Int. J. Anal. Appl. (2025), 23:324

International Journal of Analysis and Applications

A Notion of Fractional Slice Monogenic Functions with Respect to a Pair of Real Valued Functions

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Abstract. This work presents the basic elements and results of a Clifford algebra valued fractional slice monogenic functions theory defined from the null-solutions of a suitably fractional Cauchy-Riemann operator in the Riemann-Liouville and Caputo sense with respect to a pair of real valued functions on certain domains of Euclidean spaces.

1. Introduction

Slice monogenic functions are defined on domains of Euclidean spaces having values in a Clifford algebra. The literature is very rich of results and the studies on the topic are ongoing, see for instance [1, 3–9, 11, 33, 34].

Fractional integrals and derivatives are mathematical tools widely used in several branches of science and engineering. Fractional calculus, which addresses the derivatives and integrals with arbitrary real or complex order, is nowadays an extensively developed topic which the reader can approach in the classical references [23, 24, 26–28, 30, 31].

Clifford analysis is a function theory studying null solutions of the Cauchy-Riemann or Dirac systems, called monogenic functions, which are defined on domains of Euclidean spaces and with values in a Clifford algebra. Standard reference books are [2, 10, 15, 21].

In recent years, various extension of the Fractional calculus into Clifford analysis (in particular, into quaternionic analysis) has attracted more and more attention in the literature. Some examples of effective establishments can be found in [12, 13, 17–20, 22, 29, 32].

Inspired by [20], where the notion of fractional slice regular functions of a quaternionic variable defined as null-solutions of a fractional Cauchy–Riemann operator is introduced, we further

Received: Jan. 18, 2025.

2020 *Mathematics Subject Classification.* 26A33, 30G30, 30G35, 32A30, 45P05.

Key words and phrases. slice monogenic functions; Clifford algebras; fractional calculus.

https://doi.org/10.28924/2291-8639-23-2025-324

ISSN: 2291-8639

generalize these ideas into twofold directions: one is a notion of fractional slice monogenic function theory and the other is the study of null-solutions of a fractional Cauchy–Riemann operator with respect to two real valued functions, which can be exponential or lineal functions. In any case we expand the theory presented in [20].

2. Preliminaries

In this section, we collect some preliminary results on Fractional calculus and on Slice monogenic functions theory.

2.1. **Fractional integral and derivative in Riemann-Liouville and Caputo sense with respect to another function.** Suppose that $-\infty < a < b < \infty$ and $\alpha \in \mathbb{C}$ with $\Re\alpha > 0$. Let $g \in C^1([a,b],\mathbb{R})$ be such that $g'(t) > 0$ for all $t \in [a,b]$. Let us recall that the Riemann-Liouville integrals of order α of $f \in L^1([a,b],\mathbb{R})$, with respect to g, on the left and on the right, are defined by

$$(\mathbf{I}_{a^+}^{\alpha,g} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(\tau)}{(g(x)-g(\tau))^{1-\alpha}} g'(\tau)d\tau, \quad \text{with} \quad x > a$$

and

$$(\mathbf{I}_{b^-}^{\alpha,g} f)(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(\tau)}{(g(\tau)-g(x))^{1-\alpha}} g'(\tau)d\tau, \quad \text{with} \quad x < b,$$

respectively.

The fractional derivatives in the Riemann-Liouville sense, on the left and on the right, with respect to g, are defined by

$$({}_{RL}D_{a^+}^{\alpha,g} f)(x) := \frac{1}{g'(x)} \frac{d}{dx}\left[(\mathbf{I}_{a^+}^{1-\alpha,g} f)(x)\right] \tag{2.1}$$

and

$$({}_{RL}D_{b^-}^{\alpha,g} f)(x) := \frac{1}{g'(x)} \frac{d}{dx}\left[(\mathbf{I}_{b^-}^{1-\alpha,g} f)(x)\right] \tag{2.2}$$

respectively. It is worth noting that the derivatives in (2.1) and (2.2) exist for $f \in AC^1([a,b],\mathbb{R})$. Fractional Riemann-Liouville integral and derivative are linear operators.

Fundamental theorem for Riemann-Liouville fractional calculus [24,25] shows that

$$({}_{RL}D_{a^+}^{\alpha,g}\mathbf{I}_{a^+}^{\alpha,g} f)(x) = f(x) \quad \text{and} \quad ({}_{RL}D_{b^-}^{\alpha,g}\mathbf{I}_{b^-}^{\alpha,g} f)(x) = f(x). \tag{2.3}$$

From [25, Definition 2.7] we see that given $f \in C^1([a,b],\mathbb{R})$, the fractional derivatives in the Caputo sense, on the left and on the right, with respect to g, are defined by

$$\begin{aligned}({}_{C}D_{a^+}^{\alpha,g} f)(x) &:= (\mathbf{I}_{a^+}^{1-\alpha,g}\frac{f'}{g'})(x),\\ ({}_{C}D_{b^-}^{\alpha,g} f)(x) &:= (\mathbf{I}_{b^-}^{1-\alpha,g}\frac{f'}{g'})(x),\end{aligned} \tag{2.4}$$

respectively.

Finally, if $f \in C^1([a,b],\mathbb{R})$ we have

$$({}_C D^{\alpha,g}_{a^+}(\mathbf{I}^{1-\alpha,g}_{a^+} f))(x) :=(\mathbf{I}^{1-\alpha,g}_{a^+} \frac{1}{g'}\frac{d}{dx}(\mathbf{I}^{1-\alpha,g}_{a^+} f))(x) = (\mathbf{I}^{1-\alpha,g}_{a^+} {}_{RL}D^{\alpha,g}_{a^+} f)(x), \tag{2.5}$$

for all $x \in [a,b]$.

Similarly,

$$({}_C D^{\alpha,g}_{b^-}(\mathbf{I}^{1-\alpha,g}_{a^+} f))(x) :=(\mathbf{I}^{1-\alpha,g}_{b^-} \frac{1}{g'} {}_{RL}D^{\alpha,g}_{b^-} f)(x), \tag{2.6}$$

for all $x \in [a,b]$.

2.2. **Slice monogenic functions.** We first give some basic knowledge in relation to Clifford algebra [2, 14]. Let $\{e_1,\dots,e_n\}$ imaginary units satisfying $e_ie_j + e_je_i = -2\delta_{ij}$. Clifford algebra $\mathbb{R}_n$ is formed by $\sum_A e_A x_A$, where $A = \{i_1,\dots,i_r\}$, $i_\ell \in \{1,2,...,n\}$ and $i_1 < \dots < i_r$ is a multi-index, $e_A = e_{i_1}e_{i_2}\cdots e_{i_r}$ and $e_\emptyset = 1$, $x_A \in \mathbb{R}$. An element $(x_1,\dots,x_n) \in \mathbb{R}_n$ is identified with a 1-vector in $\mathbb{R}_n$ through the map

$$(x_1,\dots,x_n) \mapsto \underline{x} = x_1e_1 + \cdot + x_ne_n.$$

In addition, $(x_0, x_1,\dots,x_n) \in \mathbb{R}^{n+1}$ will be identified with $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_je_j$, which is called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is $|x|^2 = x_0^2 + x_1^2 + \dots + x_n^2$. The real part x_0 of x is written as $Re[x]$. Given $x \in \mathbb{R}^{n+1}$ and $r > 0$ denote $\mathbb{D}(x,r) := \{y \in \mathbb{R}^{n+1} \mid |x-y| < r\}$.

The sphere of unit 1-vectors in $\mathbb{R}^n$ will be given by $\mathbb{S} = \{\underline{x} = e_1x_1 + \dots + e_nx_n \mid x_1^2 + \dots + x_n^2 = 1\}$ and given $\mathcal{I} \in \mathbb{S}$ we define $\mathbb{C}_\mathcal{I} := \{u + \mathcal{I}v \mid u, v \in \mathbb{R}\}$, which is a 2-dimensional real subspace of $\mathbb{R}^{n+1}$ isomorphic to $\mathbb{C}$ as fields.

In addition, given a paravector $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ denote

$$\mathcal{I}_x = \begin{cases} \frac{\underline{x}}{|\underline{x}|} & if \ \ \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} \text{ otherwise.} \end{cases} \tag{2.7}$$

Therefore, $\mathbb{R}^{n+1} = \bigcup_{\mathcal{I}\in\mathbb{S}} \mathbb{C}_\mathcal{I}$.

Definition 2.1. *(Slice monogenic functions) [5, 7]. Suppose that $U \subset \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $\mathcal{I} \in \mathbb{S}$ and let $f_\mathcal{I}$ be the restriction of f to $U_\mathcal{I} := \mathbb{C}_\mathcal{I} \cap U$. Therefore, f is called a (left) slice monogenic function, or s-monogenic function, if*

$$\overline{\partial}_\mathcal{I} f_\mathcal{I}(u + \mathcal{I}v) := \frac{1}{2}\left(\frac{\partial}{\partial u} + \mathcal{I}\frac{\partial}{\partial v}\right) f_\mathcal{I}(u + \mathcal{I}v) = 0, \quad on\ U_\mathcal{I},$$

for every $\mathcal{I} \in \mathbb{S}$.

In what follows, $\mathcal{SM}(U)$ stands for the set of slice monogenic functions defined on U.

Definition 2.2. *(Slice domains). A domain $U \subset \mathbb{R}^{n+1}$ is called slice domain (s-domain for short) if $U \cap R$ is nonempty and if $U_\mathcal{I}$ is a domain in $\mathbb{C}_\mathcal{I}$ for all $\mathcal{I} \in \mathbb{S}$.*

Definition 2.3. *(Axially symmetric domains). A domain $U \subset \mathbb{R}^{n+1}$ is called axially symmetric if, for all $x = x_0 + \mathcal{I}_x|\underline{x}| \in U$ we have that $\{x_0 + \mathcal{I}|\underline{x}|\} \subset U$ for all $\mathcal{I} \in \mathbb{S}^2$.*

Theorem 2.1. *(Representation formula) [5,7]. Let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric s-domain. If $f \in \mathcal{SM}(U)$ and $x = u + \mathcal{I}_x v \in U$, where $u, v \in \mathbb{R}$, then*

$$f(x) = \frac{1}{2}\left(1 - \mathcal{I}_x \mathcal{I}\right) f(u + \mathcal{I} v) + \frac{1}{2}\left(1 + \mathcal{I}_x \mathcal{I}\right) f(u - \mathcal{I} v), \quad \forall \mathcal{I} \in \mathbb{S}.$$

Lemma 2.1. *(Splitting Lemma). Let $U \subset \mathbb{R}^{n+1}$ be an open set and $f \in \mathcal{SM}(U)$. For every $\mathcal{I} = \mathcal{I}_1 \in \mathbb{S}$, let $\mathcal{I}_2, \ldots, \mathcal{I}_n$ be a completion to a basis of $\mathbb{R}_n$ satisfying the defining relations $\mathcal{I}_r \mathcal{I}_s + \mathcal{I}_s \mathcal{I}_r = -2\delta_{rs}$. Then there exist 2^{n-1} many holomorphic functions $F_A : U_{\mathcal{I}} \to \mathbb{C}_{\mathcal{I}}$ such that*

$$f_{\mathcal{I}}(z) = \sum_{|A|=0}^{n-1} F_A(z) \mathcal{I}_A,$$

where $\mathcal{I}_A = \mathcal{I}_{i_1} \cdots \mathcal{I}_{i_s}$ and $A = \{i_1, \ldots, i_s\}$ is a subset of $\{2, \ldots, n\}$, with $i_1 < \cdots < i_s$, and $I_\emptyset = 1$.

Remark 2.1. *If $U \subset \mathbb{R}^{n+1}$, we will use the symbol $Hol(U_{\mathcal{I}})$ to denote the complex linear space of holomorphic functions from $U_{\mathcal{I}}$ to $\mathbb{C}_{\mathcal{I}}$.*

Let $x = x_0 + \underline{x} = u + \mathcal{I}_x v \in \mathbb{R}^{n+1}$ be a nonzero paravector, where $u = x_0$, $v = |\underline{x}|$ and $\mathcal{I}_x = v^{-1}\underline{x}$, with $v \neq 0$ and for $v = 0$, see (2.7). Moreover, if $\mathcal{I}_x = (\zeta_1, \ldots, \zeta_n)$ then $x_k = v\zeta_k$ for $k = 1, \ldots, n$, see [5,7] for more details.

From now on, $U \subset \mathbb{R}^{n+1}$ denotes an axially symmetric s-domain.

Proposition 2.1. *Suppose that $f \in \mathcal{SM}(U)$ and $\mathcal{I} \in \mathbb{S}$. If $\overline{\mathbb{D}(y,r)} \subset U$, then*

$$f(z) = \frac{1}{2\pi} \int_{\partial \mathbb{D}(y,r)_{\mathcal{I}}} (w - z)^{-1} d_w \sigma(\mathcal{I}) f(w), \quad \forall z \in \mathbb{D}(y,r)_{\mathcal{I}},$$

where $d_w\sigma(\mathcal{I}) = -(d_w\sigma)\mathcal{I}$ and $\partial\mathbb{D}(y,r)_{\mathcal{I}}$ is the boundary of $\mathbb{D}(y,r)_{\mathcal{I}}$ in $U_{\mathcal{I}}$. In addition,

$$\int_\Gamma d_w \sigma(\mathcal{I}) f(w) = 0,$$

for all $\mathcal{I} \in \mathbb{S}$ and for any closed, homotopic to a point and piecewise C^1 curve $\Gamma \subset U_{\mathcal{I}}$.

On the other hand, if the set $Z_f \cap U_{\mathcal{I}} = \{z \in U_{\mathcal{I}} \mid f(z) = 0\}$ has an accumulation point in $U_{\mathcal{I}}$, then $f \equiv 0$ on U.

Finally, if $\ell \in C(U, \mathbb{R}_n)$ satisfies

$$\int_\Gamma d_w \sigma(\mathcal{I}) \ell(w) = 0,$$

for any closed, homotopic to a point and piecewise C^1 curve $\Gamma \subset U_{\mathcal{I}}$ and for all $\mathcal{I} \in \mathbb{S}$, then $\ell \in \mathcal{SM}(U)$.

Proof. The previous facts are direct consequence of the Cauchy formula, Cauchy theorem, identity principle and Morera's theorem respectively in the complex plane $\mathbb{C}_{\mathcal{I}}$ using also Representation Formula and Splitting Lemma, see [4–9]. □

It should be pointed out that previous formulas may be extended on all axially symmetric s-domain U but we will not develop this point here.

The following definition is inspired in the work [16], which presents a generalization of the quaternionic slice regular functions.

Definition 2.4. *Suppose that $\lambda \in \mathbb{R}$. If $f \in C^1(U, \mathbb{R}_n)$ satisfies*

$$\overline{\partial}_{\mathcal{I}} f_{\mathcal{I}}(u + \mathcal{I}v) + \lambda f_{\mathcal{I}} = 0, \quad \text{on } U_{\mathcal{I}}$$

for every $\mathcal{I} \in \mathbb{S}$, then f is called λ-slice monogenic functions defined on U and we write $\mathcal{SM}_\lambda(U)$ for the set of all λ-slice monogenic functions on U.

Clearly $\mathcal{SM}_0(U) = \mathcal{SM}(U)$. In the remainder of this section we require $\lambda \in \mathbb{R}$.

A direct calculation shows

Proposition 2.2. *If $f \in C^1(U, \mathbb{R}_n)$, then*

$$\overline{\partial}_{\mathcal{I}} \left[e^{u\lambda} f_{\mathcal{I}}(u + \mathcal{I}v) \right] = e^{u\lambda} \left[\overline{\partial}_{\mathcal{I}} f_{\mathcal{I}}(u + \mathcal{I}v) + \lambda f_{\mathcal{I}} \right], \quad \forall u + \mathcal{I}v \in U_{\mathcal{I}},$$

for all $\mathcal{I} \in \mathbb{S}$.

As a consequence one gets

Corollary 2.1. *Let $f \in C^1(U, \mathbb{R}_n)$. The mapping $u + \mathcal{I}v \mapsto e^{u\lambda} f_{\mathcal{I}}(u + \mathcal{I}v)$, for all $u + \mathcal{I}v \in U$, where $u, v \in \mathbb{R}$ and $\mathcal{I} \in \mathbb{S}$, belongs to $\mathcal{SM}(U)$ if and only if $f \in \mathcal{SM}_\lambda(U)$.*

From now on we make the assumption: $f \in \mathcal{SM}_\lambda(U)$.

Proposition 2.3.

I. (Representation Theorem) Let $x = u + \mathcal{I}_x v \in U$, where $u, v \in \mathbb{R}$ and $\mathcal{I}_x \in \mathbb{S}$, then

$$f(x) = \frac{1}{2}\left(1 - \mathcal{I}_x \mathcal{I}\right) f(u + \mathcal{I}v) + \frac{1}{2}\left(1 + \mathcal{I}_x \mathcal{I}\right) f(u - \mathcal{I}v), \quad \forall \mathcal{I} \in \mathbb{S}. \tag{2.8}$$

II. (Splitting Lemma) For every $\mathcal{I} = \mathcal{I}_1 \in \mathbb{S}$, let $\mathcal{I}_2, \ldots, \mathcal{I}_n$ be a completion to a basis of $\mathbb{R}_n$ satisfying $\mathcal{I}_r \mathcal{I}_s + \mathcal{I}_s \mathcal{I}_r = -2\delta_{rs}$. Then there exist 2^{n-1} many holomorphic functions $F_A : U_{\mathcal{I}} \to \mathbb{C}_{\mathcal{I}}$ such that

$$f_{\mathcal{I}}(z) = \sum_{|A|=0}^{n-1} e^{-\lambda \Re z} F_A(z) \mathcal{I}_A, \tag{2.9}$$

where $\mathcal{I}_A = \mathcal{I}_{i_1} \cdots \mathcal{I}_{i_s}$ and $A = \{i_1, \ldots, i_s\}$ is a subset of $\{2, \ldots, n\}$, with $i_1 < \cdots < i_s$, or $I_\emptyset = 1$.

Proof. It follows from Corollary 2.1, Theorem 2.1 and Lemma 2.1. □

Proposition 2.4. *Given $\mathcal{I} \in \mathbb{S}$. If $\overline{\mathbb{D}(y, r)} \subset U$, then*

$$f(z) = \frac{1}{2\pi} \int_{\partial \mathbb{D}(y,r)_{\mathcal{I}}} e^{\lambda(\Re w - \Re z)} (w - z)^{-1} d_w \sigma(\mathcal{I}) f(w), \quad \forall z \in \mathbb{D}(y, r)_{\mathcal{I}}.$$

In addition,

$$\int_\Gamma e^{\lambda \Re w} d_w \sigma(\mathcal{I}) f(w) = 0,$$

for all $\mathcal{I} \in \mathbb{S}$ and for any closed, homotopic to a point and piecewise C^1 curve $\Gamma \subset U_{\mathcal{I}}$.

If $\{z \in U_{\mathcal{I}} \mid e^{\lambda \Re z} f(z) = 0\}$ *has an accumulation point in* $U_{\mathcal{I}}$ *then* $f \equiv 0$ *on* U*. On the other hand, if* $\ell \in C(U, \mathbb{R}_n)$ *satisfies*

$$\int_{\Gamma} e^{\lambda \Re w} d_w \sigma(\mathcal{I}) \ell(w) = 0,$$

for any closed, homotopic to a point and piecewise C^1 *curve* $\Gamma \subset U_{\mathcal{I}}$ *and for all* $\mathcal{I} \in \mathbb{S}$*, then* $\ell \in \mathcal{SM}_{\lambda}(U)$*.*

Proof. Corollary 2.1 shows that the mapping $u + \mathcal{I}v \mapsto e^{u\lambda} f_{\mathcal{I}}(u + \mathcal{I}v)$, for all $u + \mathcal{I}v \in U$, where $u, v \in \mathbb{R}$ and $\mathcal{I} \in \mathbb{S}$, satisfies the first three facts of Proposition 2.1. Finally the last property follows from Corollary 2.1 and the last fact of Proposition 2.1. □

On account of the Representation Theorem given in Proposition 2.3 we can extend some formulas of Proposition 2.4 on all axially symmetric s-domain U.

3. Fractional slice monogenic functions with respect to a pair of real valued functions

3.1. **Riemann-Liouville fractional slice monogenic functions with respect to a pair of real valued functions.** Let $a, b, c \in \mathbb{R}$ with $a < b$, $c > 0$ and let us consider the special class of axially symmetric s-domains in $\mathbb{R}^{n+1}$ defined by

$$\mathcal{S}_{a,b,c} = \{u + \mathcal{I}v \in \mathbb{R}^{n+1} \mid u \in [a, b],\ v \in [0, c],\ \mathcal{I} \in \mathbb{S}\}.$$

Given $\mathcal{I} \in \mathbb{S}$, denote $\mathcal{S}_{a,b,c,\mathcal{I}} := \mathcal{S}_{a,b,c} \cap \mathbb{C}_{\mathcal{I}} \subset \{u + \mathcal{I}v; \mid v \geq 0\}$. Consider $r \in [a, b]$, $s \in [0, c]$ and $\alpha, \beta \in (0, 1)$.

Let us denote by $AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$ the class of $\mathbb{R}_n$-valued continuously functions f on $\mathcal{S}_{a,b,c}$ such that the mapings $u \mapsto f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (u + \mathcal{I}s)$ and $v \mapsto f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (r + \mathcal{I}v)$ are absolutely continuous for all $u \in [a, b]$, $v \in [0, c]$. Given $g \in C^1([a, b], \mathbb{R})$ and $h \in C^1([0, c], \mathbb{R})$ such that $g'(t) > 0$ for all $t \in [a, b]$ and $h'(t) > 0$ for all $t \in [0, c]$.

Write

$$\begin{aligned}
[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) :=& \frac{1}{\Gamma(1-\alpha)} \int_a^u \frac{f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (\tau + \mathcal{I}s)}{(g(u) - g(\tau))^{\alpha}} g'(\tau) d\tau, \text{ with } u > a, \\
[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v) :=& \frac{1}{\Gamma(1-\beta)} \int_0^v \frac{f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (r + \mathcal{I}\tau)}{(h(v) - h(\tau))^{\beta}} h'(\tau) d\tau, \text{ with } v > 0, \\
[\mathbf{I}_{b^-}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) :=& \frac{1}{\Gamma(1-\alpha)} \int_u^b \frac{f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (\tau + \mathcal{I}s)}{(g(\tau) - g(u))^{\alpha}} g'(\tau) d\tau, \text{ with } u < b, \\
[\mathbf{I}_{c^-}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v) :=& \frac{1}{\Gamma(1-\beta)} \int_v^c \frac{f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} (r + \mathcal{I}\tau)}{(h(\tau) - h(v))^{\beta}} h'(\tau) d\tau, \text{ with } v < c.
\end{aligned} \tag{3.1}$$

Let us introduce natural extensions of the real fractal derivatives (2.1) and (2.2) preserving their structure.

Definition 3.1. *The fractional derivatives on the left and on the right of* $f \in AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$*, in the Riemann-Liouville sense, associated with the slice* $\mathbb{C}_{\mathcal{I}}$ *of order* (α, β) *and with respect to* (g, h) *are defined by*

$$({}_{RL}D_{a^+0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u + \mathcal{I}v, r, s) := ({}_{RL}D_{a^+}^{\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u + \mathcal{I}s) + \mathcal{I}({}_{RL}D_{0^+}^{\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r + \mathcal{I}v)$$

$$= \frac{1}{g'(u)}\frac{\partial}{\partial u}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}\frac{\partial}{\partial v}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v),$$

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-c^-,\mathcal{I}} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_{RL}D^{\alpha,g}_{b^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s) + \mathcal{I}({}_{RL}D^{\beta,h}_{c^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v)$$

$$= \frac{1}{g'(u)}\frac{\partial}{\partial u}[\mathbf{I}^{1-\alpha,g}_{b^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}\frac{\partial}{\partial v}[\mathbf{I}^{1-\beta,h}_{c^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v),$$

respectively, where u and v are the variables of partial fractional derivation. Similarly, reader has no difficulty in assigning a meaning to ${}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I}}$ *and* ${}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I}}$*:*

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I}} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_{RL}D^{\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s) + \mathcal{I}({}_{RL}D^{\beta,h}_{c^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),$$

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I}} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_{RL}D^{\alpha,g}_{b^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s) + \mathcal{I}({}_{RL}D^{\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),$$

In particular, if $(u,v) = (r,s)$*, then*

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I}} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s,r,s) = ({}_{RL}D^{\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s) + \mathcal{I}({}_{RL}D^{\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s),$$

$$=: ({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I}} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s)$$

and similar notation is followed for other operators.

In general, there exist two options:

a) If $\alpha,\beta \in \mathbb{C}$ with $0 < \Re\alpha, \Re\beta < 1$ then our fractal derivatives give us functions with values in the complex Clifford algebra $\mathbb{R}_n(\mathbb{C})$ which consist of $\sum_A e_A x_A$, where $x_A \in \mathbb{C}$.

b) If $\alpha = (\alpha_1,\alpha_2)$ and $\beta = (\beta_1,\beta_2)$ then use the mapping $\alpha = (\alpha_1,\alpha_2) \mapsto \alpha_1 + \mathcal{I}\alpha_2 \in \mathbb{C}_{\mathcal{I}}$, $\beta = (\beta_1,\beta_2) \mapsto \beta_1 + \mathcal{I}\beta_2 \in \mathbb{C}_{\mathcal{I}}$ for each $\mathcal{I} \in \mathbb{S}$ with $0 < \alpha_1, \beta_1 < 1$.

For abbreviation, we will restrict ourselves to $\alpha,\beta \in (0,1)$.

These slice monogenic fractional operators are $\mathbb{R}_n$ right-linear operators. In addition, the right versions of the previous operators are given by

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I},r} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_{RL}D^{\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s) + ({}_{RL}D^{\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v)\mathcal{I},$$

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-c^-,\mathcal{I},r} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_{RL}D^{\alpha,g}_{b^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s) + ({}_{RL}D^{\beta,h}_{c^-} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v)\mathcal{I},$$

where u and v are the variables of partial derivation and ${}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I},r}$ and ${}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I},r}$ can be defined similarly.

Let $\lambda \in \mathbb{R}$ and let $\delta_1,\delta_2 \in C^2(\mathbb{R},\mathbb{R})$ be two solutions of the following differential equation:

$$y'' + 2\lambda y' = 0.$$

So from now consider $g \in C^2([a,b],\mathbb{R})$ and $h \in C^2([0,c],\mathbb{R})$ given by $g(u) = \delta_1(u)$ and $g'(u) > 0$ for all $u \in [a,b]$. Similarly, $h(v) = \delta_2(v)$ and $h'(v) > 0$ for all $v \in [0,c]$.

Proposition 3.1. *Let $f \in AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$. Then*

$$
\begin{aligned}
&2\lambda\left[\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right] + ({}_{RL}D_{a^+0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)\\
=&2\overline{\partial}_{\mathcal{I}}\left\{\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right\},\\
&2\lambda\left[\frac{1}{g'(u)}[\mathbf{I}_{b^-}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}[\mathbf{I}_{c^-}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right] + ({}_{RL}D_{b^-c^-,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)\\
=&-2\overline{\partial}_{\mathcal{I}}\left\{\frac{1}{g'(u)}[\mathbf{I}_{b^-}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}_{c^-}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](s+\mathcal{I}v)\right\},\\
&2\lambda\left[\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}[\mathbf{I}_{c^-}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right] + ({}_{RL}D_{a^+c^-,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)\\
=&2\overline{\partial}_{\mathcal{I}}\left\{\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}_{c^-}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right\},\\
&2\lambda\left[\frac{1}{g'(u)}[\mathbf{I}_{b^-}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right] + ({}_{RL}D_{b^-0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)\\
=&-2\overline{\partial}_{\mathcal{I}}\left\{\frac{1}{g'(u)}[\mathbf{I}_{b^-}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](s+\mathcal{I}v)\right\},
\end{aligned}
$$

for all $u+\mathcal{I}v \in \mathbb{R}^{n+1}$, where u and v are the variables of partial derivation of $\overline{\partial}_{\mathcal{I}}$.

Proof. We give only the main ideas of the proof for the first identity, the other cases run along similar lines.

$$
\begin{aligned}
&2\overline{\partial}_{\mathcal{I}}\left\{\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right\}\\
=&\frac{\partial}{\partial u}\left(\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s)\right) + \mathcal{I}\frac{\partial}{\partial u}\left(\frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right)\\
=&\frac{\partial}{\partial u}\left(\frac{1}{g'(u)}\right)[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{\partial}{\partial u}\left(\frac{1}{h'(v)}\right)[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\\
&+\frac{1}{g'(u)}\frac{\partial}{\partial u}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}\frac{\partial}{\partial u}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\\
=&2\lambda\left[\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \mathcal{I}\frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v)\right]\\
&+({}_{RL}D_{a^+0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s),
\end{aligned}
$$

since $\dfrac{\partial}{\partial u}\left(\dfrac{1}{g'(u)}\right) = \dfrac{g''(u)}{(g'(u))^2} = 2\lambda\dfrac{1}{g'(u)}$ and $\dfrac{\partial}{\partial v}\left(\dfrac{1}{h'(v)}\right) = \dfrac{h''(u)}{(h'(u))^2} = 2\lambda\dfrac{1}{h'(u)}$. □

Let us make the definition of Riemann-Liouville fractional slice monogenic functions, of order (α,β) and with respect to (g,h) on $\mathcal{S}_{a,b,c}$

Definition 3.2. *Let $f \in AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$ such that*

$$
\begin{aligned}
&u \mapsto [\mathbf{I}_{a^+}^{1-\alpha,g} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s), \quad u \in [a,b],\\
&v \mapsto [\mathbf{I}_{0^+}^{1-\beta,h} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v), \quad v \in [0,c],
\end{aligned} \tag{3.2}
$$

are mappings of C^1−class. Then f will be called a Riemann-Liouville fractional slice monogenic function, of order (α,β) and with respect to (g,h) on $\mathcal{S}_{a,b,c}$ if

$$({}_{RL}D_{a^+0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) = 0, \quad \forall u+\mathcal{I}v \in \mathcal{S}_{a,b,c,\mathcal{I}},$$

for all $\mathcal{I} \in \mathbb{S}$.

The $\mathbb{R}_n$*-right linear space of (left) Riemann-Liouville fractional slice monogenic functions, of order* (α, β) *and with respect to* (g, h) *on* $\mathcal{S}_{a,b,c}$ *is denoted by* ${}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

The $\mathbb{R}_n$-left linear space of (right) Riemann-Liouville fractional slice monogenic functions, of order (α, β) and with respect to (g, h) on $\mathcal{S}_{a,b,c}$ is denoted by ${}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h),r}(\mathcal{S}_{a,b,c})$ which consists of $f \in AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$ satisfying (3.2) and

$$({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I},r} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u + \mathcal{I}v, r, s) = 0, \quad \forall u + \mathcal{I}v \in \mathcal{S}_{a,b,c,\mathcal{I}},$$

for all $\mathcal{I} \in \mathbb{S}$.

Remark 3.1. *The kernels of the operators* ${}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I}}$, ${}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I}}$, ${}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I},r}$ *and* ${}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I},r}$ *induce fractional slice monogenic function theories similar to* ${}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

On the other hand, the extension of [20, Example 1] to the fractional slice monogenic function theory can be used to give a non trivial element of $\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

Remark 3.2. *If* $\lambda = 0$ *then functions* g *and* h *are given by* $g(u) = \delta_1 u + \delta_2$ *for all* $u \in [a, b]$ *and* $h(v) = \rho_1 v + \rho_2$ *for all* $v \in [0, c]$*, where* $\delta_k, \rho_k \in \mathbb{R}$ *for* $k = 1, 2$ *and* $\delta_1, \rho_1 > 0$. *This case extends in the theory of slice monogenic functions the theory presented in [20].*

On the other hand, the case $\lambda \neq 0$ *introduce a new fractional approach to a notion of slice monogenic functions with respect to the pair* (g, h) *given by* $g(u) = \delta_1 e^{-2\lambda u} + \delta_2$ *for all* $u \in [a, b]$ *and* $h(v) = \rho_1 e^{-2\lambda v} + \rho_2$ *for all* $v \in [0, c]$ *where* $\delta_k, \rho_k \in \mathbb{R}$ *for* $k = 1, 2$ *with* $-2\delta_1\lambda, -2\rho_1\lambda > 0$.

Proposition 3.2. *Suppose that* $f \in AC^1(S_{a,b,c}, \mathbb{R}_n)$ *satisfies (3.2). Then* $f \in {}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$ *if and only if the mapping*

$$q = u + \mathcal{I}v \mapsto \frac{1}{g'(u)}[\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v)$$

belongs to $\mathcal{SM}_\lambda(\mathcal{S}_{a,b,c})$.

Proof. Use Definition 3.2 and Proposition 3.1. □

Corollary 3.1. *Given* $f \in AC^1(S_{a,b,c}, \mathbb{R}_n)$ *satisfying (3.2) we see that* $f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} \in Ker({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+c^-,\mathcal{I}})$ *for all* $\mathcal{I} \in \mathbb{S}^2$ *if and only if*

$$u + \mathcal{I}v \mapsto [\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) + [\mathbf{I}^{1-\beta,h}_{c^-} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v), \quad \forall u + \mathcal{I}v \in \mathcal{S}_{a,b,c,\mathcal{I}}$$

belongs to $\mathcal{SM}_\lambda(\mathcal{S}_{a,b,c})$. *In addition,* $f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} \in Ker({}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-0^+,\mathcal{I}})$ *for all* $\mathcal{I} \in \mathbb{S}^2$ *if and only if*

$$u + \mathcal{I}v \mapsto [\mathbf{I}^{1-\alpha,g}_{b^-} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) + [\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v), \quad \forall u + \mathcal{I}v \in \mathcal{S}_{a,b,c,\mathcal{I}}$$

belongs to $\mathcal{SM}_\lambda(\mathcal{S}_{a,b,c})$. *Similarly,* $f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} \in Ker$ $({}_{RL}D^{(\alpha,\beta),(g,h)}_{b^-c^-,\mathcal{I}})$ *for all* $\mathcal{I} \in \mathbb{S}^2$ *iff*

$$u + \mathcal{I}v \mapsto [\mathbf{I}^{1-\alpha,g}_{b^-} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) + [\mathbf{I}^{1-\beta,h}_{c^-} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v), \quad \forall u + \mathcal{I}v \in \mathcal{S}_{a,b,c,\mathcal{I}}$$

belongs to $\mathcal{SM}_\lambda(\mathcal{S}_{a,b,c})$.

Proof. Similar to the previous proof. □

Proposition 3.3. *(Representation Formula) Let $f \in {}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$, then for every $x = u + \mathcal{I}_x v \in \mathcal{S}_{a,b,c}$ and $\mathcal{I} \in \mathbb{S}$ we have that*

$$\begin{aligned}&\frac{1}{g'(u)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}_x}}](u+\mathcal{I}_x s) + \frac{1}{h'(v)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}_x}}](r+\mathcal{I}_x v)\\ =&\frac{1}{2}\left\{\frac{1}{g'}\Big[\,\mathbf{I}^{1-\alpha,g}_{a^+}[f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} - \mathcal{I}_x\mathcal{I} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}] + \mathbf{I}^{1-\alpha,g}_{a^+}[f \mid_{\mathcal{S}_{a,b,c,-\mathcal{I}}} + \mathcal{I}_x\mathcal{I} f \mid_{\mathcal{S}_{a,b,c,-\mathcal{I}}}]\,\Big]\right\}(u+\mathcal{I}s)\\ &+\frac{1}{2}\left\{\frac{1}{h'}\Big[\,\mathbf{I}^{1-\beta,h}_{0^+}[f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}} - \mathcal{I}_x\mathcal{I} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}] + \mathbf{I}^{1-\beta,h}_{0^+}[f \mid_{\mathcal{S}_{a,b,c,-\mathcal{I}}} + \mathcal{I}_x\mathcal{I} f \mid_{S_{a,b,c,-\mathcal{I}}}]\Big]\right\}(r+\mathcal{I}v).\end{aligned}$$

Proof. Use the mappings

$$\begin{aligned}u+\mathcal{I}v \mapsto &\frac{1}{g'(u)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}v),\\ u-\mathcal{I}v \mapsto &\frac{1}{g'(u)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,-\mathcal{I}}}](u+\mathcal{I}s) + \frac{1}{h'(v)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,-\mathcal{I}}}](r+\mathcal{I}v),\end{aligned}$$

Proposition 3.2 and identity (2.8) of Proposition 2.3. □

Proposition 3.4. *(Splitting lemma) Given $f \in {}_{RL}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$ and $\mathcal{I} = \mathcal{I}_1 \in \mathbb{S}$. Let $\mathcal{I}_2, \dots, \mathcal{I}_n \in \mathbb{S}$ such that $\mathcal{I}_r\mathcal{I}_s + \mathcal{I}_s\mathcal{I}_r = -2\delta_{rs}$. Then there exist 2^{n-1} many holomorphic functions $F_A : U_{\mathcal{I}} \to \mathbb{C}_{\mathcal{I}}$ such that*

$$\frac{1}{g'(\Re z)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z+\mathcal{I}s) + \frac{1}{h'(\Im z)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}\Im z) = \sum_{|A|=0}^{n-1} e^{-\lambda \Re z} F_A(z)\mathcal{I}_A,$$

where $\mathcal{I}_A = \mathcal{I}_{i_1}\cdots\mathcal{I}_{i_s}$ and $A = \{i_1 \cdots i_s\}$ is a subset of $\{2, \dots, n\}$, with $i_1 < \cdots < i_s$, or, $I_\emptyset = 1$.

Proof. Use Proposition 3.2 and identity (2.9) of Proposition 2.3. □

Corollary 3.2. *Some properties of $\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$ Let $f \in \mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$. Suppose $u_0 + \mathcal{I}v_0 \in \mathcal{S}_{a,b,c}$ and $r > 0$ such that $D_r := \{u + \mathcal{I}v \in \mathbb{C}_{\mathcal{I}} \mid (u-u_0)^2 + (v-v_0)^2 \le r^2\} \subset \mathcal{S}_{a,b,c,\mathcal{I}}$.*
i) There exist a sequence $(C_n)_{n\ge 0}$ of elements of $\mathbb{R}_n$ such that

$$\begin{aligned}&\frac{1}{g'(\Re z)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z+\mathcal{I}s) + \frac{1}{h'(\Im z)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}\Im z)\\ =&\sum_{n=0}^{\infty} e^{-\lambda\Re z}(z-u_0-\mathcal{I}v_0)^n C_n,\end{aligned} \tag{3.3}$$

for all $z \in int(D_r)$, the interior of D_r contained in $\mathbb{C}_{\mathcal{I}}$.
ii) For any $z \in int(D_r)$ we see that

$$\begin{aligned}&\frac{1}{g'(\Re z)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z+\mathcal{I}s) + \frac{1}{h'(\Im z)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}\Im z)\\ =&\frac{1}{2\pi i}\int_{\partial D_r} e^{\lambda(\Re w-\Re z)}(w-z)^{-1} d_w\sigma(\mathcal{I})\left\{\frac{1}{g'(\Re w)}[\mathbf{I}^{1-\alpha,g}_{a^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re w+\mathcal{I}s)\right.\\ &\left.+\frac{1}{h'(\Im w)}[\mathbf{I}^{1-\beta,h}_{0^+} f \mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r+\mathcal{I}\Im w)\right\},\end{aligned}$$

where ∂D_r the boundary of D_r is contained in $\mathbb{C}_{\mathcal{I}}$.

iii) Let Γ be a closed piecewise C^1-curve, homotopic to a point, contained in $int(\mathcal{S}_{a,b,c,\mathcal{I}})$. Then

$$\int_\Gamma e^{\lambda \Re w} d_w\sigma(\mathcal{I}) \left\{ \frac{1}{g'(\Re w)} [\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re w + \mathcal{I}s) + \frac{1}{h'(\Im w)} [\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im w) \right\}$$
$$= 0.$$

iv) If the set point

$$\{z \in \mathcal{S}_{a,b,c,\mathcal{I}} \mid h'(\Im z)[\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z + \mathcal{I}s) = -g'(\Re z)[\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im z)\}$$

has an accumulation point in $\mathcal{S}_{a,b,c,\mathcal{I}}$ then

$$h'(\Im z)[\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z + \mathcal{I}s) = -g'(\Re z)[\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im z), \quad \forall z \in \mathcal{S}_{a,b,c,\mathcal{I}}.$$

v) Given $\ell \in AC^1(\mathcal{S}_{a,b,c}, \mathbb{R}_n)$. Suppose that the mapping

$$w \mapsto \frac{1}{g'(\Re w)} [\mathbf{I}^{1-\alpha,g}_{a^+} \ell\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re w + \mathcal{I}s) + \frac{1}{h'(\Im w)} [\mathbf{I}^{1-\beta,h}_{0^+} \ell\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im w),$$

for all $w \in \mathcal{S}_{a,b,c,\mathcal{I}}$, belongs to $C(\mathcal{S}_{a,b,c,\mathcal{I}}, \mathbb{R}_n)$ and satisfies that

$$\int_\Gamma d_w\sigma(\mathcal{I}) e^{\lambda \Re w} \left\{ \frac{1}{g'(\Re w)} [\mathbf{I}^{1-\alpha,g}_{a^+} \ell\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re w + \mathcal{I}s) + \frac{1}{h'(\Im w)} [\mathbf{I}^{1-\beta,h}_{0^+} \ell\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im w) \right\} = 0,$$

for any closed, homotopic to a point and piecewise C^1 curve $\Gamma \subset \mathcal{S}_{a,b,c,\mathcal{I}}$ and for all $\mathcal{I} \in \mathbb{S}$. Then $\ell \in \mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

Proof. In (2.9) of Proposition 2.3 and in Proposition 2.4 use the mapping

$$u + \mathcal{I}v \mapsto \frac{1}{g'(u)} [\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](u + \mathcal{I}s) + \frac{1}{h'(v)} [\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}v)$$

and Proposition 3.2. □

Remark 3.3. *Given $\mathcal{I} \in \mathbb{S}$, the previous corollary shows how some properties of the slice monogenic functions are extending to mappings given by*

$$\frac{1}{g'(\Re z)} [\mathbf{I}^{1-\alpha,g}_{a^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](\Re z + \mathcal{I}s) + \frac{1}{h'(\Im z)} [\mathbf{I}^{1-\beta,h}_{0^+} f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}}](r + \mathcal{I}\Im z), \quad z \in \mathcal{S}_{a,b,c,\mathcal{I}},$$

for all $f \in {}_{RL}\mathcal{SM}^{\alpha,\beta}_{(g,h)}(\mathcal{S}_{a,b,c})$, based on well-known results of the theory of slice monogenic functions. Moreover, using Representation formula the previous formulas are extended to

$$\frac{1}{g'(u)} [\mathbf{I}^{1-\alpha,g}_{a^+} f](u + \mathcal{I}_x s) + \frac{1}{h'(v)} [\mathbf{I}^{1-\beta,h}_{0^+} f](r + \mathcal{I}_x v),$$

for all $u + \mathcal{I}_x v \in \mathcal{S}_{a,b,c}$, and $f \in \mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

Taking into account the necessary hypothesis for the fundamental theorem of fractional calculus, applying the fractional derivatives $D^{1-\alpha,g}_{a^+}$ and $D^{1-\beta,h}_{0^+}$ on $h'(v)[\mathbf{I}^{1-\alpha,g}_{a^+} f](u + \mathcal{I}s) + g'(u)[\mathbf{I}^{1-\beta,h}_{0^+} f](r + \mathcal{I}v)$, we obtain formulas for $h'(v)f(u + \mathcal{I}s) + g'(u)f(r + \mathcal{I}v)$. For example, we will see the consequence of (3.3) *for $h'(v)f(u + \mathcal{I}s) + g'(u)f(r + \mathcal{I}v)$ repeating the previous ideas.*

From (3.3) *and Representation Theorem we have that*

$$\frac{1}{g'(u)}[\mathbf{I}_{a^+}^{1-\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}_x}}](u+\mathcal{I}_x s) + \frac{1}{h'(v)}[\mathbf{I}_{0^+}^{1-\beta,h} f \mid_{S_{a,b,c,\mathcal{I}_x}}](r+\mathcal{I}_x u)$$
$$=\frac{1}{2}(1-\mathcal{I}_x\mathcal{I})\sum_{n=0}^{\infty}(u+\mathcal{I}v-u_0-\mathcal{I}v_0)^n C_n + \frac{1}{2}(1+\mathcal{I}_x\mathcal{I})\sum_{n=0}^{\infty}(u-\mathcal{I}v-u_0-\mathcal{I}v_0)^n C_n,$$

where $(C_n)_{n\geq 0}$ *is a sequence of elements of* $\mathbb{R}_n$*, for every* $(u-u_0)^2+(v-v_0)^2<r^2$ *and* $\mathcal{I}_x \in \mathbb{S}$.

Finally, writing the factor $g'(u)h'(v)$ *and applying the real fractional derivatives* $D_{a^+}^{1-\alpha,g} \circ D_{0^+}^{1-\beta,h}$ *on both sides, then the fundamental theorem of fractional calculus,* (2.3)*, allows to see that*

$$h'(v)f(u+\mathcal{I}_x s)+g'(u)f(r+\mathcal{I}_x v)$$
$$=\frac{1}{2}(1-\mathcal{I}_x\mathcal{I})D_{a^+}^{1-\alpha,g}\circ D_{0^+}^{1-\beta,h}\left[\sum_{n=0}^{\infty} g'(u)h'(v)(u+\mathcal{I}v-u_0-\mathcal{I}v_0)^n C_n\right]$$
$$+\frac{1}{2}(1+\mathcal{I}_x\mathcal{I})D_{a^+}^{1-\alpha,g}\circ D_{0^+}^{1-\beta,h}\left[\sum_{n=0}^{\infty} g'(u)h'(v)(u-\mathcal{I}v-u_0-\mathcal{I}v_0)^n C_n\right]$$
$$-D_{0^+}^{1-\beta,h}[h'(v)][\mathbf{I}_{a^+}^{1-\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}_x}}](u+\mathcal{I}_x s) - D_{a^+}^{1-\alpha,g}[g'(u)][\mathbf{I}_{0^+}^{1-\beta,h} f \mid_{S_{a,b,c,\mathcal{I}_x}}](r+\mathcal{I}_x v),$$

for all $(u-u_0)^2+(v-v_0)^2<r^2$ *and* $\mathcal{I}_x\in\mathbb{S}$*, where* u *is the real variable of the fractional derivative* $D_{a^+}^{1-\alpha,g}$ *and* v *is the variable of* $\circ D_{0^+}^{1-\beta,h}$.

So the previous computations can be repeated for the missing statements presented in Corollary 3.2 to obtain some formulas for the mapping $(u+\mathcal{I}_x v)\mapsto h'(v)f(u+\mathcal{I}_x s)+g'(u)f(r+\mathcal{I}_x v)$ *for all* $(u-u_0)^2+(v-v_0)^2<r^2$ *and* $\mathcal{I}_x\in\mathbb{S}$.

3.2. **Caputo fractional slice monogenic functions with respect to a pair of real valued functions.** With the notations introduced in the previous section, and partially relying on similar reasoning, in this section we study Caputo fractional slice monogenic functions with respect to a pair of real valued functions.

Definition 3.3. *Let* $r\in[a,b]$ *and* $s\in[0,c]$*. The fractional derivatives in the Caputo sense of* $f\in AC^1(S_{a,b,c},\mathbb{H})$*, satisfying* (3.2)*, on the left (on the right) with order induced by* (α,β) *with respect to* (g,h) *and associated to the slice* $\mathbb{C}_{\mathcal{I}}$ *are*

$$({}_C D_{a^+0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_C D_{a^+}^{\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+\mathcal{I}({}_C D_{0^+}^{\beta,h} f \mid_{S_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),$$
$$({}_C D_{b^-c^-,\mathcal{I}}^{(\alpha,\beta),(g,h)} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_C D_{b^-}^{\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+\mathcal{I}({}_C D_{c^-}^{\beta,h} f \mid_{S_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),$$

respectively, where u *and* v *are the variables of partial fractional derivation. The right versions of the previous operators are given by*

$$({}_C D_{a^+0^+,\mathcal{I},r}^{(\alpha,\beta),(g,h)} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_C D_{a^+}^{\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+({}_C D_{0^+}^{\beta,h} f \mid_{S_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v)\mathcal{I},$$
$$({}_C D_{b^-c^-,\mathcal{I},r}^{(\alpha,\beta),(g,h)} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s) := ({}_C D_{b^-}^{\alpha,g} f \mid_{S_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+({}_C D_{c^-}^{\beta,h} f \mid_{S_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v)\mathcal{I}.$$

Operators ${}_C D_{a^+c^-,\mathcal{I}}^{(\alpha,\beta),(g,h)}$, ${}_C D_{b^-0^+,\mathcal{I}}^{(\alpha,\beta),(g,h)}$, ${}_C D_{a^+c^-,\mathcal{I},r}^{(\alpha,\beta),(g,h)}$ *and* ${}_C D_{b^-0^+,\mathcal{I},r}^{(\alpha,\beta),(g,h)}$ *are defined in similar way.*

Note that if $(u,v)=(r,s)$ *then*

$$
\begin{aligned}
&({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I}}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s,r,s)=({}_{C}D^{\alpha,g}_{a^+}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s)+\mathcal{I}({}_{C}D^{\beta,h}_{0^+}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s),\\
&=:({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+0^+,\mathcal{I}}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}s),
\end{aligned}
$$

and the missing operators follow the same notation.

Remark 3.4. *The previous operators are quaternionic extensions of* (2.4). *Even more, an interesting combination of fractional derivatives in the Riemann-Liouville and Caputo sense such that*

$$
\begin{aligned}
&({}_{RL}D^{\alpha,g}_{a^+}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+\mathcal{I}({}_{C}D^{\beta,h}_{0^+}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),\\
&({}_{RL}D^{\alpha,g}_{b^-}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+\mathcal{I}({}_{C}D^{\beta,h}_{0^+}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),\\
&({}_{C}D^{\alpha,g}_{b^-}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}s)+\mathcal{I}({}_{RL}D^{\beta,h}_{c^-}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(r+\mathcal{I}v),
\end{aligned}
$$

and some others similar operators can be considered.

Definition 3.4. *Let* $f\in AC^1(\mathcal{S}_{a,b,c},\mathbb{R}_n)$ *satisfying* (3.2). *Then* f *will be called a Caputo fractional slice monogenic function of order* (α,β)*, with respect to* (g,h) *on* $\mathcal{S}_{a,b,c}$ *if*

$$({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+,\mathcal{I}}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)=0,\quad \forall u+\mathcal{I}v\in\mathcal{S}_{a,b,c,\mathcal{I}},$$

for all $\mathcal{I}\in\mathbb{S}$.

The $\mathbb{R}_n$*-right linear space of fractional monogenic functions, in Caputo sense is denoted by* ${}_{C}\mathcal{SM}^{(\alpha,\beta)}_{(g,h)}(\mathcal{S}_{a,b,c})$.

On the other hand, by ${}_{C}\mathcal{SM}^{(\alpha,\beta)}_{(g,h),r}(\mathcal{S}_{a,b,c})$ *we denote the* $\mathbb{R}_n$*-left linear space of right-fractional slice monogenic functions, in Caputo sense, on* $\mathcal{S}_{a,b,c}$ *and it consists of* $f\in AC^1(\mathcal{S}_{a,b,c},\mathbb{R}_n)$ *satisfying* (3.2) *and*

$$({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+,\mathcal{I},r}f\mid_{\mathcal{S}_{a,b,c,\mathcal{I}}})(u+\mathcal{I}v,r,s)=0,\quad \forall u+\mathcal{I}v\in\mathcal{S}_{a,b,c,\mathcal{I}},$$

for all $\mathcal{I}\in\mathbb{S}$.

Proposition 3.5. *Suppose that* $f\in AC^1(\mathcal{S}_{a,b,c},\mathbb{R}_n)$ *satisfies* (3.2). *Define the operator*

$$\mathcal{H}(f)(u+\mathcal{I}v,r,s):=(\mathbf{I}^{1-\alpha,g}_{a^+}f)(u+\mathcal{I}s)+(\mathbf{I}^{1-\beta,h}_{0^+}f)(r+\mathcal{I}v).$$

Then

$$
\begin{aligned}
({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+,0^+}\mathcal{H}(f)(u+\mathcal{I}v,r,s)=&\mathcal{H}\left[({}_{RL}D^{(\alpha,\beta),(g,h)}_{a^+,0^+}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\right]\\
&-\mathbf{I}^{1-\beta,h}_{0^+}[1]({}_{RL}D^{\alpha,g}_{a^+}f)(u+\mathcal{I}s)-\mathbf{I}^{1-\alpha,g}_{a^+}[1]\mathcal{I}({}_{RL}D^{\beta,h}_{0^+}f)(r+\mathcal{I}v),
\end{aligned}
$$

for all $u+\mathcal{I}v\in\mathcal{S}_{a,b,c}$ *and* $\mathcal{I}\in\mathbb{S}$.

Proof. From direct computation and making use of the identities (2.5) and (2.6) we have that

$$
\begin{aligned}
&({}_{C}D^{(\alpha,\beta),(g,h)}_{a^+,0^+}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\\
=&\left[\mathbf{I}^{1-\alpha,g}_{a^+}\frac{1}{g'}\frac{\partial}{\partial u}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\right]+\mathcal{I}\left[\mathbf{I}^{1-\beta,h}_{0^+}\frac{1}{h'}\frac{\partial}{\partial v}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\right]
\end{aligned}
$$

$$=\left[\mathbf{I}_{a^+}^{1-\alpha,g}\frac{1}{g'}\frac{\partial}{\partial u}(\mathbf{I}_{a^+}^{1-\alpha,g}f)(u+\mathcal{I}s)\right]+\mathcal{I}\left[\mathbf{I}_{0^+}^{1-\beta,h}\frac{1}{h'}\frac{\partial}{\partial v}(\mathbf{I}_{0^+}^{1-\beta,h}f)(r+\mathcal{I}v)\right]$$
$$=\mathbf{I}_{a^+}^{1-\alpha,g}({}_{RL}D_{a^+}^{\alpha,g}f)(u+\mathcal{I}s)+\mathcal{I}\mathbf{I}_{0^+}^{1-\beta,h}({}_{RL}D_{0^+}^{\beta,h}f)(r+\mathcal{I}v),$$

for all $u+\mathcal{I}v\in\mathcal{S}_{a,b,c}$ and $\mathcal{I}\in\mathbb{S}$. Therefore,

$$\begin{aligned}({}_{C}D_{a^+,0^+}^{(\alpha,\beta),(g,h)}\mathcal{H}(f)(u+\mathcal{I}v,r,s)=&\mathcal{H}\left[({}_{RL}D_{a^+}^{\alpha,g}f)(u+\mathcal{I}s)+\mathcal{I}({}_{RL}D_{0^+}^{\beta,h}f)(r+\mathcal{I}v)\right]\\&-\mathbf{I}_{0^+}^{1-\beta,h}[1]({}_{RL}D_{a^+}^{\alpha,g}f)(u+\mathcal{I}s)-\mathbf{I}_{a^+}^{1-\alpha,g}[1]\mathcal{I}({}_{RL}D_{0^+}^{\beta,h}f)(r+\mathcal{I}v),\end{aligned}$$

for all $u+\mathcal{I}v\in\mathcal{S}_{a,b,c}$ and $\mathcal{I}\in\mathbb{S}$. □

So we obtain a characterization of the elements of ${}_{C}\mathcal{SM}_{(g,h)}^{(\alpha,\beta)}(\mathcal{S}_{a,b,c})$, in terms of the slice monogenic operator in the Riemann-Loiuville sense as follows:

Corollary 3.3. *$f\in{}_{C}\mathcal{SM}_{(g,h)}^{(\alpha,\beta)}(\mathcal{S}_{a,b,c})$ if and only if*

$$\mathcal{H}\left[({}_{RL}D_{a^+,0^+}^{(\alpha,\beta),(g,h)}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\right]=\mathbf{I}_{0^+}^{1-\beta,h}[1]({}_{RL}D_{a^+}^{\alpha,g}f)(u+\mathcal{I}s)+\mathbf{I}_{a^+}^{1-\alpha,g}[1]\mathcal{I}({}_{RL}D_{0^+}^{\beta,h}f)(r+\mathcal{I}v),\tag{3.4}$$

for all $u+\mathcal{I}v\in\mathcal{S}_{a,b,c}$ and $\mathcal{I}\in\mathbb{S}$.

On the other hand, if $f\in{}_{C}\mathcal{SM}_{(g,h)}^{(\alpha,\beta)}(\mathcal{S}_{a,b,c})$ we have

$$\begin{aligned}&{}_{RL}D_{a^+}^{1-\alpha,g}\circ{}_{RL}D_{0^+}^{1-\beta,h}\left\{\mathcal{H}\left[({}_{RL}D_{a^+,0^+}^{(\alpha,\beta),(g,h)}\mathcal{H}(f)(u+\mathcal{I}v,r,s)\right]\right\}\\&={}_{RL}D_{a^+}^{1-\alpha,g}({}_{RL}D_{a^+}^{\alpha,g}f)(u+\mathcal{I}s)+\mathcal{I}{}_{RL}D_{0^+}^{1-\beta,h}({}_{RL}D_{0^+}^{\beta,h}f)(r+\mathcal{I}v),\end{aligned}$$

for all $u+\mathcal{I}v\in\mathcal{S}_{a,b,c}$ and $\mathcal{I}\in\mathbb{S}$.

Proof. The first fact follows from the previous proposition. If we apply $D_{a^+}^{1-\alpha,g}\circ D_{0^+}^{1-\beta,h}$ on both sides of (3.4) the second fact follows. □

Funding: This work was partially supported by Instituto Politécnico Nacional (grant numbers 20253512, 20253412).

Author Contributions: All authors contributed equally to the study, read and approved the final version of the submitted manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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