdot F\cdot N$s) $\widetilde{\mathcal{D}} = (\text{ƫ};\ \text{л},\mu)$, where $\text{ƫ}\in\mathbb{R}$, and $\text{л},\mu\in\mathbb{R}$, thus

$$\widetilde{\mathcal{D}}(\text{ȥ}) = \begin{cases} \frac{\upsilon-\text{ƫ}+\text{л}}{\text{л}}, & \upsilon\in[\text{ƫ}-\text{л},\text{ƫ}] \\ \frac{\text{ƫ}+\mu-\upsilon}{\mu}, & \upsilon\in(\text{ƫ},\text{ƫ}+\mu] \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

The following is the geometric representation of $T\cdot F\cdot N$s:

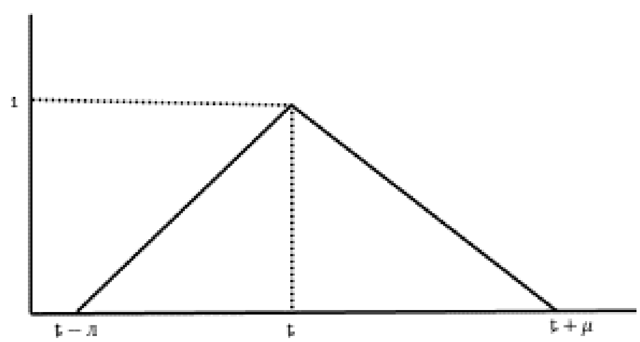

Figure 2: Trapezoidal fuzzy number

with a parametrized form of $[\widetilde{\mathcal{D}}]^{⋋} = [\text{ƫ}-\text{л}(1-⋋), \text{ƫ}+\mu(1-⋋)]$, for all $⋋\in[0,1]$. Then inequality (3.4) simplifies to the Hölder's-like inequality over $T\cdot F\cdot N$ $\widetilde{\mathcal{D}}$ such that

$$\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ƫ}+\mu(1-⋋)}|\mathcal{Q}(\text{ȥ})\mathcal{G}(\text{ȥ})|\,d\text{ȥ} \le \left[\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ƫ}+\mu(1-⋋)}|\mathcal{Q}(\text{ȥ})|^{p}\,d\text{ȥ}\right]^{\frac{1}{p}}\left[\int_{\text{ƫ}-\text{л}(1-⋋)}^{\text{ƫ}+\mu(1-⋋)}|\mathcal{G}(\text{ȥ})|^{q}\,d\text{ȥ}\right]^{\frac{1}{q}}. \tag{3.6}$$

Remark 4.1. If $\widetilde{\mathcal{D}} = \widetilde{[\mathfrak{a},\mathfrak{b}]}$, then from (3.4), we get the classical Hölder's-like inequality (2.3) for real-valued mappings.

4. Applications

When we write $|\mathcal{Q}||\mathcal{G}| = \left(|\mathcal{Q}|^{\frac{1}{p}}\right)\left(|\mathcal{Q}|^{\frac{1}{q}}|\mathcal{G}|\right)$, as a straightforward outcome of the Hölder's Inequality, we have the trapezoidal fuzzy Hölder's power-mean-like integral inequality that follows:

Theorem 4.1. Suppose $p > 1$. If $\mathcal{Q}$ and $\mathcal{G}$ are two real functions stated on $F{\cdot}N$ $\left[\widetilde{\mathcal{D}}\right]^{⋋} = [ŧ - л(1-⋋), ɳ + \mu(1-⋋)]$ such that $|\mathcal{Q}|$ and $|\mathcal{Q}||\mathcal{G}|^q$ are integrable functions on $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, then

$$\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)\mathcal{G}(z)|\, dz \leq \left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)|\, dz\right)^{1-\frac{1}{q}} \left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)||\mathcal{G}(z)|^q\, dz\right)^{\frac{1}{q}}. \tag{4.1}$$

Proof. By using similar arguments as in Theorem 3.1, it can be proved. ∎

If $p = 2 = q$, then we attain the following outcome:

Corollary 1. (Trapezoidal fuzzy Cauchy-Schwarz's inequality) In accordance with the premises of Theorem 3.1, if $p = 2 = q$, then, it is evident that

$$\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)\mathcal{G}(z)|\, dz \leq \left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)|^2\, dz\right)^{\frac{1}{2}} \left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{G}(z)|^2\, dz\right)^{\frac{1}{2}}, \tag{4.2}$$

for each $⋋ \in [0,1]$.

4.1 Trapezoidal fuzzy Minkowski's inequality

In this subsection, we propose Minkowski's inequality over $T{\cdot}F{\cdot}N$.

Theorem 4.2. (Minkowski's inequality) Suppose $p \geq 1$. If $\mathcal{Q}$ and $\mathcal{G}$ are two real functions stated on $\left[\widetilde{\mathcal{D}}\right]^{⋋} = [ŧ - л(1-⋋), ɳ + \mu(1-⋋)]$ such that $|\mathcal{Q}|^p$ and $|\mathcal{G}|^p$ are integrable functions on $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, then, for each $⋋ \in [0,1]$

$$\left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\, dz\right)^{\frac{1}{p}} \leq \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)|^p\, dz\right]^{\frac{1}{p}} + \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{G}(z)|^p\, dz\right]^{\frac{1}{p}}, \tag{4.3}$$

with equality if and only if $\mathcal{Q}$ and $\mathcal{G}$ are proportional.

If $1 > p > 0$, then

$$\left(\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\, dz\right)^{\frac{1}{p}} \geq \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)|^p\, dz\right]^{\frac{1}{p}} + \left[\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{G}(z)|^p\, dz\right]^{\frac{1}{p}}. \tag{4.4}$$

Proof. ***Case I.*** Suppose that $p = 1$. We know that

$$|\mathcal{Q}(z)+\mathcal{G}(z)| \leq |\mathcal{Q}(z)| + |\mathcal{G}(z)|.$$

Considering integration on both sides over $\left[\widetilde{\mathcal{D}}\right]^0 = [ŧ - л, ɳ + \mu]$ (and consequently every $\left[\widetilde{\mathcal{D}}\right]^{⋋}$, for all $⋋ \in [0,1]$), we have

$$\int_{ŧ-л}^{ɳ+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|\, dz \leq \int_{ŧ-л}^{ɳ+\mu} |\mathcal{Q}(z)|\, dz + \int_{ŧ-л}^{ɳ+\mu} |\mathcal{G}(z)|\, dz,$$

which implies that

$$\int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)+\mathcal{G}(z)|\, dz \leq \int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{Q}(z)|\, dz + \int_{ŧ-л(1-⋋)}^{ɳ+\mu(1-⋋)} |\mathcal{G}(z)|\, dz.$$

Case II. Consider that $p > 1$ and that p and q are conjugate indices. Then,

$$\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz = \int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|\,|\mathcal{Q}(z)+\mathcal{G}(z)|^{p-1}\,dz$$
$$= \int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)||\mathcal{Q}(z)+\mathcal{G}(z)|^{p-1}\,dz + \int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)||\mathcal{Q}(z)+\mathcal{G}(z)|^{p-1}\,dz.$$

By using Hölder's-like inequality, we have

$$\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz \le \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{(p-1)q}\,dz\right]^{\frac{1}{q}}$$
$$+\left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}} \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{(p-1)q}\,dz\right]^{\frac{1}{q}}$$
$$=\left(\left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}\right) \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{(p-1)q}\,dz\right]^{\frac{1}{q}},$$

which implies,

$$\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz \le \left(\left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}\right) \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{p}\,dz\right]^{\frac{1}{q}}$$
$$=\left(\left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}\right) \left[\left(\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{p}\,dz\right)^{\frac{1}{p}}\right]^{\frac{p}{q}}.$$

From the above inequality, we have

$$\left(\left(\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz\right)^{\frac{1}{p}}\right)^{p} \left[\left(\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^{p}\,dz\right)^{\frac{1}{p}}\right]^{-\frac{p}{q}}$$
$$\le \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}},$$

implies that

$$\left(\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz\right)^{\frac{1}{p}} \le \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}}^{\text{ռ}+\mu} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}},$$

and consequently every $[\widetilde{\mathcal{D}}]^{\leftthreetimes} = [\text{ե}-\text{л}(1-\leftthreetimes), \text{ռ}+\mu(1-\leftthreetimes)]$, for all $\leftthreetimes \in [0,1]$.
Hence,

$$\left(\int_{\text{ե}-\text{л}(1-\leftthreetimes)}^{\text{ռ}+\mu(1-\leftthreetimes)} |\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz\right)^{\frac{1}{p}} \le \left[\int_{\text{ե}-\text{л}(1-\leftthreetimes)}^{\text{ռ}+\mu(1-\leftthreetimes)} |\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ե}-\text{л}(1-\leftthreetimes)}^{\text{ռ}+\mu(1-\leftthreetimes)} |\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}.$$

For the proof of inequality (4.4), the demonstration is similar to the proof of inequality (4.3).

Remark 4.2. Considering $T \cdot F \cdot N$ such that

$$[\widetilde{\mathcal{D}}]^{\leftthreetimes} = [\text{ե}-\text{л}(1-\leftthreetimes), \text{ե}+\mu(1-\leftthreetimes)],$$

then inequalities (4.3) and (4.4) simplify to the Minkowski's-like inequalities over $[\text{ե}-\text{л}(1-\leftthreetimes), \text{ե}+\mu(1-\leftthreetimes)]$ such that for $p \ge 1$,

$$\left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz\right)^{\frac{1}{p}} \leq \left[\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}. \tag{4.5}$$

For $1 > p > 0$, we have

$$\left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{Q}(z)+\mathcal{G}(z)|^p\,dz\right)^{\frac{1}{p}} \geq \left[\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{Q}(z)|^p\,dz\right]^{\frac{1}{p}} + \left[\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ҍ}+\mu(1-\text{⅄})}|\mathcal{G}(z)|^p\,dz\right]^{\frac{1}{p}}. \tag{4.6}$$

Remark 4.3. If $\widetilde{\mathcal{D}} = \widetilde{[\mathfrak{a},\mathfrak{b}]}$, then from (4.3) and (4.4), we get the classical Minkowski's inequality for real-valued mappings.

4.2 Trapezoidal fuzzy Beckenbach's inequality

In this subsection, we propose Beckenbach's inequality over $T{\cdot}F{\cdot}N$.

Theorem 4.3. (Beckenbach's inequality) Suppose $1 > p > 0$. If $\mathcal{Q}$ and $\mathcal{G}$ are two real functions stated on $[\widetilde{\mathcal{D}}]^{\text{⅄}} = [\text{ҍ}-\text{л}(1-\text{⅄}), \text{ꞑ}+\mu(1-\text{⅄})]$ and $\mathcal{Q}(z) > 0, \mathcal{G}(z) > 0$, then

$$\frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p}dz} \leq \frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)\right)^{p}dz} + \frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{G}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{G}(z)\right)^{p}dz}, \tag{4.7}$$

with equality if $\mathcal{Q}$ and $\mathcal{G}$ are proportional.

If $p \geq 1$, then

$$\frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p}dz} \geq \frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{Q}(z)\right)^{p}dz} + \frac{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{G}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}\left(\mathcal{G}(z)\right)^{p}dz}. \tag{4.8}$$

Proof. Let

$$l_1 = \left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}(\mathcal{Q})^{p+1}\,dz\right)^{\frac{1}{p+1}}, \qquad l_2 = \left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}(\mathcal{G})^{p+1}\,dz\right)^{\frac{1}{p+1}}$$

and

$$J_1 = \left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}(\mathcal{Q})^{p}\,dz\right)^{\frac{1}{p}}, \qquad J_2 = \left(\int_{\text{ҍ}-\text{л}(1-\text{⅄})}^{\text{ꞑ}+\mu(1-\text{⅄})}(\mathcal{G})^{p}\,dz\right)^{\frac{1}{p}}.$$

Now by using Radon's inequality for real number, we have

$$\frac{{l_1}^{p+1}}{{J_1}^p} + \frac{{l_2}^{p+1}}{{J_2}^p} \geq \frac{(l_1+l_2)^{p+1}}{(J_1+J_2)^p},$$

that is to say

$$\frac{\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p}dz} + \frac{\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p+1}dz}{\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p}dz} \geq \frac{\left(\left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p+1}dz\right)^{\frac{1}{p+1}} + \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p+1}dz\right)^{\frac{1}{p+1}}\right)^{p+1}}{\left(\left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p}dz\right)^{\frac{1}{p}} + \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p}dz\right)^{\frac{1}{p}}\right)^{p}} \tag{4.9}$$

and consequently every $[\widetilde{\mathcal{D}}]^{\text{⅄}} = [\text{ҍ}-\text{л}(1-\text{⅄}), \text{ꞑ}+\mu(1-\text{⅄})]$, for all $\text{⅄} \in [0,1]$.

Now because $1 > p > 0$, then $2 > p+1 > 1$, from (4.3) and (4.4), we achieve

$$\left[\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p+1}dz\right]^{\frac{1}{p+1}} \leq \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p+1}dz\right)^{\frac{1}{p+1}} + \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p+1}dz\right)^{\frac{1}{p+1}}, \tag{4.10}$$

and

$$\left[\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p}dz\right]^{\frac{1}{p}} \geq \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{Q}(z)\right)^{p}dz\right)^{\frac{1}{p}} + \left(\int_{\text{ҍ}-\text{л}}^{\text{ꞑ}+\mu}\left(\mathcal{G}(z)\right)^{p}dz\right)^{\frac{1}{p}}. \tag{4.11}$$

As we know that, if $a, b, c > 0$, then we have

$$a \geq c \Leftrightarrow \frac{a}{b} \geq \frac{c}{b}, \tag{4.12}$$

$$b \leq c \Leftrightarrow \frac{a}{b} \geq \frac{a}{c}. \tag{4.13}$$

Finally, from (4.10), (4.11), (4.12) and (4.13), we have

$$\frac{\left(\left(\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{Q}(z))^{p+1}dz\right)^{\frac{1}{p+1}}+\left(\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{G}(z))^{p+1}dz\right)^{\frac{1}{p+1}}\right)^{p+1}}{\left(\left(\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{Q}(z))^{p}dz\right)^{\frac{1}{p}}+\left(\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{G}(z))^{p}dz\right)^{\frac{1}{p}}\right)^{p}} \geq \frac{\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{Q}(z)+\mathcal{G}(z))^{p+1}dz}{\int_{\mathfrak{t}-л}^{\mathfrak{n}+\mu}(\mathcal{Q}(z)+\mathcal{G}(z))^{p}dz}, \tag{4.14}$$

and consequently every $[\widetilde{\mathcal{D}}]^{\lambda} = [\mathfrak{t} - л(1-\lambda), \mathfrak{n} + \mu(1-\lambda)]$, for all $\lambda \in [0,1]$. Hence, from (4.9) and (4.14), we obtain the required result (4.7).

For inequality (4.8), it can be proved similarly as in the proof of inequality (4.7).

Remark 4.4. When $\mathfrak{n} = \mathfrak{t}$, then we acquire the following result for triangular $F \cdot N$:

$$[\widetilde{\mathcal{D}}]^{\lambda} = [\mathfrak{t} - л(1-\lambda), \mathfrak{t} + \mu(1-\lambda)], \tag{4.15}$$

then inequality (4.7) simplifies to the Beckenbach's-like inequality over $T \cdot F \cdot N$ $\widetilde{\mathcal{D}}$ such that

$$\frac{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{Q}(z)+\mathcal{G}(z))^{p+1}dz}{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{Q}(z)+\mathcal{G}(z))^{p}dz} \leq \frac{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{Q}(z))^{p+1}dz}{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{Q}(z))^{p}dz} + \frac{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{G}(z))^{p+1}dz}{\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{t}+\mu(1-\lambda)}(\mathcal{G}(z))^{p}dz}. \tag{4.16}$$

Remark 4.5. If $\widetilde{\mathcal{D}} = \widetilde{[\mathfrak{a}, \mathfrak{b}]}$, subsequently, from (4.7), we obtain the traditional Beckenbach's inequality for real-valued functions.

Example 4.1. Let the $T \cdot F \cdot N$ be denoted as $\widetilde{\mathcal{D}} = \left(1,2;\frac{1}{2},2\right)$, which represents.

$$\widetilde{\mathcal{D}}(z) = \begin{cases} 1, & \upsilon \in [1,2] \\ \frac{\upsilon-\frac{1}{2}}{\frac{1}{2}}, & \upsilon \in \left[1-\frac{1}{2}, 1\right] \\ \frac{4-\upsilon}{2}, & \upsilon \in [2, 2+2] \\ 0, & \text{otherwise}, \end{cases} \tag{4.17}$$

with a parametrized form of $[\widetilde{\mathcal{D}}]^{\lambda} = \left[1 - \frac{1}{2}(1-\lambda), 2 + 2(1-\lambda)\right]$, for all $\lambda \in [0,1]$. Let $p = \frac{1}{2}$, and $\mathcal{Q}(z) = z$ and $\mathcal{G}(z) = z^2$ be the real-valued mappings on fuzzy domain $\widetilde{\mathcal{D}}$. By direct computation, we find

$$\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{n}+\mu(1-\lambda)}(\mathcal{Q}(z))^{p+1}dz = \frac{1}{20}\left(8(4-2\lambda)^{\frac{5}{2}} - \sqrt{2}(\lambda+1)^{\frac{5}{2}}\right),$$

$$\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{n}+\mu(1-\lambda)}(\mathcal{Q}(z))^{p}dz = \frac{1}{6}\left(4(4-2\lambda)^{\frac{3}{2}} - \sqrt{2}(\lambda+1)^{\frac{3}{2}}\right),$$

$$\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{n}+\mu(1-\lambda)}(\mathcal{G}(z))^{p+1}dz = \frac{1}{4}\left(16(\lambda-2)^{4} - \frac{1}{16}(\lambda+1)^{4}\right),$$

$$\int_{\mathfrak{t}-л(1-\lambda)}^{\mathfrak{n}+\mu(1-\lambda)}(\mathcal{G}(z))^{p}dz = \frac{3}{8}(5\lambda^{2} - 22\lambda + 21),$$

$$\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p+1}dz$$
$$=\frac{1}{64}\left\{\begin{array}{c}\sqrt{(4-2\lambda)(5-2\lambda)}(9-4\lambda))(16(2-\lambda)(5-2\lambda)-3)\\ -\sqrt{\frac{1+\lambda}{2}\left(\frac{3+\lambda}{2}\right)}(2+\lambda)\left(4(1+\lambda)\left(\frac{3+\lambda}{2}\right)-3\right)+3\ln\left(\frac{\sqrt{4-2\lambda}+\sqrt{5-2\lambda}}{\sqrt{\frac{1+\lambda}{2}}+\sqrt{\frac{3+\lambda}{2}}}\right)\end{array}\right\},$$

$$\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p}dz$$
$$=\frac{1}{4}\left\{\sqrt{(4-2\lambda)(5-2\lambda)}\left(9-4\lambda-\frac{\sinh^{-1}\left(\sqrt{4-2\lambda}\right)}{\sqrt{(4-2\lambda)(5-2\lambda)}}\right)-\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}\left(2+\lambda-\frac{\sinh^{-1}\left(\sqrt{\frac{1+\lambda}{2}}\right)}{\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}}\right)\right\}.$$

Now

$$\left[\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{Q}(z)\right)^{p+1}dz\right]^{\frac{1}{p+1}}=\left(\frac{1}{20}\left(8(4-2\lambda)^{\frac{5}{2}}-\sqrt{2}(\lambda+1)^{\frac{5}{2}}\right)\right)^{\frac{2}{3}},$$

$$\left[\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{G}(z)\right)^{p+1}dz\right]^{\frac{1}{p+1}}=\left(\frac{1}{4}\left(16(\lambda-2)^{4}-\frac{1}{16}(\lambda+1)^{4}\right)\right)^{\frac{2}{3}},$$

$$\left(\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p+1}dz\right)^{\frac{1}{p+1}}$$
$$=\left[\frac{1}{64}\left\{\begin{array}{c}\sqrt{(4-2\lambda)(5-2\lambda)}(9-4\lambda))(16(2-\lambda)(5-2\lambda)-3)\\ -\sqrt{\frac{1+\lambda}{2}\left(\frac{3+\lambda}{2}\right)}(2+\lambda)\left(4(1+\lambda)\left(\frac{3+\lambda}{2}\right)-3\right)+3\ln\left(\frac{\sqrt{4-2\lambda}+\sqrt{5-2\lambda}}{\sqrt{\frac{1+\lambda}{2}}+\sqrt{\frac{3+\lambda}{2}}}\right)\end{array}\right\}\right]^{\frac{2}{3}}.$$

Then, from Minkowski's inequality (4.3), we establish the following

$$\left[\frac{1}{64}\left\{\begin{array}{c}\sqrt{(4-2\lambda)(5-2\lambda)}(9-4\lambda))(16(2-\lambda)(5-2\lambda)-3)\\ -\sqrt{\frac{1+\lambda}{2}\left(\frac{3+\lambda}{2}\right)}(2+\lambda)\left(4(1+\lambda)\left(\frac{3+\lambda}{2}\right)-3\right)+3\ln\left(\frac{\sqrt{4-2\lambda}+\sqrt{5-2\lambda}}{\sqrt{\frac{1+\lambda}{2}}+\sqrt{\frac{3+\lambda}{2}}}\right)\end{array}\right\}\right]^{\frac{2}{3}}$$
$$\leq\left(\frac{1}{20}\left(8(4-2\lambda)^{\frac{5}{2}}-\sqrt{2}(\lambda+1)^{\frac{5}{2}}\right)\right)^{\frac{2}{3}}+\left(\frac{1}{4}\left(16(\lambda-2)^{4}-\frac{1}{16}(\lambda+1)^{4}\right)\right)^{\frac{2}{3}}, \tag{4.18}$$

for each $\lambda\in[0,1]$. In Figure 3, we validate the inequality (4.18) by plotting the difference = (right side - left side) against λ. It is clear from the graph that (4.18) indeed holds for each $\lambda\in[0,1]$.

Next, to apply Minkowski's inequality (4.4), we have

$$\left[\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{Q}(z)\right)^{p}dz\right]^{\frac{1}{p}}=\left(\frac{1}{6}\left(4(4-2\lambda)^{\frac{3}{2}}-\sqrt{2}(\lambda+1)^{\frac{3}{2}}\right)\right)^{2},$$

$$\left[\int_{\flat-\text{л}(1-\lambda)}^{\eta+\mu(1-\lambda)}\left(\mathcal{G}(z)\right)^{p}dz\right]^{\frac{1}{p}}=\left(\frac{3}{8}\left(5\lambda^{2}-22\lambda+21\right)\right)^{2},$$

$$\left(\int_{\text{ȶ}-\text{л}(1-\lambda)}^{\text{ɳ}+\mu(1-\lambda)}\left(\mathcal{Q}(z)+\mathcal{G}(z)\right)^{p}\,dz\right)^{\frac{1}{p}}$$

$$=\left[\frac{1}{4}\left\{\sqrt{(4-2\lambda)(5-2\lambda)}\left(9-4\lambda-\frac{\sinh^{-1}\left(\sqrt{4-2\lambda}\right)}{\sqrt{(4-2\lambda)(5-2\lambda)}}\right)-\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}\left(2+\lambda-\frac{\sinh^{-1}\left(\sqrt{\frac{1+\lambda}{2}}\right)}{\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}}\right)\right\}\right]^{2}.$$

Then, Minkowski's inequality (4.4) leads to the following

$$\left[\frac{1}{4}\left\{\sqrt{(4-2\lambda)(5-2\lambda)}\left(9-4\lambda-\frac{\sinh^{-1}\left(\sqrt{4-2\lambda}\right)}{\sqrt{(4-2\lambda)(5-2\lambda)}}\right)-\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}\left(2+\lambda-\frac{\sinh^{-1}\left(\sqrt{\frac{1+\lambda}{2}}\right)}{\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}}\right)\right\}\right]^{2}$$

$$\geq\left(\frac{1}{6}\left(4(4-2\lambda)^{\frac{3}{2}}-\sqrt{2}(\lambda+1)^{\frac{3}{2}}\right)\right)^{2}+\left(\frac{3}{8}(5\lambda^{2}-22\lambda+21)\right)^{2}, \quad (4.19)$$

for each $\lambda\in[0,1]$. In Figure 4, we plot the difference = (left side - right side) of (4.19) against λ. From the graph, it is obvious that the inequality (4.19) is validated for each $\lambda\in[0,1]$.

Finally, using Beckenbach's inequality (4.7), we derive the following

$$\frac{\frac{1}{64}\left\{\sqrt{(4-2\lambda)(5-2\lambda)}(9-4\lambda))(16(2-\lambda)(5-2\lambda)-3)-\sqrt{\frac{1+\lambda}{2}\left(\frac{3+\lambda}{2}\right)}(2+\lambda)\left(4(1+\lambda)\left(\frac{3+\lambda}{2}\right)-3\right)+3\ln\left(\frac{\sqrt{4-2\lambda}+\sqrt{5-2\lambda}}{\sqrt{\frac{1+\lambda}{2}}+\sqrt{\frac{3+\lambda}{2}}}\right)\right\}}{\frac{1}{4}\left\{\sqrt{(4-2\lambda)(5-2\lambda)}\left(9-4\lambda-\frac{\sinh^{-1}\left(\sqrt{4-2\lambda}\right)}{\sqrt{(4-2\lambda)(5-2\lambda)}}\right)-\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}\left(2+\lambda-\frac{\sinh^{-1}\left(\sqrt{\frac{1+\lambda}{2}}\right)}{\sqrt{\left(\frac{1+\lambda}{2}\right)\left(\frac{3+\lambda}{2}\right)}}\right)\right\}}$$

$$\leq\frac{\frac{1}{20}\left(8(4-2\lambda)^{\frac{5}{2}}-\sqrt{2}(\lambda+1)^{\frac{5}{2}}\right)}{\frac{1}{6}\left(4(4-2\lambda)^{\frac{3}{2}}-\sqrt{2}(\lambda+1)^{\frac{3}{2}}\right)}+\frac{\frac{1}{4}\left(16(\lambda-2)^{4}-\frac{1}{16}(\lambda+1)^{4}\right)}{\frac{3}{8}(5\lambda^{2}-22\lambda+21)}, \quad (4.20)$$

for each $\lambda\in[0,1]$. In Figure 5, to check that (4.20) is true, we plot the difference = (right side - left side) against λ. From the graph, it is evident that the inequality (4.20) holds for each $\lambda\in[0,1]$.

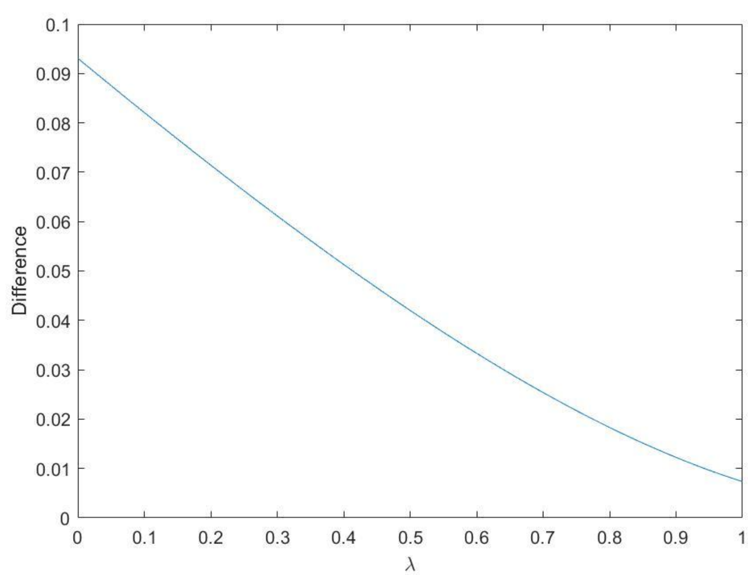

Figure 3: Validation of inequality (4.18)

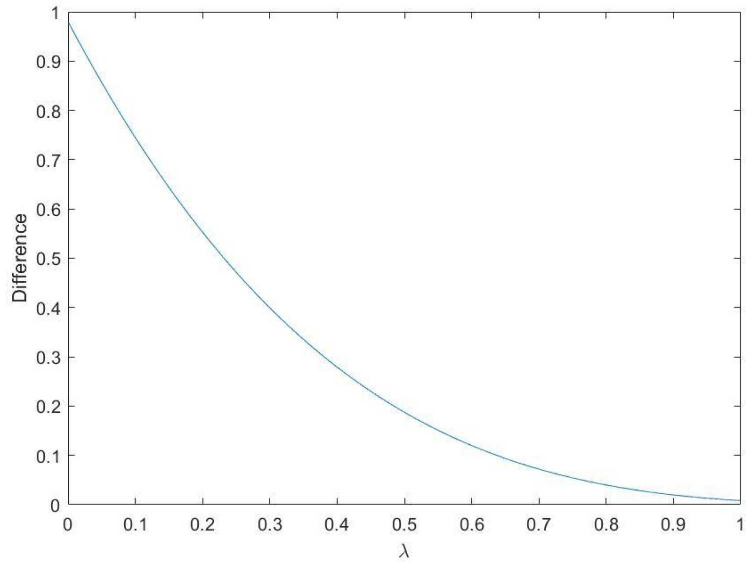

Figure 4: Validation of inequality (4.19)

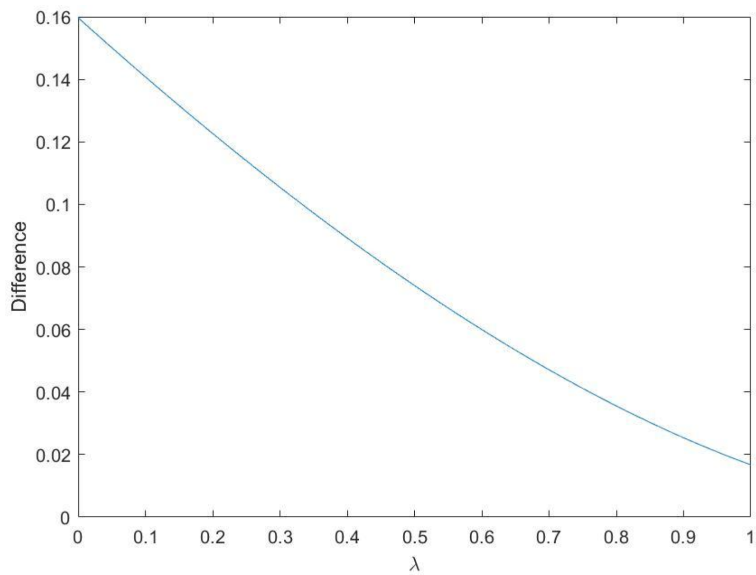

Figure 5: Validation of inequality (4.20)

5. Conclusion

By employing a simple proof technique within the space of trapezoidal fuzzy numbers, we have developed several original enhancements for the integral representations of both classical and recently introduced Hölder's inequality in this research. Applications such as trapezoidal fuzzy Beckenbach's inequality over trapezoidal fuzzy number, trapezoidal fuzzy Cauchy-Schwarz's inequality, trapezoidal fuzzy Minkowski's inequality, and trapezoidal fuzzy Hölder's power-mean integral inequality demonstrate how various existing inequalities related to trapezoidal fuzzy Hölder's inequality can be improved through the newly obtained ones. Our findings are important advances in inequalities that are potential tools in carrying out analysis in many branches of mathematics.

Funding: This research did not obtain any outside financial assistance.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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