Int. J. Anal. Appl. (2025), 23:97

International Journal of Analysis and Applications

On Some Ciric Type Cyclic Coupled F-Contractions in Complete Metric Spaces

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Abstract. In this paper, the notions of cyclic coupled Wardowski's F-contraction and generalized Ciric type mappings in complete metric space are discussed. Some coupled cyclic F-contractions of generalized Ciric type mappings are defined, and existence results for coupled fixed point, coupled coincidence point, strong coupled fixed point, and coupled best proximity point are established in the framework of complete metric space. An existence result for a coupled fixed point for generalized Ciric-type cyclic coupled F-contractive multivalued mapping is established. Further, an application of our result with regard to the existence of a system of functional equations is discussed.

1. Introduction

The exploration of coupled fixed points constitutes a prominent avenue within metric fixed point theory [1–3]. This branch of study originated in 1987 with the pioneering effors of Guo and Laksmikantham [4]. Nevertheless, it garnered widespread recognition and interest through the subsequent contributions of Bhaskar and Lakshmikantham [5]. They brought forth the concept of mixed monotone property which was later extended to mixed $\hbar$-monotone property by Lakshmikantham and Ciric [6]. Following that, various extensions of theorems related to coupled fixed points and coupled coincidence points were documented in diverse contexts. One may refer the following papers [5] and the references therein. In the year 2003, Kirk et.al [7] introduced the innovative concept of cyclic mapping, contributing a noteworthy addition to the field with

Received: Feb. 5, 2025.

2020 *Mathematics Subject Classification.* 47H09, 47H10, 54E50.

Key words and phrases. coupled fixed point; strong coupled fixed point; complete metric space with property uc; best proximity points; generalized ciric type multivalued mapping; F-contraction; cyclic contraction.

https://doi.org/10.28924/2291-8639-23-2025-97

ISSN: 2291-8639

their exploration and establishment of this novel idea. Blending it with the notion of coupled fixed points, Choudhury and Maity [8] introduced the notion of coupled cyclic mappings and established strong coupled fixed point results. Motivated by Choudhury and Maity, Udo-Utun [9] define a cyclic Ciric operator and established strong coupled fixed point results. Henceforth, many authors used the idea of coupling in different directions. Some of the works can be seen ([10–12], etc).

Wardowski [13] in 2012, brought forth a novel class of contraction called F-contraction and established fixed point results for such contractions. Wardowski's F-contraction was widely studied extended in various set ups. Recently, in 2021, Ganvir et al. [14] defined cyclic F-contraction and established a result for strong coupled fixed point.

In this paper, motivated by Ganvir et al. [14], we have blended the cyclic ciric operator and F-contraction to define some generalized cyclic Ciric type F-contractive mappings. We have also established some existence results for coupled and strong coupled fixed points, best proximity points and coupled coincidence points for such contractions in the framework of complete metric space. An application of our result with regard to the existence of a system of functional equations has also been discussed.

2. Preliminaries

Next, we present some of the preliminary definitions.
Let $\mathcal{F}$ represents the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

($F1$) F is strictly increasing.
($F2$) For each sequence $(\delta_\flat)$ of positive numbers, $\lim_{\flat\to\infty} \delta_\flat = 0$ if and only if $\lim_{\flat\to\infty} F(\delta_\flat) = -\infty$.
($F3$) there exists $k \in (0,1)$ such that $\lim_{\flat\to\infty} (\delta_\flat)^k F(\delta_\flat) = 0$

Definition 2.1. *[13] Let $\mathscr{U} : \mathscr{M} \to \mathscr{M}$ be an operator defined on a metric space $(\mathscr{M}, \rho)$. Then, $\mathscr{U}$ is said to an F-contraction if there exists a positive real number τ such that for all $\ell, \lambda \in \mathscr{M}$,*

$$\rho(\mathscr{U}(\ell), \mathscr{U}(\lambda)) > 0 \implies \tau + F(\rho(\mathscr{U}\ell, \mathscr{U}\lambda)) \leq F(\rho(\ell, \lambda)).$$

Definition 2.2. *[8] Let $\mathscr{A}$ and $\mathscr{O}$ are two non-empty subsets of a set $\mathscr{M}$. Then, a mapping $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is said to be cyclic mapping or coupling with reference to the sets $\mathscr{A}$ and $\mathscr{O}$ if*

(1) *$\mathscr{U}(\ell, \lambda) \in \mathscr{O}$ when $\ell \in \mathscr{A}$ and $\lambda \in \mathscr{O}$.*
(2) *$\mathscr{U}(\ell, \lambda) \in \mathscr{A}$ when $\ell \in \mathscr{O}$ and $\lambda \in \mathscr{A}$.*

Definition 2.3. *[5] An element $(\ell, \lambda) \in \mathscr{M} \times \mathscr{M}$ is called a coupled fixed point of the mapping $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ if $\mathscr{U}(\ell, \lambda) = \ell$ and $\mathscr{U}(\lambda, \ell) = \lambda$. If $\mathscr{U}(\ell, \ell) = \ell$, then u is called strong coupled fixed point.*

Definition 2.4. *[9] Let $\mathscr{A}$ and $\mathscr{O}$ are two nonempty subsets of a metric space $(\mathscr{M}, \rho)$. Then, a mapping $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is said to be cyclic coupled Ciric type mapping with reference to the sets $\mathscr{A}$ and $\mathscr{O}$ if $\mathscr{U}$ is cyclic with reference to the sets $\mathscr{A}$ and $\mathscr{O}$ such that for some $k \in (0, 1/2)$, the following inequality holds:*

For $b, \ell \in \mathscr{A}$ *and* $a, \lambda \in \mathscr{O}$, $\rho(\mathscr{U}(a,b), \mathscr{U}(\ell,\lambda)) \leq k\mathfrak{M}(a,\ell)$.
$\mathfrak{M}(a,\ell) = \max\{\rho(a,\ell), \frac{1}{2}\{\rho(a,\mathscr{U}(a,b)) + \rho(\ell,\mathscr{U}(\ell,\lambda))\}, \frac{1}{2}\rho(b,\mathscr{U}(\ell,\lambda)), \frac{1}{2}\rho(\ell,\mathscr{U}(a,b))\}$.

Definition 2.5. *[14] Let* $\mathscr{A}$ *and* $\mathscr{O}$ *be two nonempty subsets of a metric space* $(\mathscr{M}, \rho)$. *Then, a mapping* $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ *is said to be cyclic* F*-contraction if*

1. $\mathscr{U}(\ell,\lambda) \in \mathscr{O}$ *for all* $\ell \in \mathscr{A}$ *and* $\lambda \in \mathscr{O}$; $\mathscr{U}(\lambda,\ell) \in \mathscr{A}$ *for all* $\ell \in \mathscr{A}$ *and* $\ell \in \mathscr{O}$;
2. *a positive real number* τ *and a function* $F \in \mathcal{F}$ *exists such that for all* $(\ell,\lambda),(a,b) \in \mathscr{M} \times \mathscr{M}$, $\rho(\mathscr{U}(a,b),\mathscr{U}(\ell,\lambda)) > 0 \implies \tau + F(\rho(\mathscr{U}\ell,\mathscr{U}\lambda)) \leq F(\rho(b,\lambda))$.

3. Coupled fixed point results for single valued F-contractions of Ciric type

In this section, we proceed to articulate the definition of some cyclic Ciric F-type contractions and establish coupled fixed point results for single valued operators.

Definition 3.1. *Let* $\mathscr{A}$ *and* $\mathscr{O}$ *be two nonempty closed subsets of a metric space* $(\mathscr{M}, \rho)$ *and let* $\mathcal{Z} = \mathscr{A} \cup \mathscr{O}$. *Let* $\mathscr{U} : \mathcal{Z} \to \mathcal{Z}$ *be an operator. Then,* $\mathscr{U}$ *is said to be cyclic* F*-contraction of generalized Ciric type, if*

1. $\mathscr{U}$ *is cyclic with respect to* $\mathscr{A}$ *and* $\mathscr{O}$;
2. *a positive real number* τ *and a function* $F \in \mathcal{F}$ *exist such that for all* $\ell \in \mathscr{A}$ *and* $\lambda \in \mathscr{O}$, $\rho(\mathscr{U}(\ell),\mathscr{U}(\lambda)) > 0 \implies \tau + F(\rho(\mathscr{U}\ell,\mathscr{U}\lambda)) \leq F(\mathcal{M}(\ell,\lambda))$, *where* $\mathcal{M}(\ell,\lambda) = \max\{\rho(\ell,\lambda), \rho(\lambda,\mathscr{U}\lambda), \rho(\ell,\mathscr{U}\ell), \frac{1}{2}\rho(\ell,\mathscr{U}\lambda), \frac{1}{2}\rho(\lambda,\mathscr{U}\ell)\}$.

Theorem 3.1. *Let* $\mathscr{A}$ *and* $\mathscr{O}$ *be two nonempty and closed subsets of a complete metric space* $(\mathscr{M}, \rho)$. *Let* $\mathcal{Z} = \mathscr{A} \cup \mathscr{O}$. *Let* $\mathscr{U} : \mathcal{Z} \to \mathcal{Z}$ *be a cyclic* F*-contraction of generalized Ciric type mapping. If* $\mathscr{U}$ *is continuous, then* $\mathscr{U}$ *has a unique fixed point in* $\mathscr{A} \cap \mathscr{O}$.

Proof. Let $\ell_0 \in \mathcal{Z}$. Starting with ℓ_0, define a sequence $(\ell_\flat)$ such that $\ell_{\flat+1} = \mathscr{U}\ell_\flat$, for $\flat \in \mathbb{N}$.

Now,

$$\begin{aligned} F(\rho(\ell_{\flat+1},\ell_\flat)) &= F(\rho(\mathscr{U}\ell_\flat,\mathscr{U}\ell_{\flat-1})) \\ &\leq F(\mathcal{M}(\ell_\flat,\ell_{\flat-1})) - \tau \end{aligned} \tag{3.1}$$

Therefore

$$\begin{aligned} \mathcal{M}(\ell_\flat,\ell_{\flat-1}) &= \max\{\rho(\ell_\flat,\ell_{\flat-1}), \rho(\ell_\flat,\mathscr{U}\ell_\flat), \rho(\ell_{\flat-1},\mathscr{U}\ell_{\flat-1}), \frac{1}{2}\rho(\ell_\flat,\mathscr{U}\ell_{\flat-1}), \frac{1}{2}\rho(\ell_{\flat-1},\mathscr{U}\ell_\flat)\} \\ &= \max\{\rho(\ell_\flat,\ell_{\flat-1}), \rho(\ell_\flat,\ell_{\flat+1})\}. \end{aligned}$$

Now, if $\rho(\ell_\flat,\ell_{\flat-1}) < \rho(\ell_\flat,\ell_{\flat+1})$, then (3.1) gives

$$F(\rho(\ell_{\flat+1},\ell_\flat)) \leq F(\rho(\ell_\flat,\ell_{\flat+1})) - \tau$$

which is a contradiction. So, $\rho(\ell_\flat, \ell_{\flat-1}) > \rho(\ell_\flat, \ell_{\flat+1})$.
Therefore,

$$\begin{aligned}
F(\rho(\ell_{\flat+1}, \ell_\flat)) &\leq F(\rho(\ell_\flat, \ell_{\flat-1})) - \tau \\
&= F(\rho(\mathscr{U}\ell_{\flat-1}, \mathscr{U}\ell_{\flat-2})) - \tau \\
&\leq F(\mathcal{M}(\ell_{\flat-1}, \ell_{\flat-2})) - 2\tau \\
&\leq F(\rho(\ell_\flat, \ell_{\flat-1})) - \tau \\
&\dots \\
&\leq F(\rho(\ell_1, \ell_0)) - \flat\tau
\end{aligned} \tag{3.2}$$

So, as $\flat \to \infty$, we get

$$\begin{aligned}
&\lim_{\flat\to\infty} F(\rho(\ell_{\flat+1}, \ell_\flat)) = -\infty \\
\implies &\lim_{\flat\to\infty} (\rho(\ell_{\flat+1}, \ell_\flat)) = 0.
\end{aligned}$$

Let $\delta_\flat = \rho(\ell_{\flat+1}, \ell_\flat)$. Also, by virtue of the properties of F, a real $k \in (0,1)$ exists such that $\lim_{\flat\to\infty} (\delta_\flat)^k F(\delta_\flat) = 0$. Thus, from (3.2), we have

$$\begin{aligned}
&F(\delta_\flat) \leq F(\delta_0) - m\tau \\
\implies &(\delta_\flat)^k F(\delta_\flat) \leq (\delta_\flat)^k F(\delta_0) - m\tau(\delta_\flat)^k \\
\implies &(\delta_\flat)^k F(\delta_\flat) - (\delta_\flat)^k F(\delta_0) \leq -m\tau(\delta_\flat)^k.
\end{aligned}$$

Letting m tends to ∞, the above gives $\lim_{\flat\to\infty} m(\delta_\flat)^k = 0$. Therefore, a natural number N exists in a manner that for all $m \geq N$, $\flat(\delta_\flat)^k \leq 1$.
That is, for $\flat \geq N$, $\delta_\flat \leq \frac{1}{\flat^{\frac{1}{k}}}$.
Now, for $n > \flat \geq N$, we have

$$\begin{aligned}
\rho(\ell_\flat, \ell_n) &\leq \rho(\ell_\flat, \ell_{\flat+1}) + \rho(\ell_{\flat+1}, \ell_{\flat+2}) + \dots + \rho(\ell_{n-1}, \ell_n) \\
&= \delta_\flat + \delta_{\flat+1} + \dots + \delta_{n-1} \\
&\leq \sum_{i=\flat}^{n-1} \delta_i \\
&\leq \sum_{i=\flat}^{\infty} \delta_i \\
&\leq \sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

Since the series $\sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges, $\rho(\ell_\flat, \ell_n) \to 0$ as $m \to \infty$. Thus, $(\ell_\flat)$ is a cauchy sequence in a complete subspace $\mathcal{Z}$. Therefore, $(\ell_\flat)$ converges to some $\varrho \in \mathcal{Z}$. Moreover, the sequence $(\ell_\flat)$ has an infinitely many terms in each $\mathscr{A}$ and $\mathscr{O}$. So, each $\mathscr{A}$ and $\mathscr{O}$ have subsequences of $(\ell_\flat)$ that

converges to ϱ. Hence, $\varrho \in \mathscr{A} \cap \mathscr{O}$.
Now, using the continuity of $\mathscr{U}$, $\varrho = \lim_{\mathfrak{b}\to\infty} \ell_{\mathfrak{b}+1} = \lim_{\mathfrak{b}\to\infty} \mathscr{U}\ell_{\mathfrak{b}} = H\varrho$. Thus, ϱ is a fixed point of $\mathscr{U}$.
To see uniqueness, let us assume that ς is another fixed point of $\mathscr{U}$.
Then, $F(\rho(\varrho,\varsigma)) = F(\rho(\mathscr{U}\varrho, \mathscr{U}\varsigma)) \leq F(\mathcal{M}(\varrho,\varsigma)) - \tau \leq F(\rho(\varrho,\varsigma)) - \tau$,
which is a contradiction. Thus, $\mathscr{U}$ has a unique fixed point in $\mathscr{A} \cap \mathscr{O}$. □

Now, we define cyclic coupled F-contractive mapping of generalized Ciric type and establish a coupled fixed point theorem.

Definition 3.2. *Let $\mathscr{A}$ and $\mathscr{O}$ are two non-empty closed subsets of a metric space $(\mathscr{M},\rho)$ and $\mathscr{U} : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ be a mapping defined on $\mathscr{M}$. Then $\mathscr{U}$ is cyclic coupled F- contractive mapping of generalized Ciric type, if*

(1) *$\mathscr{U}(\ell,\lambda) \in \mathscr{O}$ for all $\ell \in \mathscr{A}$ and $\lambda \in \mathscr{O}$; $\mathscr{U}(\lambda,\ell) \in \mathscr{A}$ for all $\ell \in \mathscr{A}$ and $\ell \in \mathscr{O}$.*
(2) *a positive real number τ and a function $F \in \mathcal{F}$ exist such that for all $\ell,\lambda,a,b \in \mathscr{M}$, $\rho(\mathscr{U}(a,b),\mathscr{U}(\ell,\lambda)) > 0 \implies \tau + F(\rho(\mathscr{U}(a,b),\mathscr{U}(\ell,\lambda))) \leq F(\mathcal{M}_c(b,\lambda))$, where $\mathcal{M}_c(b,\lambda) = \max\{\rho(b,\lambda), \rho(b,\mathscr{U}(a,b)), \rho(\lambda,\mathscr{U}(\ell,\lambda)), \frac{1}{2}\rho(b,\mathscr{U}(\ell,\lambda)), \frac{1}{2}\rho(\lambda,\mathscr{U}(a,b))\}$.*

Theorem 3.2. *Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty and closed subsets of a complete metric space $(\mathscr{M},\rho)$. Let $\mathscr{U} : \mathscr{M}\times\mathscr{M} \to \mathscr{M}$ be a cyclic coupled F-contractive mapping of generalized Ciric type. If $\mathscr{U}$ is continuous, then $\mathscr{U}$ has a unique strong coupled fixed point in $\mathscr{A} \cap \mathscr{O}$.*

Proof. Let $\ell_0 \in \mathscr{A}$ and $\lambda_0 \in \mathscr{O}$ such that $\ell_{\mathfrak{b}+1} = \mathscr{U}(\lambda_{\mathfrak{b}},\ell_{\mathfrak{b}})$ and $\lambda_{\mathfrak{b}+1} = \mathscr{U}(\ell_{\mathfrak{b}},\lambda_{\mathfrak{b}})$.
Then, $(\ell_{\mathfrak{b}}) \subset \mathscr{A}$ and $(\lambda_{\mathfrak{b}}) \subset \mathscr{O}$.
Now,

$$\begin{aligned} F(\rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})) &= F(\rho(\mathscr{U}(\lambda_{\mathfrak{b}-1},\ell_{\mathfrak{b}-1}),\mathscr{U}(\lambda_{\mathfrak{b}},\ell_{\mathfrak{b}}))) \\ &\leq F(\mathcal{M}_c(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}})) - \tau. \end{aligned} \tag{3.3}$$

Therefore

$$\begin{aligned} \mathcal{M}_c(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}}) = max\{&\rho(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}}), \rho(\ell_{\mathfrak{b}},\mathscr{U}(\lambda_{\mathfrak{b}},\ell_{\mathfrak{b}})), \rho(\ell_{\mathfrak{b}-1},\mathscr{U}(\lambda_{\mathfrak{b}-1},\ell_{\mathfrak{b}-1})), \\ &\frac{1}{2}\rho(\ell_{\mathfrak{b}},\mathscr{U}(\lambda_{\mathfrak{b}-1},\ell_{\mathfrak{b}-1})), \frac{1}{2}\rho(\ell_{\mathfrak{b}-1},\mathscr{U}(\lambda_{\mathfrak{b}},\ell_{\mathfrak{b}}))\} \\ = max\{&\rho(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}}), \rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})\}. \end{aligned}$$

Now, if $\rho(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}}) < \rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})$, then (3.3) gives
$F(\rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})) \leq F(\rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})) - \tau$,
which is a contradiction. So, $\rho(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}}) > \rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})$.
Therefore,

$$\begin{aligned} F(\rho(\ell_{\mathfrak{b}},\ell_{\mathfrak{b}+1})) &\leq F(\rho(\ell_{\mathfrak{b}-1},\ell_{\mathfrak{b}})) - \tau \\ &= F(\rho(\mathscr{U}(\lambda_{\mathfrak{b}-2},\ell_{\mathfrak{b}-2}),\mathscr{U}(\lambda_{\mathfrak{b}-1},\ell_{\mathfrak{b}-1}))) - \tau \\ &\leq \ldots \end{aligned}$$

$$\begin{aligned} &\leq F(\mathcal{M}_c(\ell_0, \ell_1)) - m\tau \\ &= F(\rho(\ell_\flat, \lambda_{\flat+1})) - m\tau \end{aligned} \tag{3.4}$$

So, as $\flat \to \infty$, we get

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\ell_{\flat+1}, \ell_\flat)) = -\infty \\ \Longrightarrow &\lim_{\flat\to\infty} (\rho(\ell_{\flat+1}, \ell_\flat)) = 0 \end{aligned}$$

Let $\delta_\flat = \rho(\ell_{\flat+1}, \ell_\flat)$.
Also, by virtue of the properties of F, a real $k \in (0, 1)$ exists such that $\lim\limits_{\flat\to\infty} (\delta_\flat)^k F(\delta_\flat) = 0$.
Thus, from (3.4), we have

$$\begin{aligned} &F(\delta_\flat) \leq F(\delta_0) - m\tau \\ \Longrightarrow &(\delta_\flat)^k F(\delta_\flat) \leq (\delta_\flat)^k F(\delta_0) - \flat\tau(\delta_\flat)^k \\ \Longrightarrow &(\delta_\flat)^k F(\delta_\flat) - (\delta_\flat)^k F(\delta_0) \leq -\flat\tau(\delta_\flat)^k \end{aligned}$$

Letting $\flat$ tends to ∞, the above gives $\lim\limits_{\flat\to\infty} \flat(\delta_\flat)^k = 0$. Therefore, a natural number N exists such that for all $\flat \geq N$, $\flat(\delta_\flat)^k \leq 1$.
That is, for $\flat \geq N$, $\delta_\flat \leq \frac{1}{\flat^{\frac{1}{k}}}$.
Now, for $n > \flat \geq N$, we have

$$\begin{aligned} \rho(\ell_\flat, \ell_n) &\leq \rho(\ell_\flat, \ell_{\flat+1}) + \rho(\ell_{\flat+1}, \ell_{\flat+2}) + ... + \rho(\ell_{n-1}, \ell_n) \\ &= \delta_\flat + \delta_{\flat+1} + ... + \delta_{n-1} \\ &\leq \sum_{i=\flat}^{n-1} \delta_i \\ &\leq \sum_{i=\flat}^{\infty} \delta_i \\ &\leq \sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum\limits_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges, $\rho(\ell_\flat, \ell_n) \to 0$ as $\flat \to \infty$. Thus, $(\ell_\flat)$ is a cauchy sequence. Therefore, $(\ell_\flat)$ converges to some $\varrho \in \mathscr{A}$.
The same argument may be applied to obtain $\zeta \in \mathscr{O}$ such that $\lim\limits_{\flat\to\infty} \lambda_\flat = \zeta$.
Now,

$$\begin{aligned} F(\rho(\ell_\flat, \lambda_\flat)) &= F(\rho(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\ell_{\flat-1}, \lambda_{\flat-1}))) \\ &\leq F(\mathcal{M}_c(\ell_{\flat-1}, \lambda_{\flat-1})) - \tau. \end{aligned} \tag{3.5}$$

Thus

$$\begin{aligned}\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) &= max\{\rho(\ell_{\flat-1},\lambda_{\flat-1}),\rho(\ell_{\flat-1},\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1})),\rho(\lambda_{\flat-1},\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})),\\ &\quad \frac{1}{2}\rho(\ell_{\flat-1},\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})),\frac{1}{2}\rho(\lambda_{\flat-1},\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}))\}\\ &= max\{\rho(\ell_{\flat-1},\lambda_{\flat-1}),\rho(\ell_{\flat-1},\ell_\flat),\rho(\lambda_{\flat-1},\lambda_\flat)\}.\end{aligned}$$

Here, three cases arise:

Case 1: $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\ell_{\flat-1},\ell_\flat)$.
Then, (3.5) gives
$F(\rho(\ell_\flat,\lambda_\flat)) \leq F(\rho(\ell_{\flat-1},\ell_\flat)) - \tau \leq ... \leq F(\rho(\ell_0,\ell_1)) - m\tau$
That is, $F(\rho(\varrho,\zeta)) \to -\infty$ as $\flat \to \infty$.

Case 2: $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\lambda_{\flat-1},\lambda_\flat)$.
Then, (3.5) gives
$F(\rho(\ell_\flat,\lambda_\flat)) \leq F(\rho(\lambda_{\flat-1},\lambda_\flat)) - \tau \leq ... \leq F(\rho(\lambda_0,\lambda_1)) - m\tau$
That is, $F(\rho(\varrho,\zeta)) \to -\infty$ as $\flat \to \infty$.

Case 3: $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\ell_{\flat-1},\lambda_{\flat-1})$.
Then,

$$\begin{aligned}F(\rho(\ell_\flat,\lambda_\flat)) &\leq F(\rho(\ell_{\flat-1},\lambda_{\flat-1})) - \tau\\ &= F(\rho(\mathscr{U}(\lambda_{\flat-2},\ell_{\flat-2}),\mathscr{U}(\ell_{\flat-2},\lambda_{\flat-2}))) - \tau\\ &\leq F(\mathcal{M}_c(\ell_{\flat-2},\lambda_{\flat-2})) - 2\tau\end{aligned}$$

it is easy to see that
$\mathcal{M}_c(\ell_{\flat-2},\lambda_{\flat-2}) = max\{\rho(\ell_{\flat-2},\lambda_{\flat-2}),\rho(\ell_{\flat-2},\ell_{\flat-1}),\rho(\lambda_{\flat-2},\lambda_{\flat-1})\}$.
As seen in Case 1 and Case 2, $\rho(\ell_{\flat-2},\ell_{\flat-1})$ and $\rho(\lambda_{\flat-2},\lambda_{\flat-1})$ as maximums will eventually give $F(\rho(\varrho,\zeta)) \to -\infty$ as $\flat \to \infty$.
While $\rho(\ell_{\flat-2},\lambda_{\flat-2})$ as maximum will give
$F(\rho(\ell_\flat,\lambda_\flat)) \leq F(\rho(\ell_{\flat-2},\lambda_{\flat-2})) - 2\tau \leq F(\mathcal{M}_c(\ell_{\flat-3},\lambda_{\flat-3})) - 3\tau$ continuing the above argument repeatedly, we will eventually get $\lim_{\flat\to\infty} F(\rho(\varrho,\zeta)) = -\infty$.

Thus, in all the cases

$$\begin{aligned}&\lim_{\flat\to\infty} F(\rho(\varrho,\zeta)) = -\infty\\ &\Longrightarrow \lim_{\flat\to\infty} \rho(\varrho,\zeta) = 0\\ &\Longrightarrow \varrho = \zeta.\end{aligned}$$

So, $\mathscr{A} \cap \mathscr{O} \neq \emptyset$. Now,

$$\begin{aligned}F(\rho(\ell_\flat,\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}),\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})))\\ &\leq F(\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1})) - \tau\\ &\quad ....\end{aligned}$$

$$\cdots$$

$$\to -\infty \text{ as } \flat \to \infty$$

That is, $\mathscr{U}$ being continuous, letting $\flat$ tends to ∞, we get
$\lim_{\flat\to\infty} F(\rho(\varrho, \mathscr{U}(\varrho, \zeta)) = -\infty$.
This gives, $\rho(\varrho, \mathscr{U}(\varrho, \zeta) = 0$.
Thus, $\varrho = \mathscr{U}(\varrho, \zeta) \implies \varrho = \mathscr{U}(\varrho, \varrho)$ as $\varrho = \zeta$. Hence, ϱ is a strong coupled fixed point in $\mathscr{A} \cap \mathscr{O}$.
To see uniqueness, let us assume that ς is another strong coupled fixed point of $\mathscr{U}$.
Then, $F(\rho(\varrho, \varsigma)) = F(\rho(\mathscr{U}(\varrho, \varrho), \mathscr{U}(\varsigma, \varsigma))) \le F(\mathcal{M}_c(\varrho, \varsigma)) - \tau \le F(\rho(\varrho, \varsigma)) - \tau$,
leading to a contradiction. Hence, $\mathscr{U}$ has a unique strong coupled fixed point in $\mathscr{A} \cap \mathscr{O}$. □

Now, we define the generalized ciric type cyclic $\hbar$-coupled F-contraction mapping.

Definition 3.3. *Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty subsets of a metric space $(\mathscr{M}, \rho)$. Let $\hbar : \mathscr{M} \to \mathscr{M}$ be a mapping. Then, the mapping $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is considered to be generalized ciric type cyclic $\hbar$-coupled F-contraction mapping, if*

(1) *$\mathscr{U}(\ell, \lambda) \in \hbar(\mathscr{O})$ for all $\ell \in \mathscr{A}$ and $\lambda \in \mathscr{O}$; $\mathscr{U}(\lambda, \ell) \in \hbar(\mathscr{A})$ for all $\ell \in \mathscr{A}$ and $\ell \in \mathscr{O}$;*
(2) *a positive real number τ and a function $F \in \mathcal{F}$ exist such that for all $\ell, \lambda, a, b \in \mathscr{M}$, $\rho(\mathscr{U}(a,b), \mathscr{U}(\ell, \lambda)) > 0 \implies \tau + F(\rho(\mathscr{U}(a,b), \mathscr{U}(\ell, \lambda))) \le F(\mathcal{M}_{cc}(\hbar b, \hbar\lambda))$, where $\mathcal{M}_{cc}(b, \lambda) = \max\{\rho(\hbar b, \hbar\lambda), \rho(\hbar b, \mathscr{U}(a,b)), \rho(\hbar\lambda, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\rho(\hbar b, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\rho(\hbar\lambda, \mathscr{U}(a,b))\}$.*

Theorem 3.3. *Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty and closed subsets of a complete metric space $(\mathscr{M}, \rho)$. Let $\hbar : \mathscr{M} \to \mathscr{M}$ be a continuous, sequentially convergent mapping such that $\mathscr{A}$ and $\mathscr{O}$ are invariant under $\hbar$. Let $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ be a generalized ciric type cyclic $\hbar$-coupled F-contraction mapping. If $\mathscr{U}$ is continuous, then $\mathscr{A} \cap \mathscr{O} \neq \emptyset$ and $\mathscr{U}, \hbar$ have a coupled coincidence point in $\mathscr{A} \cap \mathscr{O}$.*

Proof. Let $\ell_0 \in \mathscr{A}$ and $\lambda_0 \in \mathscr{O}$ such that $\hbar\ell_{\flat+1} = \mathscr{U}(\lambda_\flat, \ell_\flat)$ and $\hbar\lambda_{\flat+1} = \mathscr{U}(\ell_\flat, \lambda_\flat)$.
Then, $(\hbar\ell_\flat) \subset \mathscr{A}$ and $(\hbar\lambda_\flat) \subset \mathscr{O}$.
Now,

$$\begin{aligned} F(\rho(\hbar\ell_\flat, \hbar\ell_{\flat+1})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\lambda_\flat, \ell_\flat))) \\ &\le F(\mathcal{M}_{cc}(\hbar\ell_{\flat-1}, \hbar\ell_\flat)) - \tau. \end{aligned} \tag{3.6}$$

Therefore

$$\begin{aligned} \mathcal{M}_{cc}(\hbar\ell_{\flat-1}, \hbar\ell_\flat) = \max\{&\rho(\hbar\ell_{\flat-1}, \hbar\ell_\flat), \rho(\hbar\ell_\flat, \mathscr{U}(\lambda_\flat, \ell_\flat)), \rho(\hbar\ell_{\flat-1}, \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1})), \\ &\frac{1}{2}\rho(\hbar\ell_\flat, \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1})), \frac{1}{2}\rho(\hbar\ell_{\flat-1}, \mathscr{U}(\lambda_\flat, \ell_\flat))\} \\ = \max\{&\rho(\hbar\ell_{\flat-1}, \hbar\ell_\flat), \rho(\hbar\ell_\flat, \hbar\ell_{\flat+1})\}. \end{aligned}$$

Now, if $\rho(\hbar\ell_{\flat-1}, \hbar\ell_\flat) < \rho(\hbar\ell_\flat, \hbar\ell_{\flat+1})$, then (3.6) gives
$F(\rho(\hbar\ell_\flat, \hbar\ell_{\flat+1})) \le F(\rho(\hbar\ell_\flat, \hbar\ell_{\flat+1})) - \tau$, leading to a contradiction.

So, $\rho(\hbar\ell_{\flat-1},\hbar\ell_{\flat}) > \rho(\hbar\ell_{\flat},\hbar\ell_{\flat+1})$.
Therefore,

$$\begin{aligned} F(\rho(\hbar\ell_{\flat},\hbar\ell_{\flat+1})) &\leq F(\rho(\hbar\ell_{\flat-1},\hbar\ell_{\flat})) - \tau \\ &= F(\rho(\mathscr{U}(\lambda_{\flat-2},\ell_{\flat-2}),\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}))) - \tau \\ &\leq \dots \\ &= F(\rho(\hbar\ell_0,\hbar\ell_1)) - \flat\tau. \end{aligned} \tag{3.7}$$

So, as $\flat \to \infty$, we get

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\hbar\ell_{\flat+1},\hbar\ell_{\flat})) = -\infty \\ \Longrightarrow &\lim_{\flat\to\infty} (\rho(\ell_{\flat+1},\ell_{\flat})) = 0. \end{aligned}$$

Let $\delta_{\flat} = \rho(\hbar\ell_{\flat+1},\hbar\ell_{\flat})$.
Also, by virtue of the properties of F, a real $k \in (0,1)$ exists such that $\lim_{\flat\to\infty}(\delta_{\flat})^k F(\delta_{\flat}) = 0$.
Thus, from (3.7), we have

$$\begin{aligned} &F(\delta_{\flat}) \leq F(\delta_0) - \flat\tau \\ \Longrightarrow &(\delta_{\flat})^k F(\delta_{\flat}) \leq (\delta_{\flat})^k F(\delta_0) - \flat\tau(\delta_{\flat})^k \\ \Longrightarrow &(\delta_{\flat})^k F(\delta_{\flat}) - (\delta_{\flat})^k F(\delta_0) \leq -\flat\tau(\delta_{\flat})^k. \end{aligned}$$

Letting m tends to ∞, the above gives $\lim_{\flat\to\infty} \flat(\delta_{\flat})^k = 0$. Therefore, a natural number N exists such that for all $\flat \geq N$, $\flat(\delta_{\flat})^k \leq 1$.
That is, for $\flat \geq N$, $\delta_{\flat} \leq \frac{1}{\flat^{\frac{1}{k}}}$.
Now, for $n > \flat \geq N$, we have

$$\begin{aligned} \rho(\hbar\ell_{\flat},\hbar\ell_n) &\leq \rho(\hbar\ell_{\flat},\hbar\ell_{\flat+1}) + \rho(\hbar\ell_{\flat+1},\hbar\ell_{\flat+2}) + \dots + \rho(\hbar\ell_{n-1},\hbar\ell_n) \\ &= \delta_{\flat} + \delta_{\flat+1} + \dots + \delta_{n-1} \\ &\leq \sum_{i=\flat}^{n-1} \delta_i \\ &\leq \sum_{i=\flat}^{\infty} \delta_i \\ &\leq \sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}. \end{aligned}$$

Since the series $\sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges, $\rho(\hbar\ell_{\flat},\hbar\ell_n) \to 0$ as $\flat \to \infty$. Thus, $(\hbar\ell_{\flat})$ is a cauchy sequence. So, $(\hbar\ell_{\flat})$ is convergent which, in turn, implies that $(\ell_{\flat})$ is convergent as $\hbar$ is sequentially convergent. Let $(\ell_{\flat})$ converges to some ϱ in $\mathscr{A}$. Then, $\hbar$ being continuous, $(\hbar\ell_{\flat})$ converges to $\hbar\varrho$ in $\mathscr{A}$.
Similarly, the same argument may be applied to obtain $\zeta \in \mathscr{O}$ such that $\lim_{\flat\to\infty} \lambda_{\flat} = \zeta$ and $\lim_{\flat\to\infty} \hbar\lambda_{\flat} =$

$\hbar\zeta$. Now,

$$F(\rho(\hbar\ell_\flat,\hbar\lambda_\flat)) = F(\rho(\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}),\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1}))) \leq F(\mathcal{M}_{cc}(\ell_{\flat-1},\lambda_{\flat-1})) - \tau \tag{3.8}$$

Now,

$$\mathcal{M}_{cc}(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1}) = \max\{\rho(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1}),\rho(\hbar\ell_{\flat-1},\hbar\ell_\flat),\rho(\hbar\lambda_{\flat-1},\hbar\lambda_\flat)\}.$$

So, three cases arise:

Case 1: $\mathcal{M}_{cc}(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1}) = \rho(\hbar\ell_{\flat-1},\hbar\ell_\flat)$.
Then, (3.8) gives
$F(\rho(\hbar\ell_\flat,\hbar\lambda_\flat)) \leq F(\rho(\hbar\ell_{\flat-1},\hbar\ell_\flat)) - \tau \leq \ldots \leq F(\rho(\hbar\ell_0,\hbar\ell_1)) - \flat\tau$.
That is, $F(\rho(\hbar\varrho,\hbar\zeta)) \to -\infty$ as $\flat \to \infty$.

Case 2: $\mathcal{M}_{cc}(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1}) = \rho(\hbar\lambda_{\flat-1},\hbar\lambda_\flat)$.
Then, (3.8) gives
$F(\rho(\hbar\ell_\flat,\hbar\lambda_\flat)) \leq F(\rho(\hbar\lambda_{\flat-1},\hbar\lambda_\flat)) - \tau \leq \ldots \leq F(\rho(\hbar\lambda_0,\hbar\lambda_1)) - \flat\tau$.
That is, $F(\rho(\hbar\varrho,\hbar\zeta)) \to -\infty$ as $\flat \to \infty$.

Case 3: $\mathcal{M}_{cc}(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1}) = \rho(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1})$.
Then,

$$\begin{aligned} F(\rho(\hbar\ell_\flat,\hbar\lambda_\flat)) &\leq F(\rho(\hbar\ell_{\flat-1},\hbar\lambda_{\flat-1})) - \tau \\ &= F(\rho(\mathscr{U}(\lambda_{\flat-2},\ell_{\flat-2}),\mathscr{U}(\ell_{\flat-2},\lambda_{\flat-2}))) - \tau \\ &\leq F(\mathcal{M}_{cc}(\hbar\ell_{\flat-2},\hbar\lambda_{\flat-2})) - 2\tau. \end{aligned}$$

It is easy to see that
$\mathcal{M}_{cc}(\hbar\ell_{\flat-2},\hbar\lambda_{\flat-2}) = \max\{\rho(\hbar\ell_{\flat-2},\hbar\lambda_{\flat-2}),\rho(\hbar\ell_{\flat-2},\hbar\ell_{\flat-1}),\rho(\hbar\lambda_{\flat-2},\hbar\lambda_{\flat-1})\}$.
As seen in Case 1 and Case 2, $\rho(\hbar\ell_{\flat-2},\hbar\ell_{\flat-1})$ and $\rho(\hbar\lambda_{\flat-2},\hbar\lambda_{\flat-1})$ as maximums will eventually give
$F(\rho(\hbar\varrho,\hbar\zeta)) \to -\infty$ as $\flat \to \infty$.
While $\rho(\hbar\ell_{\flat-2},\hbar\lambda_{\flat-2})$ as maximum will give
$F(\rho(\hbar\ell_\flat,\hbar\lambda_\flat)) \leq F(\rho(\hbar\ell_{\flat-2},\hbar\lambda_{\flat-2})) - 2\tau \leq F(\mathcal{M}_{cc}(\hbar\ell_{\flat-3},\hbar\lambda_{\flat-3})) - 3\tau$.
Continuing the above argument repeatedly, we will eventually get
$\lim_{\flat\to\infty} F(\rho(\hbar\varrho,\hbar\zeta)) = -\infty$.

Thus, in all the cases

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\hbar\varrho,\hbar\zeta)) = -\infty \\ &\Longrightarrow \lim_{\flat\to\infty} \rho(\hbar\varrho,\hbar\zeta) = 0 \\ &\Longrightarrow \hbar\varrho = \hbar\zeta. \end{aligned}$$

So, $\mathscr{A} \cap \mathscr{O} \neq \emptyset$. Now,

$$\begin{aligned} F(\rho(\hbar\ell_{\flat}, \mathscr{U}(\ell_{\flat-1}, \lambda_{\flat-1})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\ell_{\flat-1}, \lambda_{\flat-1}))) \\ &\leq F(\mathcal{M}_{cc}(\ell_{\flat-1}, \lambda_{\flat-1})) - \tau \\ &.... \\ &.... \\ &\to -\infty \text{ as } \flat \to \infty. \end{aligned}$$

That is, $\mathscr{U}$ being continuous, letting m tends to ∞, we get
$\lim_{\flat\to\infty} F(\rho(\hbar\varrho, \mathscr{U}(\varrho, \zeta)) = -\infty$.
This gives, $\rho(\hbar\varrho, \mathscr{U}(\varrho, \zeta) = 0$.
Thus, $\hbar\varrho = \mathscr{U}(\varrho, \zeta)$.
Similarly, we can get $\hbar\zeta = \mathscr{U}(\zeta, \varrho)$. Thus, (ϱ, ζ) is a coupled coincidence point of $\hbar$ and$\mathscr{U}$. □

Theorem 3.4. *In addition to hypothesis in Theorem 3.3, it is further postulated that the mapping $\hbar$ is injective. Then, $\mathscr{U}$, $\hbar$ have a unique strong coupled coincidence point.*

Proof. As we have seen that $\hbar\varrho = \hbar\zeta$. This implies $\varrho = \zeta$, $\hbar$ being injective. Hence, $\hbar\varrho = \mathscr{U}(\varrho, \varrho)$. To see uniqueness, we may consider ς to be another strong coupled coincidence point of $\mathscr{U}$ and $\hbar$. Then, $F(\rho(\hbar\varrho, \hbar\varsigma)) = F(\rho(\mathscr{U}(\varrho, \varrho), \mathscr{U}(\varsigma, \varsigma))) \leq F(\mathcal{M}_{cc}(\hbar\varrho, \hbar\varsigma)) - \tau \leq F(\rho(\hbar\varrho, \hbar\varsigma)) - \tau$,
which is a contradiction as $\tau > 0$. Thus, $\mathscr{U}$, $\hbar$ have a unique strong coupled coincidence point. □

4. Coupled best proximity result for generalized Ciric type cyclic coupled F-contractive mapping

In the subsequent steps of our discourse, we will proceed to articulate and define a mapping of the cyclic coupled F-contractive type, thereby laying the foundation for the establishment of a consequential coupled best proximity result. It is noteworthy to mention that the inception of the concept concerning coupled best proximity can be attributed to the scholarly work presented by Sintunavarat and Kumam [15] in 2012.

Definition 4.1. *[15] Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty subsets of a metric space $(\mathscr{M}, \rho)$ and let $\mathcal{Z} = \mathscr{A} \cup \mathscr{O}$. An element $(\ell, \lambda) \in \mathscr{A} \times \mathscr{O}$ is called a coupled best proximity point of the mapping $\mathscr{U} : \mathcal{Z} \times \mathcal{Z} \to \mathcal{Z}$ if $\rho(\ell, \mathscr{U}(\ell, \lambda)) = \rho(\mathscr{A}, \mathscr{O}) = \rho(\lambda, \mathscr{U}(\lambda, \ell))$, where $\rho(\mathscr{A}, \mathscr{O}) = \inf\{\rho(\ell, \lambda) | \ell \in \mathscr{A}, \lambda \in \mathscr{O}\}$.*

Definition 4.2. *Let $\mathscr{A}$ and $\mathscr{O}$ are two non-empty subsets of a metric space $(\mathscr{M}, \rho)$. Then, the mapping $\mathscr{U} : \mathscr{M} \times \mathscr{M} \to \mathscr{M}$ is said to be generalized Ciric type cyclic coupled F-contractive mapping, if*

1. *$\mathscr{U}(\mathscr{A} \times \mathscr{O}) \subset \mathscr{O}$ and $\mathscr{U}(\mathscr{O} \times \mathscr{A}) \subset \mathscr{A}$;*
2. *a positive real number τ and a function $F \in \mathcal{F}$ exists such that for all $\ell, \lambda, a, b \in \mathscr{M}$,*
   $\rho(\mathscr{U}(a, b), \mathscr{U}(\ell, \lambda)) - \rho(\mathscr{A}, \mathscr{O}) > 0 \implies \tau + F(\rho(\mathscr{U}(a, b), \mathscr{U}(\ell, \lambda)) - \rho(\mathscr{A}, \mathscr{O})) \leq F(\mathcal{M}_c(b, \lambda) - \rho(\mathscr{A}, \mathscr{O}))$,
   where $\mathcal{M}_c(b, \lambda) = \max\{\rho(b, \lambda), \rho(b, \mathscr{U}(a, b)), \rho(\lambda, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\rho(b, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\rho(\lambda, \mathscr{U}(a, b))\}$.

In 2009, Suzuki et al. [16] introduced property UC as follows:

Definition 4.3. *[16] Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty subsets of a metric space $(\mathscr{M},\rho)$. Then, $(\mathscr{A},\mathscr{O})$ is considered to satisfy property UC in the condition that the following holds:*
if $(\ell_\flat)$,$(z_\flat)$ are sequences in $\mathscr{A}$ and sequence $(\lambda_\flat)$ in $\mathscr{O}$ such that $\lim_{\flat\to\infty}\rho(\ell_\flat,\lambda_\flat)=\rho(\mathscr{A},\mathscr{O})$, $\lim_{\flat\to\infty}\rho(z_\flat,\lambda_\flat)=\rho(\mathscr{A},\mathscr{O})$, then $\lim_{\flat\to\infty}\rho(\ell_\flat,z_\flat)=0$.

To establish our result in this section, we will use the following lemma which was proved by Neammanee et al. [17], in 2011.

Lemma 4.1. *[17] Let $\mathscr{A},\mathscr{O}$ be a pair of nonempty subsets of a metric space $(\mathscr{M},\rho)$ satisfying property UC. Let $(\ell_\flat)$ be a sequence in $\mathscr{A}$. If there exists a sequence $(\lambda_\flat)$ in $\mathscr{O}$ such that $\rho(\ell_\flat,\lambda_\flat)\to\rho(\mathscr{A},\mathscr{O})$ and $\rho(\ell_{\flat+1},\lambda_\flat)\to\rho(\mathscr{A},\mathscr{O})$, then $(\ell_\flat)$ is a cauchy sequence.*

Now, we present our result.

Theorem 4.1. *Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty, closed and bounded subsets of a complete metric space $(\mathscr{M},\rho)$ the pair $(\mathscr{A},\mathscr{O})$ satisfies property UC. Let $\mathscr{U}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ be a generalized Ciric type cyclic coupled F-contractive mapping. If $\mathscr{U}$ is continuous, then $\mathscr{U}$ has a coupled best proximity point.*

Proof. Let $\ell_0\in\mathscr{A}$ and $\lambda_0\in\mathscr{O}$ such that $\ell_{\flat+1}=\mathscr{U}(\lambda_\flat,\ell_\flat)$ and $\lambda_{\flat+1}=\mathscr{U}(\ell_\flat,\lambda_\flat)$.
Then, $(\ell_\flat)\subset\mathscr{A}$ and $(\lambda_\flat)\subset\mathscr{O}$.
Now,

$$\begin{aligned} F(\rho(\ell_\flat,\ell_{\flat+1})-\rho(\mathscr{A},\mathscr{O})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}),\mathscr{U}(\lambda_\flat,\ell_\flat))-\rho(\mathscr{A},\mathscr{O})) \\ &\leq F(\mathcal{M}_c(\ell_{\flat-1},\ell_\flat)-\rho(\mathscr{A},\mathscr{O}))-\tau. \end{aligned} \tag{4.1}$$

Therefore

$$\begin{aligned} \mathcal{M}_c(\ell_{\flat-1},\ell_\flat) &= \max\{\rho(\ell_{\flat-1},\ell_\flat),\rho(\ell_\flat,\mathscr{U}(\lambda_\flat,\ell_\flat)),\rho(\ell_{\flat-1},\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1})), \\ &\quad \frac{1}{2}\rho(\ell_\flat,\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1})),\frac{1}{2}\rho(\ell_{\flat-1},\mathscr{U}(\lambda_\flat,\ell_\flat))\} \\ &= \max\{\rho(\ell_{\flat-1},\ell_\flat),\rho(\ell_\flat,\ell_{\flat+1})\}. \end{aligned}$$

Now, if $\rho(\ell_{\flat-1},\ell_\flat)<\rho(\ell_\flat,\ell_{\flat+1})$, then (4.1) gives
$F(\rho(\ell_\flat,\ell_{\flat+1})-\rho(\mathscr{A},\mathscr{O}))\leq F(\rho(\ell_\flat,\ell_{\flat+1})-\rho(\mathscr{A},\mathscr{O}))-\tau$, leading to a contradiction. So, $\rho(\ell_{\flat-1},\ell_\flat)>\rho(\ell_\flat,\ell_{\flat+1})$.
Therefore,

$$\begin{aligned} F(\rho(\ell_\flat,\ell_{\flat+1})-\rho(\mathscr{A},\mathscr{O})) &\leq F(\rho(\ell_{\flat-1},\ell_\flat)-\rho(\mathscr{A},\mathscr{O}))-\tau \\ &= F(\rho(\mathscr{U}(\lambda_{\flat-2},\ell_{\flat-2}),\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}))-\rho(\mathscr{A},\mathscr{O}))-\tau \\ &\leq \dots \\ &\leq F(\mathcal{M}_c(\ell_0,\ell_1)-\rho(\mathscr{A},\mathscr{O}))-\flat\tau \\ &= F(\rho(\ell_\flat,\lambda_{\flat+1})-\rho(\mathscr{A},\mathscr{O}))-\flat\tau. \end{aligned}$$

So, as $\flat \to \infty$, we get

$$\lim_{\flat\to\infty} F(\rho(\ell_{\flat+1},\ell_{\flat}) - \rho(\mathscr{A},\mathscr{O})) = -\infty$$

Similarly, it can be shown that
$\lim_{\flat\to\infty} F(\rho(\lambda_{\flat+1},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) = -\infty.$

Now,

$$\begin{aligned} F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}),\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})) - \rho(\mathscr{A},\mathscr{O})) \\ &\leq F(\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) - \rho(\mathscr{A},\mathscr{O})) - \tau \end{aligned} \tag{4.2}$$

Therefore

$$\begin{aligned} \mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \max\{&\rho(\ell_{\flat-1},\lambda_{\flat-1}), \rho(\ell_{\flat-1},\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1})), \rho(\lambda_{\flat-1},\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})), \\ &\frac{1}{2}\rho(\ell_{\flat-1},\mathscr{U}(\ell_{\flat-1},\lambda_{\flat-1})), \frac{1}{2}\rho(\lambda_{\flat-1},\mathscr{U}(\lambda_{\flat-1},\ell_{\flat-1}))\} \\ = \max\{&\rho(\ell_{\flat-1},\lambda_{\flat-1}), \rho(\ell_{\flat-1},\ell_{\flat}), \rho(\lambda_{\flat-1},\lambda_{\flat})\}. \end{aligned}$$

Here, three cases arise:

Case 1. $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\ell_{\flat-1},\ell_{\flat})$.
Then, (4.2) gives
$F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) \leq F(\rho(\ell_{\flat-1},\ell_{\flat})\rho(\mathscr{A},\mathscr{O})) - \tau \leq \ldots \leq F(\rho(\ell_0,\ell_1)) - \flat\tau.$
That is, $F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) \to -\infty$ as $\flat \to \infty$.

Case 2. $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\lambda_{\flat-1},\lambda_{\flat})$.
Then, (4.2) gives
$F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) \leq F(\rho(\lambda_{\flat-1},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) - \tau \leq \ldots \leq F(\rho(\lambda_0,\lambda_1) - \rho(\mathscr{A},\mathscr{O})) - \flat\tau.$
That is, $F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) \to -\infty$ as $\flat \to \infty$.

Case 3. $\mathcal{M}_c(\ell_{\flat-1},\lambda_{\flat-1}) = \rho(\ell_{\flat-1},\lambda_{\flat-1})$.
Then,

$$\begin{aligned} F(\rho(\ell_{\flat},\lambda_{\flat}) - \rho(\mathscr{A},\mathscr{O})) &\leq F(\rho(\ell_{\flat-1},\lambda_{\flat-1}) - \rho(\mathscr{A},\mathscr{O})) - \tau \\ &= F(\rho(\mathscr{U}(\lambda_{\flat-2},\ell_{\flat-2}),\mathscr{U}(\ell_{\flat-2},\lambda_{\flat-2})) - \rho(\mathscr{A},\mathscr{O})) - \tau \\ &\leq F(M(\ell_{\flat-2},\lambda_{\flat-2}) - \rho(\mathscr{A},\mathscr{O})) - 2\tau. \end{aligned}$$

Here, it is easy to see that
$\mathcal{M}_c(\ell_{\flat-2},\lambda_{\flat-2}) = \max\{\rho(\ell_{\flat-2},\lambda_{\flat-2}), \rho(\ell_{\flat-2},\ell_{\flat-1}), \rho(\lambda_{\flat-2},\lambda_{\flat-1})\}.$
So, as seen in Case 1 and Case 2, $\rho(\ell_{\flat-2},\ell_{\flat-1})$ and $\rho(\lambda_{\flat-2},\lambda_{\flat-1})$ as maximums will eventually give

$\lim_{\flat\to\infty} F(\rho(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) = -\infty.$

While $\rho(\ell_{\flat-2}, \lambda_{\flat-2})$ as maximum will give

$$\begin{aligned} F(\rho(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) &\leq F(\rho(\ell_{\flat-2}, \lambda_{\flat-2}) - \rho(\mathscr{A}, \mathscr{O})) - 2\tau \\ &\leq F(\mathcal{M}_c(\ell_{\flat-3}, \lambda_{\flat-3}) - \rho(\mathscr{A}, \mathscr{O})) - 3\tau. \end{aligned}$$

Continuing the above argument repeatedly, we will eventually get

$\lim_{\flat\to\infty} F(\rho(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) = -\infty.$

Thus, in all the cases, we have

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) = -\infty \\ &\Longrightarrow \lim_{\flat\to\infty} (\rho(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) = 0 \\ &\Longrightarrow \lim_{\flat\to\infty} \rho(\ell_\flat, \lambda_\flat) = \rho(\mathscr{A}, \mathscr{O}) \end{aligned}$$

Using similar arguments, it can be proved that $\lim_{\flat\to\infty} \rho(\ell_{\flat+1}, \lambda_\flat) = \rho(\mathscr{A}, \mathscr{O})$. Thus, using Lemma (4.1), $(\ell_\flat)$ is a cauchy sequence in $\mathscr{A}$ as $(\mathscr{A}, \mathscr{O})$ satisfies property UC. Hence, there exists some $\varrho \in \mathscr{A}$ such that $\lim_{\flat\to\infty} \ell_\flat = \varrho$, as $\mathscr{A}$ is closed.

Similarly, the same argument may be applied to obtain $\zeta \in \mathscr{O}$ such that $\lim_{\flat\to\infty} \lambda_\flat = \zeta$.

Now,

$$\begin{aligned} F(\rho(\ell_\flat, \mathscr{U}(\ell_\flat, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) &= F(\rho(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\ell_\flat, \lambda_\flat)) - \rho(\mathscr{A}, \mathscr{O})) \\ &\leq F(\mathcal{M}_c(\ell_{\flat-1}, \lambda_\flat) - \rho(\mathscr{A}, \mathscr{O})) - \tau \\ &\quad .... \\ &\quad .... \\ &\to -\infty \text{ as } \flat \to \infty. \end{aligned}$$

So, as $\flat \to \infty$, using continuity of $\mathscr{U}$, we get

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\varrho, \mathscr{U}(\varrho, \zeta) - \rho(\mathscr{A}, \mathscr{O})) = -\infty \\ \Longrightarrow &\lim_{\flat\to\infty} (\rho(\varrho, \mathscr{U}(\varrho, \zeta) - \rho(\mathscr{A}, \mathscr{O})) = 0 \\ \Longrightarrow &\rho(\varrho, \mathscr{U}(\varrho, \zeta)) = \rho(\mathscr{A}, \mathscr{O}). \end{aligned}$$

Similarly we can get $\rho(\zeta, \mathscr{U}(\zeta, \varrho)) = \rho(\mathscr{A}, \mathscr{O})$ Thus, (ϱ, ζ) is a coupled best proximity point of $\mathscr{U}$. □

5. Coupled fixed point for generalized Ciric type cyclic coupled F-contractive multivalued mapping

Let $(\mathscr{M}, \rho)$ be a metric space. Let $P(\mathscr{M})$ denotes the set of all non empty subsets of $\mathscr{M}$ and $CB(\mathscr{M})$ denotes the set of all non empty closed and bounded subsets in $\mathscr{M}$. Then, Hausdorff metric induced by d, $\Psi : CB(\mathscr{M}) \times CB(\mathscr{M}) \to \mathbb{R}$, is defined by

$$\Psi(\mathscr{A}, \mathscr{O}) = \max\{\sup_{\ell \in \mathscr{A}} \mathfrak{D}(\ell, \mathscr{O}), \sup_{\lambda \in \mathscr{O}} \mathfrak{D}(\lambda, \mathscr{A})\},$$

where $\mathscr{A}$, $\mathscr{O} \in CB(\mathscr{M})$ and $\mathfrak{D}(\ell, \mathscr{O}) = \inf\{\rho(\ell, \lambda) | \lambda \in \mathscr{O}\}$. In 1969, Nadler [18] used this concept of Hausdorff metric and put forth the notion of multivalued contraction. A mapping $\mathscr{U} : \mathscr{M} \to CB(\mathscr{M})$ is a multivalued contraction if there exists $L \in [0, 1)$ such that

$$\Psi(\mathscr{U}\ell, \mathscr{U}\lambda) \leq L\rho(\ell, \lambda), \textit{ for all } \ell, \lambda \in \mathscr{M}.$$

Here, we present the following multivalued contraction mapping which is in fact a blend of F-contraction, cyclic ciric type contraction in multivalued version.

Definition 5.1. *Let $\mathscr{A}$ and $\mathscr{O}$ be two nonempty subsets of a metric space $(\mathscr{M}, \rho)$ and $\mathcal{W} = \mathscr{A} \cup \mathscr{O}$. Then, the mapping $\mathscr{U} : \mathcal{W} \times \mathcal{W} \to P(\mathcal{W})$ is said to be generalized Ciric type cyclic coupled F-contractive multivalued mapping, if*

1. *$\mathscr{U}(\mathscr{A} \times \mathscr{O}) \subset \mathscr{O}$ and $\mathscr{U}(\mathscr{O} \times \mathscr{A}) \subset \mathscr{A}$;*
2. *there exists $F \in \mathcal{F}$ with $F(a + b) \leq F(a) + F(b)$ and a positive real number τ such that for all $\ell, \lambda, a, b \in \mathcal{W}$,*
   $\Psi(\mathscr{U}(\ell, \lambda), \mathscr{U}(a, b)) > 0 \implies \tau + F(\Psi(\mathscr{U}(a, b), \mathscr{U}(\ell, \lambda))) \leq F(\mathcal{M}_{\mathfrak{bc}}(b, \ell))$,
   where, $\mathcal{M}_{\mathfrak{bc}}(b, \lambda) = \max\{\rho(b, \lambda), \mathfrak{D}(b, \mathscr{U}(a, b)), \mathfrak{D}(\lambda, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\mathfrak{D}(b, \mathscr{U}(\ell, \lambda)), \frac{1}{2}\mathfrak{D}(\lambda, \mathscr{U}(a, b))\}$.

Definition 5.2. *Let $(\mathscr{M}, \rho)$ be a metric space. Then, a subset $\mathscr{A}$ of $\mathscr{M}$ is called proximinal if for any $y \in \mathscr{M}$, there exists $uin\mathscr{A}$ such that $\rho(y, \ell) = \mathfrak{D}(y, \mathscr{A})$.*

Denote $P_{prox}(\mathscr{M}) = \{\mathscr{A} \in P(\mathscr{M}) |\ \mathscr{A}$ is proximinal$\}$.

Theorem 5.1. *Let $\mathscr{A}$ and $\mathscr{O}$ be two non-empty, closed and bounded subsets of a complete metric space $(\mathscr{M}, \rho)$. Let $\mathscr{U} : \mathcal{W} \times \mathcal{W} \to P_{prox}(\mathcal{W})$ be a Generalized Ciric type cyclic coupled F-contractive Multivalued mapping. If $\mathscr{U}$ is continuous, in that case, $\mathscr{U}$ has a coupled fixed point.*

Proof. Let $\ell_0 \in \mathscr{A}$ and $\lambda_0 \in \mathscr{O}$.
Then, $\mathscr{U}(\lambda_0, \ell_0) \in P_{prox}(\mathcal{W})$ and $\mathscr{U}(\ell_0, \lambda_0) \in P_{prox}(\mathcal{W})$.
This implies that there exists some $\ell_1 \in \mathscr{U}(\lambda_0, \ell_0) \subset \mathscr{A}$ and $\lambda_1 \in \mathscr{U}(\ell_0, \lambda_0) \subset \mathscr{O}$ such that $\rho(\ell_0, \ell_1) = \mathfrak{D}(\ell_0, \mathscr{U}(\lambda_0, \ell_0))$ and $\rho(\lambda_0, \lambda_1) = \mathfrak{D}(\lambda_0, \mathscr{U}(\ell_0, \lambda_0))$.
Again, existence of ℓ_1 and λ_1 implies the existence of $\mathscr{U}(\lambda_1, \ell_1)$ and $\mathscr{U}(\ell_1, \lambda_1)$ in $P_{prox}(\mathcal{W})$.
This further implies the existence of some $\ell_2 \in \mathscr{U}(\lambda_1, \ell_1) \subset \mathscr{A}$, $\lambda_2 \in \mathscr{U}(\ell_1, \lambda_1) \subset \mathscr{O}$,
and $\rho(\ell_1, \ell_2) = \mathfrak{D}(\ell_1, \mathscr{U}(\lambda_1, \ell_1))$, $\rho(\lambda_1, \hbar\lambda_2) = \mathfrak{D}(\lambda_1, \mathscr{U}(\ell_1, \lambda_1))$.
Thus, continuing this process, we get sequences $(\ell_{\mathfrak{b}})$ and $(\lambda_{\mathfrak{b}})$ in $\mathscr{A}$ and $\mathscr{O}$, respectively such that $\ell_{\mathfrak{b}+1} \in \mathscr{U}(\lambda_{\mathfrak{b}}, \ell_{\mathfrak{b}})$, $\lambda_{\mathfrak{b}+1} \in \mathscr{U}(\ell_{\mathfrak{b}}, \lambda_{\mathfrak{b}})$,
and
$\rho(\ell_{\mathfrak{b}}, \ell_{\mathfrak{b}+1}) = \mathfrak{D}(\ell_{\mathfrak{b}}, \mathscr{U}(\lambda_{\mathfrak{b}}, \ell_{\mathfrak{b}}))$, $\rho(\lambda_{\mathfrak{b}}, \lambda_{\mathfrak{b}+1}) = \mathfrak{D}(\lambda_{\mathfrak{b}}, \mathscr{U}(\ell_{\mathfrak{b}}, \lambda_{\mathfrak{b}}))$.
Now, take $\ell_{\mathfrak{b}} \notin \mathscr{U}(\lambda_{\mathfrak{b}}, \ell_{\mathfrak{b}})$.

Then, $\mathfrak{D}(\ell_\flat, \mathscr{U}(\lambda_\flat, \ell_\flat)) > 0$.
So,

$$\begin{aligned} F(\rho(\ell_\flat, \ell_{\flat+1})) &= F(\mathfrak{D}(\ell_\flat, \mathscr{U}(\lambda_\flat, \ell_\flat))) \\ &\leq F(\Psi(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\lambda_\flat, \ell_\flat))) \\ &\leq F(\mathcal{M}_{\flat c}(\ell_{\flat-1}, \ell_\flat)) - \tau. \end{aligned} \tag{5.1}$$

We have

$$\begin{aligned} \mathcal{M}_{\flat c}(\ell_{\flat-1}, \ell_\flat) = \max\{&\rho(\ell_{\flat-1}, \ell_\flat), \mathfrak{D}(\ell_\flat, \mathscr{U}(\lambda_\flat, \ell_\flat)), \mathfrak{D}(\ell_{\flat-1}, \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1})), \\ &\frac{1}{2}\mathfrak{D}(\ell_\flat, \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1})), \frac{1}{2}\mathfrak{D}(\ell_{\flat-1}, \mathscr{U}(\lambda_\flat, \ell_\flat))\} \\ = \max\{&\rho(\ell_{\flat-1}, \ell_\flat), \rho(\ell_\flat, \ell_{\flat+1})\}. \end{aligned}$$

Now, if $\rho(\ell_{\flat-1}, \ell_\flat) < \rho(\ell_\flat, \ell_{\flat+1})$, then (5.1) gives
$F(\rho(\ell_\flat, \ell_{\flat+1})) \leq F(\rho(\ell_\flat, \ell_{\flat+1})) - \tau$, which is a contradiction. So, $\rho(\ell_{\flat-1}, \ell_\flat) > \rho(\ell_\flat, \ell_{\flat+1})$.
Therefore,

$$\begin{aligned} F(\rho(\ell_\flat, \ell_{\flat+1})) &\leq F(\rho(\ell_{\flat-1}, \ell_\flat)) - \tau \\ &= F(\mathfrak{D}(\ell_{\flat-1}, \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}))) - \tau \\ &\leq F(\Psi(\mathscr{U}(\lambda_{\flat-2}, \ell_{\flat-2}), \mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}))) - \tau \\ &\leq F(\mathcal{M}_{\flat c}(\ell_{\flat-2}, \ell_{\flat-1})) - 2\tau \\ &\leq \dots \\ &= F(\rho(\ell_0, \ell_1)) - \flat\tau. \end{aligned} \tag{5.2}$$

So, as $\flat \to \infty$, we get

$$\begin{aligned} &\lim_{\flat\to\infty} F(\rho(\ell_\flat, \ell_{\flat+1})) = -\infty \\ \Longrightarrow &\lim_{\flat\to\infty} (\rho(\ell_\flat, \ell_{\flat+1})) = 0. \end{aligned}$$

Let $\delta_\flat = \rho(\ell_\flat, \ell_{\flat+1})$.
Also, by virtue of the properties of F, a real $k \in (0, 1)$ exists such that $\lim\limits_{\flat\to\infty} (\delta_\flat)^k F(\delta_\flat) = 0$.
Thus, from (5.2), we have

$$\begin{aligned} &F(\delta_\flat) \leq F(\delta_0) - \flat\tau \\ \Longrightarrow &(\delta_\flat)^k F(\delta_\flat) \leq (\delta_\flat)^k F(\delta_0) - \flat\tau(\delta_\flat)^k \\ \Longrightarrow &(\delta_\flat)^k F(\delta_\flat) - (\delta_\flat)^k F(\delta_0) \leq -\flat\tau(\delta_\flat)^k. \end{aligned}$$

Letting $\flat$ tends to ∞, the above gives $\lim\limits_{\flat\to\infty} m(\delta_\flat)^k = 0$. Therefore, a natural number N exists such that for all $\flat \geq N$, $\flat(\delta_\flat)^k \leq 1$.

That is, for $\flat \geq N$, $\delta_\flat \leq \frac{1}{\flat^{\frac{1}{k}}}$.
Now, for $n > \flat \geq N$, we have

$$\begin{aligned}
\rho(\ell_\flat, \ell_n) &\leq \rho(\ell_\flat, \ell_{\flat+1}) + \rho(\ell_{\flat+1}, \ell_{\flat+2}) + ... + \rho(\ell_{n-1}, \ell_n) \\
&= \delta_\flat + \delta_{\flat+1} + ... + \delta_{n-1} \\
&\leq \sum_{i=\flat}^{n-1} \delta_i \\
&\leq \sum_{i=\flat}^{\infty} \delta_i \\
&\leq \sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}.
\end{aligned}$$

Since the series $\sum_{i=\flat}^{\infty} \frac{1}{i^{\frac{1}{k}}}$ converges, $\rho(\ell_\flat, \ell_n) \to 0$ as $\flat \to \infty$. Thus, $(\ell_\flat)$ is a cauchy sequence. Therefore, $(\ell_\flat)$ converges to some $\varrho \in \mathscr{A}$, $\mathscr{A}$ being closed.
Similarly, the same argument may be applied to obtain $\zeta \in \mathscr{O}$ such that $\lim\limits_{\flat\to\infty} \lambda_\flat = \zeta$.
Now,

$$\begin{aligned}
F(\mathfrak{D}(\ell_{\flat+1}, \mathscr{U}(\ell_\flat, \lambda_\flat)) &\leq F(\rho(\ell_{\flat+1}, \ell_\flat) + \rho(\ell_\flat, \lambda_\flat) + \mathfrak{D}(\lambda_\flat, \mathscr{U}(\ell_\flat, \lambda_\flat)) \\
&\leq F(\rho(\ell_{\flat+1}, \ell_\flat)) + F(\rho(\ell_\flat, \lambda_\flat)) + F(\rho(\lambda_\flat, \lambda_{\flat+1})) \\
&= F(\mathfrak{D}(\ell_\flat, \mathscr{U}(\lambda_\flat, \ell_\flat))) + F(\rho(\ell_\flat, \lambda_\flat)) + F(\rho(\lambda_\flat, \lambda_{\flat+1})) \\
&\leq F(\Psi(\mathscr{U}(\lambda_{\flat-1}, \ell_{\flat-1}), \mathscr{U}(\lambda_\flat, \ell_\flat))) + F(\rho(\ell_\flat, \lambda_\flat)) + F(\rho(\lambda_\flat, \lambda_{\flat+1})) \\
&\leq F(\mathcal{M}_{bc}(\ell_{\flat-1}, \ell_\flat)) + F(\rho(\ell_\flat, \lambda_\flat)) + F(\rho(\lambda_\flat, \lambda_{\flat+1})) - \tau \\
&\leq ... \\
&\leq F(\rho(\ell_0, \ell_1)) + F(\rho(\ell_\flat, \lambda_\flat)) + F(\rho(\lambda_\flat, \lambda_{\flat+1})) - \flat\tau.
\end{aligned}$$

So, as $\flat \to \infty$, using continuity of $\mathscr{U}$, we get

$$\begin{aligned}
&\lim_{\flat\to\infty} F(\mathfrak{D}(\varrho, \mathscr{U}(\varrho, \zeta)) = -\infty \\
\Longrightarrow &\lim_{\flat\to\infty} (\mathfrak{D}(\varrho, \mathscr{U}(\varrho, \zeta)) = 0 \\
\Longrightarrow &\varrho \in \mathscr{U}(\varrho, \zeta).
\end{aligned}$$

Similarly, we can get $\zeta \in \mathscr{U}(\zeta, \varrho)$ Which prooves, (ϱ, ζ) is a coupled fixed point of $\mathscr{U}$. □

6. Application

The theory of Fixed point finds wide applications in the field of mathematical optimization where in dynamic programming turns out to be notably useful tool. Moreover, Finding solutions to system of functional equations often appears while studying dynamic programming problems. Here, in this section, our focus centers on examining the existence of a bounded solution for a given system of functional equations-a crucuial facet that holds prominence in the realm of papers dedicated to dynamic programming problem [19].

Consider the following system of functional equations:

$$\begin{aligned} a(\kappa) &= \sup_{s\in\mathcal{D}}[\Theta(\kappa,s)+\Psi(\kappa,s,a(\eta(\kappa,s)),b(\eta(\kappa,s)))],\ t\in\mathcal{W} \\ b(\kappa) &= \sup_{s\in\mathcal{D}}[\Theta(\kappa,s)+\Psi(\kappa,s,b(\eta(\kappa,s)),a(\eta(\kappa,s)))],\ t\in\mathcal{W}, \end{aligned} \tag{6.1}$$

where $\mathcal{W}$ is a state space, $\mathcal{D}$ is a decision space, $\eta:\mathcal{W}\times\mathcal{D}\to\mathcal{W}$, $\Theta:\mathcal{W}\times\mathcal{D}\to\mathcal{R}$ and $\Psi:\mathcal{W}\times\mathcal{D}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Let the set of all bounded real valued functions defined on $\mathcal{W}$ is denoted by $\mathcal{B}(\mathcal{W})$ and for any $a\in\mathcal{B}(\mathcal{W})$, $\|a\|=\sup_{t\in\mathcal{W}}|a(\kappa)|$.

Clearly, $\mathcal{B}(\mathcal{W})$ endowed with sup metric,

$\rho(a,b)=\sup_{t\in\mathcal{W}}|a(\kappa)-b(\kappa)|$, for all $a,b\in\mathcal{B}(\mathcal{W})$, is complete.

Let $\mathscr{A}$ and $\mathscr{O}$ be nonempty closed subsets of $\mathcal{B}(\mathcal{W})$ such that $\mathscr{M}=\mathscr{A}\cup\mathscr{O}$. Define a mapping $\mathscr{U}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ by

$$\mathscr{U}(a,b)(\kappa)=\sup_{s\in\mathcal{D}}[\Theta(\kappa,s)+\Psi(\kappa,s,a(\eta(\kappa,s)),b(\eta(\kappa,s)))] \tag{6.2}$$

for all $t\in\mathcal{W}$ and $a,b\in\mathscr{M}$.

Also, let $\mathcal{M}_c(b,\lambda)=\max\{\rho(b,\lambda),\rho(b,\mathscr{U}(a,b)),\rho(\lambda,\mathscr{U}(\ell,\lambda)),\frac{1}{2}\rho(b,\mathscr{U}(\ell,\lambda)),\frac{1}{2}\rho(\lambda,\mathscr{U}(a,b))\}$.

Theorem 6.1. *Let $\mathscr{U}:\mathscr{M}\times\mathscr{M}\to\mathscr{M}$ defined by (6.2) be a continuous mapping such that the following holds:*

1. $\Theta:\mathcal{W}\times\mathcal{D}\to\mathcal{R}$ *and* $\Psi:\mathcal{W}\times\mathcal{D}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *are continuous and bounded.*
2. *there exists* $\tau>0$ *such that* $|\Psi(\kappa,s,a,b)-\Psi(\kappa,s,\ell,\lambda)|\le e^{-\tau}\mathcal{M}_c(b,\lambda)$ *for all* $a,b,\ell,\lambda\in\mathscr{M}$, $\kappa\in\mathcal{W}$ *and* $s\in\mathcal{D}$.

Then, the system (6.1) has a unique bounded solution in $\mathscr{A}\cap\mathscr{O}$.

Proof. For any $b(\kappa)\in\mathscr{A}$ and $a(\kappa)\in\mathscr{O}$, the system (6.1) implies

$$\begin{aligned} \mathscr{U}(a,b)(\kappa) &= \sup_{s\in\mathcal{D}}[\Theta(\kappa,s)+\Psi(\kappa,s,a(\eta(\kappa,s)),b(\eta(\kappa,s)))]\,ds \\ &= a(\kappa)\in\mathscr{O} \\ \text{and } \mathscr{U}(b,a)(\kappa) &= \sup_{s\in\mathcal{D}}[\Theta(\kappa,s)+\Psi(\kappa,s,b(\eta(\kappa,s)),a(\eta(\kappa,s)))]\,ds \\ &= b(\kappa)\in\mathscr{A}. \end{aligned}$$

Now, consider any arbitrary $\lambda>0$, $t\in\mathcal{W}$ and $a,b,\ell,\lambda\in\mathscr{M}$, then there exists $s_1,s_2\in\mathcal{D}$ such that

$$\mathscr{U}(a,b)(\kappa)<\Theta(\kappa,s_1)+\Psi(\kappa,s_1,a(\eta(\kappa,s_1)),b(\eta(\kappa,s_1)))+\lambda \tag{6.3}$$

$$\mathscr{U}(\ell,\lambda)(\kappa)<\Theta(\kappa,s_2)+\Psi(\kappa,s_2,\ell(\eta(\kappa,s_2)),\lambda(\eta(\kappa,s_2)))+\lambda \tag{6.4}$$

$$\mathscr{U}(a,b)(\kappa)\ge\Theta(\kappa,s_2)+\Psi(\kappa,s_2,a(\eta(\kappa,s_2)),b(\eta(\kappa,s_2))) \tag{6.5}$$

$$\mathscr{U}(\ell,\lambda)(\kappa)\ge\Theta(\kappa,s_1)+\Psi(\kappa,s_1,\ell(\eta(\kappa,s_1)),\lambda(\eta(\kappa,s_1))). \tag{6.6}$$

So, using (6.3) and (6.6),

$$\begin{aligned}\mathscr{U}(a,b)(\kappa) - \mathscr{U}(\ell,\lambda)(\kappa) &< \Psi(\kappa,s_1,a(\eta(\kappa,s_1)),b(\eta(\kappa,s_1))) - \Psi(\kappa,s_1,\ell(\eta(\kappa,s_1)),\lambda(\eta(\kappa,s_1))) + \lambda \\ &\leq |\Psi(\kappa,s_1,a(\eta(\kappa,s_1)),b(\eta(\kappa,s_1))) - \Psi(\kappa,s_1,\ell(\eta(\kappa,s_1)),\lambda(\eta(\kappa,s_1)))| + \lambda \\ &\leq e^{-\tau}\mathcal{M}_c(b,\lambda) + \lambda.\end{aligned}$$

That is, $\mathscr{U}(a,b)(\kappa) - \mathscr{U}(\ell,\lambda)(\kappa) \leq e^{-\tau}\mathcal{M}_c(b,\lambda) + \lambda$.
Similarly, using (6.4) and (6.5), we get
$\mathscr{U}(\ell,\lambda)(\kappa) - \mathscr{U}(a,b)(\kappa) \leq e^{-\tau}\mathcal{M}_c(b,\lambda) + \lambda$.
Thus, $|\mathscr{U}(a,b)(\kappa) - \mathscr{U}(\ell,\lambda)(\kappa)| \leq e^{-\tau}\mathcal{M}_c(b,\lambda) + \lambda$.
Since λ is arbitrary,
$\rho(\mathscr{U}(a,b)(\kappa) - \mathscr{U}(\ell,\lambda)(\kappa)) \leq e^{-\tau}\mathcal{M}_c(b,\lambda)$.
Now, taking natural logarithm on both side, we get
$\ln(\rho(\mathscr{U}(a,b) - \mathscr{U}(\ell,\lambda))) \leq \ln(e^{-\tau}\mathcal{M}_c(b,\lambda))$.
That is, $\ln(\rho(\mathscr{U}(a,b) - \mathscr{U}(\ell,\lambda))) \leq \ln(\mathcal{M}_c(b,\lambda)) - \tau$.
Thus, the mapping $\mathscr{U}$ is a cyclic coupled F-contractive mapping of generalized Ciric type with $F(\delta) = \ln\delta$. As the necessary conditions of Theorem 3.1 are all being fulfilled. So, $\mathscr{U}$ has a strong coupled fixed point, say ϱ, in $\mathscr{A} \cap \mathscr{O}$.
That is, $\mathscr{U}(\varrho,\varrho) = \varrho$.
Hence, (ϱ,ϱ) is the unique solution of the given system of functional equations (6.1). □

7. Conclusion

Cyclic coupled F-contractive mappings of generalized Ciric type have been defined and results for existence of strong coupled fixed point coupled coincidence point, coupled best proximity points have been established and proved. A Generalized Ciric type cyclic coupled F-contractive Multivalued mapping was also defined and an existence result for coupled fixed point has been established. We have also explored the application of our result concerning the existence of solution of a system of functional equations.

Acknowledgments: The author D. Santina, S. Haque and N. Mlaiki would like to thank Prince Sultan University for paying the publication fees for this work through TAS LAB.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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