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Analysis of Weyl-Type Operators and the Windowed Kontorovich-Lebedev-Clifford Transform with Applications

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Abstract. In this paper, we define the windowed Kontorovich-Lebedev-Clifford transform and introduce the corresponding Weyl transform. Furthermore, we examine the boundedness of the windowed Kontorovich-Lebedev-Clifford in Lebesgue spaces and establish some of its fundamental properties. We also provide criteria for the boundedness and compactness of the Weyl transform in Lebesgue spaces.

1. Introduction

The classical Weyl transform, initially introduced by Weyl [1] within the framework of quantum mechanics, has been extensively studied and referred to as the Weyl transform in the literature, including Wong's work [2]. This operator is a specific instance of pseudo-differential operators in the context of partial differential equations [1], and it has demonstrated significant utility in addressing various mathematical problems, including regularity issues, spectral asymptotics, and elliptic theory.

For $1 \leq p \leq 2$, Wong [2] analyzed the boundedness of the Weyl transform for symbol functions residing in certain L^α spaces of integrable functions. On the other hand, Simon [3] established that for $p > 2$, the Weyl transform of a function in L^α is not typically bounded. These results, along with related findings, are discussed in [1]. Furthermore, foundational tools such as the Wigner transform and the Fourier-Wigner transform were pivotal in studying the Weyl transform. When the symbol belongs to the L^2 space, Weyl [1] characterized the transform as a Hilbert-Schmidt operator. In addition, Zhao and Peng [4] investigated the interplay between wavelet and Weyl transforms using the spherical mean operator. Rachdi and Trimeche [3] analyzed the Fourier-Wigner transform involving the spherical mean operator $\mathcal{R}$ and studied the Weyl transform in this

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context. Additionally, Verma and Prasad [5] explored the Weyl operator within the framework of the Mehler-Fock transform.

This paper is organized as follows: Section 2 introduces the essential preliminaries and notations, including the definitions of function spaces, the Kontorovich-Lebedev-Clifford transform and its inverse formula, Lebesgue spaces, Parseval and Plancherel relations, generalized translation operators, and convolution operators. Section 3 delves into the study of the Windowed Kontorovich-Lebedev-Clifford transform, incorporating the generalized translation operator and the Kontorovich-Lebedev-Clifford transform, while exploring some fundamental properties and providing estimates in Lebesgue spaces. Section 4 is dedicated to constructing the Weyl transform associated with the Windowed Kontorovich-Lebedev-Clifford transform. This section also derives estimates based on different symbol classes.

2. Kontorovich-Lebedev-Clifford Transform Harmonic Analysis

- $L^{\alpha}\left(\mathbb{R}_{+}, d\mu\right), 1 \leq \alpha \leq \infty$, is the usual Lebesgue space with measure

$$d\mu(a_1) = \frac{1}{2}{a_1}^{-1}\, da_1,$$

and satisfies the norm

$$||h_1||_{L^{\alpha}(\mathbb{R}_+, d\mu)} = \begin{cases} \left(\int_0^{\infty} |h_1(a_1)|^{\alpha} d\mu(a_1)\right)^{\frac{1}{\alpha}} < \infty, & \text{for } 1 \leq \alpha < \infty, \\ \operatorname*{ess\,sup}_{x \in I} |h_1(a_1)| < \infty, & \text{for } \alpha = \infty. \end{cases}$$

- Similarly, we define the space

$$L^{\alpha}\left(\mathbb{R}_{+}, dv\right), 1 \leq \alpha \leq \infty,$$

which is equipped with the measurable functions on $\mathbb{R}_+ = (0, \infty)$ and measure

$$dv(\lambda) = \frac{4}{\pi^2} sinh(2\pi\sqrt{\lambda})\, d\lambda,$$

and satisfies the norm

$$||h_1||_{L^{\alpha}(\mathbb{R}_+, dv)} = \begin{cases} \left(\int_0^{\infty} |h_1(\lambda)|^{\alpha} dv(\lambda)\right)^{\frac{1}{\alpha}} < \infty, & \text{for } 1 \leq \alpha < \infty, \\ \operatorname*{ess\,sup}_{\lambda \in \mathbb{R}_+} |h_1(\lambda)| < \infty, & \text{for } \alpha = \infty. \end{cases}$$

- $L^{\alpha}\left(\mathbb{R}_{+} \times \mathbb{R}_{+}, d\mu \otimes dv\right)$ is the space of measurable functions on $\mathbb{R}_+ \times \mathbb{R}_+$, satisfying

$$||h_1||_{L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)} = \left(\int_0^{\infty}\int_0^{\infty} |h_1(a_1, \lambda)|^{\alpha} d\mu(a_1) dv(\lambda)\right)^{\frac{1}{\alpha}} < \infty, \quad 1 \leq \alpha < \infty,$$

$$||h_1||_{L^{\infty}(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)} = \operatorname*{ess\,sup}_{(a_1, \lambda) \in \mathbb{R}_+ \times \mathbb{R}_+} |h_1(a_1, \lambda)| < \infty, \quad \alpha = \infty.$$

Consequently, the following notation for their respective inner products will be adopted:

$$\langle h_1, h_2 \rangle_{L^2(\mathbb{R}_+, d\mu)} = \int_0^\infty h_1(a_1)\overline{h_2}(a_1)\, d\mu(a_1), \quad h_1, g \in L^2(\mathbb{R}_+, d\mu).$$

Similarly, we define inner products of

$$L^2\left(\mathbb{R}_+, dv\right), \quad \text{and for} \quad h_1, h_2 \in L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right),$$

$$\langle h_1, h_2 \rangle_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)} = \int_0^\infty \int_0^\infty h_1(a_1, \lambda) h_2(a_1, \lambda)\, d\mu(a_1)\, dv(\lambda)$$

respectively.

- $C_c^\infty(\mathbb{R}_+)$ denotes the space of smooth compactly supported functions; it is dense in the Banach space $L^\alpha(\mathbb{R}_+, d\mu)$ for $1 \leq \alpha < \infty$.

In this paper, the Kontorovich-Lebedev-Clifford Transformof a function h_1 defined on $\mathbb{R}_+$ is given by [11]:

$$\mathbb{K}(h_1)(\lambda) = \int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{a_1}) h_1(a_1) d\mu(a_1), \lambda \in \mathbb{R}_+, \tag{2.1}$$

The inversion formula for (2.1) is given by:

$$h_1(a_1) = \int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{a_1}) \mathbb{K}(h_1)(\lambda) dv(\lambda), a_1 \in \mathbb{R}_+. \tag{2.2}$$

where $K_{ia_2}(a_1),\ y \in \mathbb{R}_+^*$, is the Macdonald function given as (see cite emt)

$$K_{ia_2}(a_1) = \int_0^\infty e^{-a_1 \cosh t} \cos(a_2 t)\, dt, \quad a_1 \in \mathbb{R}_+^*.$$

From [12], we have

$$\left|K_{ia_2}(a_1)\right| \leq \int_0^\infty e^{-a_1 \cosh t}\, dt = K_0(a_1).$$

Moreover, $K_{2i\sqrt{\lambda}}(2\sqrt{a_1})$ satisfies (see [10])

$$\left|K_{2i\sqrt{\lambda}}(2\sqrt{a_1})\right| \leq C'(a_4) a_1^{\frac{-a_4}{4}} \left[\sinh(2\pi\sqrt{\lambda})\right]^{-\frac{1}{2}}, \quad 0 < a_4 < \frac{1}{2}.$$

Remark 2.1. *Let* $\Gamma = \{\mathbb{K}(h_1) \mid h_1 \in C_c^\infty \mathbb{R}_+\}$. *Then* Γ *is dense in* $L^2\left(\mathbb{R}_+, dv\right)$.

Thus, the KLC transform acts as an isometric isomorphism operator from
$L^2\left(\mathbb{R}_+, d\mu\right)$ to $L^2\left(\mathbb{R}_+, dv\right)$.

The Parseval and Plancherel's formulas are as follows [11]:

$$\int_0^\infty \mathbb{K}\left(h_1\right)(\lambda) \overline{\mathbb{K}\left(h_2\right)}(\lambda)\, dv(\lambda) = \int_0^\infty h_1(a_1)\overline{h_2}(a_1)\, d\mu(a_1), \tag{2.3}$$

$$\int_0^\infty \left|\mathbb{K}\left(h_1\right)(\lambda)\right|^2 dv(\lambda) = \int_0^\infty \left|h_1(a_1)\right|^2 d\mu(a_1),$$

respectively.This shows that the Kontorovich-Lebedev-Clifford transform preserves the scalar product.

Theorem 2.1. *For $h_1 \in L^\alpha(\mathbb{R}_+, d\mu)$, $1 \le \alpha \le 2$, and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have $\mathbb{K}(h_1) \in L^\beta(\mathbb{R}_+, dv)$, and it holds that:*

$$\|\mathbb{K}(h_1)\|_{L^\beta(\mathbb{R}_+, dv)} \le \|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)}. \tag{2.4}$$

Also, for $F \in L^\alpha(\mathbb{R}_+, dv)$, $1 \le \alpha \le 2$, and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have $\mathbb{K}^{-1}(F) \in L^\beta(\mathbb{R}_+, d\mu)$, and the inequality

$$\left\|\mathbb{K}^{-1}(F)\right\|_{L^\beta(\mathbb{R}_+, d\mu)} \le \|F\|_{L^\alpha(\mathbb{R}_+, dv)}$$

holds true.

Proof. By Plancherel's formula (2.3), we have

$$\|\mathbb{K}(h_1)\|_{L^2(\mathbb{R}_+, dv)} = \|h_1\|_{L^2(\mathbb{R}_+, d\mu)}. \tag{2.5}$$

Next, using the definition of the Kontorovich-Lebedev-Clifford transform and From the propreties of $K_{2i\sqrt{\lambda}}(2\sqrt{a_1})$ there exits $C_1 > 0$ such that:

$$K_0(2\sqrt{a_1}) \le C_1,$$

we get

$$\|\mathbb{K}(h_1)\|_{L^\infty(\mathbb{R}_+, dv)} \le C_1 \|h_1\|_{L^1(\mathbb{R}_+, d\mu)}. \tag{2.6}$$

Thus, from (2.5), (2.6), and the Riesz-Thorin theorem [2], we arrive at

$$\|\mathbb{K}(h_1)\|_{L^\beta(\mathbb{R}_+, dv)} \le \|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)}, \quad h_1 \in C_c^\infty \mathbb{R}_+.$$

By the density of $C_c^\infty(\mathbb{R}_+)$ in $L^\alpha(\mathbb{R}_+ d\mu)$, a limiting argument can be used to complete the proof that the Kontorovich-Lebedev-Clifford transform is a bounded linear operator from $L^\alpha(\mathbb{R}_+ d\mu)$ into $L^\beta(\mathbb{R}_+, dv)$. Hence, (2.4) holds true.

Proceeding as above, for $F \in L^\alpha(\mathbb{R}_+, dv)$, $1 \le \alpha \le 2$, and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have $\mathbb{K}^{-1}(F) \in L^\beta(\mathbb{R}_+ d\mu)$, and it follows that:

$$\left\|\mathbb{K}^{-1}(F)\right\|_{L^\beta(\mathbb{R}_+, d\mu)} \le \|F\|_{L^\alpha(\mathbb{R}_+, dv)}.$$

□

The translation operator is one of the fundamental operators in time-frequency analysis. Similar to the work presented in [9] [5], the generalized translation or shift operator corresponding to the Kontorovich-Lebedev-Clifford transform for $h_1 \in C_c^\infty \mathbb{R}_+$ is defined as:

$$h_1(a_1, a_2) = (\mathcal{T}_{a_1} h_1)(a_2) = (\mathcal{T}_{a_2} h_1)(a_1) = \int_0^\infty D(a_1, a_2, a_3) h_1(a_3)\, d\mu(a_3), \tag{2.7}$$

From [11], we have

$$\frac{4}{\pi^2} \int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{a_1}) K_{2i\sqrt{\lambda}}(2\sqrt{a_2}) K_{2i\sqrt{\lambda}}(2\sqrt{a_3}) \sinh(2\pi\sqrt{\lambda})\, d\lambda = D(a_1, a_2, a_3),$$

where $D(a_1, a_2, a_3)$ is given as

$$D(a_1, a_2, a_3) = \frac{1}{2}\exp\left[-\frac{a_1a_2 + a_2a_3 + a_3a_1}{\sqrt{a_1a_2a_3}}\right], \quad a_1, a_2, a_3 \in \mathbb{R}_+,$$

which is symmetric in a_1, a_2 and a_3. we have a product of Macdonald functions as [12]

$$K_{2i\sqrt{\lambda}}(2\sqrt{a_1})K_{2i\sqrt{\lambda}}(2\sqrt{a_2}) = \frac{1}{2}\int_0^\infty K_{2i\sqrt{\lambda}}(2\sqrt{a_3})D(a_1, a_2, a_3){a_3}^{-1}\, da_3 = \mathbb{K}(D(a_1, a_2, .))(\lambda),$$

Also from [7] [11] [12], we have few estimates that will be useful in further calculations

$$\int_0^\infty D(a_1, a_2, a_3)\, d\mu(a_3) = 2K_0\left(2\sqrt{a_1 + a_2}\right) \leq K_0(2\sqrt{a_1}) \text{ or } K_0(2\sqrt{a_2}).$$

$$0 < D(a_1, a_2, a_3) \leq \frac{e^{-2\sqrt{a_1}}}{2}$$

$$\int_0^\infty D(a_1, a_2, a_3){a_3}^{-1}\, da_3 = 2K_0(2\sqrt{a_1 + a_2}).$$

By the definition of the translation operator and the above equality, we have:

$$\mathbb{K}\left(\mathcal{T}_{a_1}h_1\right)(\lambda) = K_{2i\sqrt{\lambda}}(2\sqrt{a_1})\mathbb{K}(h_1)(\lambda), \tag{2.9}$$

Theorem 2.2. *Let $h_1 \in L^\alpha(\mathbb{R}_+, t^{-1}dt)$, $1 \leq \alpha \leq \infty$. Then for all $a_1 > 0$, $\mathcal{T}_{a_1}(h_1) \in L^\alpha(\mathbb{R}_+, {a_1}^{-1}da_1)$ and*

$$\|\mathcal{T}_{a_1}(h_1)\|_{L^\alpha(\mathbb{R}_+, d\mu)} \leq K_0(2\sqrt{a_1})\|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)} \leq C_1\|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)}. \tag{2.10}$$

Proof. Using ((2.7)), Holder's inequality, we have for all $a_1 > 0$,

$$\begin{aligned}|\mathcal{T}_{a_1}(h_1)(a_2)|^\alpha &\leq \left(\frac{1}{2}\right)^\alpha \int_0^\infty |h_1(a_3)|^\alpha D(a_1, a_2, a_3){a_3}^{-1}\, da_3 \left(\int_0^\infty D(a_1, a_2, a_3){a_3}^{-1}\, da_3\right)^{\frac{\alpha}{\beta}} \\ &\leq \left(\frac{1}{2}\right)^\alpha (2K_0(2\sqrt{a_1}))^{\frac{\alpha}{\beta}} \int_0^\infty |h_1(a_3)|^\alpha D(a_1, a_2, a_3){a_3}^{-1}\, da_3, \quad a_2 > 0,\end{aligned}$$

if $1 < \alpha < \infty$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. Therefore, by the symmetry of $D(a_1, a_2, a_3)$,

$$\begin{aligned}\int_0^\infty |\mathcal{T}_{a_1}(h_1)(a_2)|^\alpha {a_2}^{-1}\, da_2 &\leq \left(\frac{1}{2}\right)^\alpha (2K_0(2\sqrt{a_1}))^{\frac{\alpha}{\beta}} \int_0^\infty |h_1(a_3)|^\alpha \int_0^\infty D(a_1, a_2, a_3){a_2}^{-1}\, da_2{a_3}^{-1}\, da_3 \\ &\leq \left(\frac{1}{2}\right)^\alpha (2K_0(2\sqrt{a_1}))^{\frac{\alpha}{\beta}+1} \int_0^\infty |h_1(a_3)|^\alpha {a_3}^{-1}\, da_3.\end{aligned}$$

Then

$$\|\mathcal{T}_{a_1}(h_1)\|_{L^\alpha(\mathbb{R}_+, d\mu)} \leq K_0(2\sqrt{a_1})\|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)} \leq C_1\|h_1\|_{L^\alpha(\mathbb{R}_+, d\mu)}.$$

□

The shift operator associated with the Kontorovich-Lebedev-Clifford transform provides a convolution structure as follows [11]:

$$(h_1 * h_2)(a_1) = \int_0^\infty \left(\mathcal{T}_{a_1}h_1\right)(a_3)(h_2a_3)\, d\mu(a_3) = \int_0^\infty\int_0^\infty D(a_1, a_2, a_3)h_1(a_2)h_2(a_3)\, d\mu(a_2)\, d\mu(a_3),$$

for $h_1, h_2 \in C_c^\infty(\mathbb{R}_+)$.

The Convolution Theorem is the mathematical tool that provides the foundation for signal filtering. The Kontorovich-Lebedev-Clifford transform of the Kontorovich-Lebedev-Clifford convolution is the product of the Kontorovich-Lebedev-Clifford transforms, that is,

$$\mathbb{K}\left(h_1 * h_2\right)(\lambda) = \mathbb{K}\left(h_1\right)(\lambda)\mathbb{K}\left(h_2\right)(\lambda).$$

3. Windowed Kontorovich-Lebedev-Clifford Transform

Following the framework established for the windowed Kontorovich-Lebedev transform [8], the windowed Mehler-Fock transform [5], and the Weyl operator associated with the index Whittaker transform [9], we introduce the windowed Kontorovich-Lebedev-Clifford transform. In this section, we present the formal definition of this transform and outline the principal results achieved in this paper.

Definition 3.1. *For $h_1, h_2 \in C_c^\infty(\mathbb{R}_+)$, the Windowed Kontorovich-Lebedev-Clifford is defined as*

$$\mathcal{WK}\left(h_1, h_2\right)\left(a_1, \lambda\right) = \int_0^\infty h_1(a_2)\left(\mathcal{T}_{a_1} h_2\right)(a_2) K_{2i\sqrt{\lambda}}(2\sqrt{a_2})\, d\mu(a_2).$$

Using the Kontorovich-Lebedev-Clifford transform and (2.7) , it is represented as

$$\mathcal{WK}\left(h_1, h_2\right)\left(a_1, \lambda\right) = \mathbb{K}\left[h_1\left(\mathcal{T}_{a_1} h_2\right)\right](\lambda). \tag{3.1}$$

By invoking (2.7), we readily obtain

$$\mathcal{WK}\left(h_1, h_2\right)\left(a_1, \lambda\right) = \left(h_2 * K_{2i\sqrt{\lambda}}(\cdot) f\right)(a_1).$$

Theorem 3.1. *For $h_1, h_2 \in C_c^\infty(\mathbb{R}_+)$, $1 \le \alpha \le 2$, and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$, we have*

$$\left\|\mathcal{WK}\left(h_1, h_2\right)\right\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \le C_1 \left\|h_1\right\|_{L^\alpha(\mathbb{R}_+, d\mu)} \left\|h_2\right\|_{L^\beta(\mathbb{R}_+, d\mu)}.$$

Moreover, the transform $\mathcal{WK}$ can be extended to a bounded operator from $L^\alpha(\mathbb{R}_+ d\mu) \times L^\beta(\mathbb{R}_+ d\mu)$ to $L^\beta\left(I \times \mathbb{R}_+, d\mu \otimes dv\right)$.

Proof. For $h_1, h_2 \in C_c^\infty(\mathbb{R}_+)$, invoking (3.1) and (2.4), we have

$$\begin{aligned}\left\|\mathcal{WK}\left(h_1, h_2\right)\right\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} &= \left(\int_0^\infty \int_0^\infty \left|\mathbb{K}\left[h_1\left(\mathcal{T}_{a_2} h_2\right)\right](\lambda)\right|^\beta d\mu(a_2)\, dv(\lambda)\right)^{\frac{1}{\beta}} \\ &\le \left(\int_0^\infty \left(\int_0^\infty \left|h_1(a_1)\left(\mathcal{T}_{a_2} h_2\right)(a_1)\right|^\alpha d\mu(a_1)\right)^{\frac{\beta}{\alpha}} d\mu(a_2)\right)^{\frac{1}{\beta}}.\end{aligned}$$

Now, applying Minkowski's inequality and theorem (2.2) , we readily obtain

$$\left\|\mathcal{WK}\left(h_1, h_2\right)\right\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \le C_1 \left\|h_1\right\|_{L^\alpha(\mathbb{R}_+, d\mu)} \left\|h_2\right\|_{L^\beta(\mathbb{R}_+, d\mu)}.$$

Thus, $\mathcal{WK}$ can be extended to a bounded operator from $L^{\alpha}(\mathbb{R}_+, d\mu) \times L^{\beta}(\mathbb{R}_+, d\mu)$ to $L^{\beta}(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)$. Therefore, the proof is complete. □

Theorem 3.2. *Let $h_2 \in L^1(\mathbb{R}_+, d\mu)$ such that $c = 2K_0(2\sqrt{a_2+a_3}) \int_0^{\infty} h_2(a_3)\, d\mu(a_3) \neq 0$. Then for all $h_1 \in (L^1 \cap L^2)(\mathbb{R}_+, d\mu)$, we have*

$$\mathbb{K}(h_1)(\lambda) = \frac{1}{c} \int_0^{\infty} \mathcal{WK}(h_1, h_2)(a_1, \lambda)\, d\mu(a_1). \tag{3.2}$$

Moreover,

$$h_1(a_2) = \frac{1}{c} \int_0^{\infty} K_{2i\sqrt{\lambda}}(2\sqrt{a_3}) \left[\int_0^{\infty} \mathcal{WK}(h_1, h_2)(a_1, \lambda)\, d\mu(a_1) \right] dv(\lambda). \tag{3.3}$$

Proof. The result follows from (3.1), Fubini's theorem, and the fact that

$$\int_0^{\infty} (\mathcal{T}_{a_1} h_2)(a_2)\, d\mu(a_1) = 2K_0(2\sqrt{a_2+a_3}) \int_0^{\infty} (h_2 a_3)\, d\mu(a_3) = c.$$

Further, using the inverse Kontorovich-Lebedev-Clifford transform (2.3) in (3.2), we obtain (3.3). □

Theorem 3.3. *For $h_1, h_2, h \in L^2(\mathbb{R}_+, d\mu)$, we have*

$$\langle \mathcal{WK}(h_1, h_2), \mathcal{WK}(h, h_2) \rangle = \int_0^{\infty} h_1(a_1) \mathfrak{S}(a_1) \bar{h}(a_1)\, d\mu(a_1) = \langle h_1 \mathfrak{S}, h \rangle.$$

where

$$\mathfrak{S}(a_1) = \int_0^{\infty} \left| (\mathcal{T}_{a_1} h_2)(a_2) \right|^2 d\mu(a_2). \tag{3.4}$$

Proof. Using Plancherel's relation (2.5) and (3.1), we have

$$\begin{aligned} \langle \mathcal{WK}(h_1, h_2), \mathcal{WK}(h, h_2) \rangle &= \int_0^{\infty} \int_0^{\infty} h_1(a_1) (\mathcal{T}_{a_2} h_2)(a_1) \overline{h(a_1) (\mathcal{T}_{a_2} h_2)(a_1)} d\mu(a_1) d\mu(a_2) \\ &= \int_0^{\infty} h_1(a_1) \overline{h(a_1)} \int_0^{\infty} \left| (\mathcal{T}_{a_2} h_2)(a_1) \right|^2 d\mu(a_2) d\mu(a_1) \\ &= \langle h_1 \mathfrak{S}, h \rangle, \end{aligned}$$

where $\mathfrak{S}(a_1)$ is defined as(3.4). This completes the proof.

□

Theorem 3.4. *If $\alpha \in [2, \infty]$, $h_1, h_2 \in C_c^{\infty} \mathbb{R}_+$, then*

$$\left\| \mathcal{WK}(h_1, h_2) \right\|_{L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)} \leq C_1 \|h_1\|_{L^2(\mathbb{R}_+, d\mu)} \|h_2\|_{L^2(\mathbb{R}_+, d\mu)}. \tag{3.5}$$

Moreover, $\mathcal{WK}$ can be extended to a bounded operator from $L^2(\mathbb{R}_+, d\mu) \times L^2(\mathbb{R}_+, d\mu)$ to $L^{\alpha}(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)$.

Proof. For $\alpha = 2$, by Theorem (3.1), we have

$$\left\|\mathcal{WK}(h_1, h_2)\right\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \leq C_1 \|h_1\|_{L^2(\mathbb{R}_+, d\mu)} \|h_2\|_{L^2(\mathbb{R}_+, d\mu)}. \tag{3.6}$$

For $\alpha = \infty$, by Theorem (2.1) , the Holder inequality, and theorem(2.2), we get

$$\left\|\mathcal{WK}(h_1, h_2)\right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \leq C_1 \|h_1\|_{L^2(\mathbb{R}_+, d\mu)} \|h_2\|_{L^2(\mathbb{R}_+, d\mu)}. \tag{3.7}$$

Then, by(3.6), (3.7), and the Riesz-Thorin theorem [2], we yield (3.5). Since $C_c^\infty\mathbb{R}_+$ is dense in $L^2(\mathbb{R}_+ d\mu)$, a limiting argument completes the proof. □

Theorem 3.5. *Let us fix β and α satisfying $\beta \geq 2$ and $\beta' \leq \alpha \leq \beta$, where β' and α' are the conjugate exponents of β and α, respectively. Then*

$$\mathcal{WK} : L^{\alpha'}(\mathbb{R}_+, d\mu) \times L^{\alpha'}(\mathbb{R}_+, d\mu) \to L^\beta(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv)$$

is bounded.
In particular,

$$\left\|\mathcal{WK}(h_1, h_2)\right\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \leq C' \|h_1\|_{L^{\alpha'}(\mathbb{R}_+, d\mu)} \|h_2\|_{L^{\alpha}(\mathbb{R}_+, d\mu)}.$$

where C' is some constant.

Proof. Invoking Theorem (3.1), we have

$$\left\|\mathcal{WK}(h_1, h_2)\right\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \leq C_1 \|h_1\|_{L^2(\mathbb{R}_+, d\mu)} \|h_2\|_{L^2(\mathbb{R}_+, d\mu)}. \tag{3.8}$$

Using the propreties of $K_{2i\sqrt{\lambda}}(2\sqrt{a_1})$ there exits $C_1 > 0$ such that:

$$\sup_{(a_1,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+} \left|K_{2i\sqrt{\lambda}}(2\sqrt{a_1})\right| \leq C_1, \quad \forall x > 1, \lambda > 0,$$

also noting theorem (2.2),we have

$$\begin{aligned}
\left\|\mathcal{WK}(h_1, h_2)\right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} &= \sup_{(y,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+} \left|\int_0^\infty h_1(a_1)\left(\mathcal{T}_{a_1}h_2\right)(a_2) K_{2i\sqrt{\lambda}}(2\sqrt{a_1}) d\mu(a_1)\right| \\
&\leq C_1 \sup_{(y,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+} \int_0^\infty \left|h_1(a_1)\left(\mathcal{T}_{a_1}h_2\right)(a_2)\right| d\mu(a_1) \\
&\leq C_1 \sup_{(y,\lambda)\in\mathbb{R}_+\times\mathbb{R}_+} \|h_1\|_{L^{k'}(\mathbb{R}_+ d\mu)} \left\|\left(\mathcal{T}_{a_2}h_2\right)\right\|_{L^k(\mathbb{R}_+ d\mu)} \left[k \geq 1; \frac{1}{k} + \frac{1}{k'} = 1\right] \\
&\leq C_1^2 \|h_1\|_{L^{k'}(\mathbb{R}_+, d\mu)} \|h_2\|_{L^k(\mathbb{R}_+, d\mu)}.
\end{aligned} \tag{3.9}$$

Now, let $k = \infty$ in (3.9). Then by (3.8) and the interpolation of multipliers of multiple linear maps [12], we obtain the desired result.

$$\left\|\mathcal{WK}\left(h_1, h_2\right)\right\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv)} \leq C' \|h_1\|_{L^{\alpha'}(\mathbb{R}_+, d\mu)} \|h_2\|_{L^\alpha(\mathbb{R}_+, d\mu)}, \alpha \geq 2 \tag{3.10}$$

Again using (3.9) with $k=\alpha$ and (3.10), by the interpolation of multiplier maps , we get

$$\left\|\mathcal{WK}\left(h_1,h_2\right)\right\|_{L^{\alpha}(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)} \leq C'\left\|h_1\right\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}\left\|h_2\right\|_{L^{\alpha}(\mathbb{R}_+,d\mu)}, p\geq 2$$

For $\alpha<2$, let $k=1$ in (3.9) . Then by (3.10) , we get

$$\left\|\mathcal{WK}\left(h_1,h_2\right)\right\|_{L^{\beta}(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)} \leq C'\left\|h_1\right\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}\left\|h_2\right\|_{L^{\alpha}(\mathbb{R}_+,d\mu)} \tag{3.11}$$

By (3.9) with $k=\alpha$ and (3.11), we obtain

$$\left\|\mathcal{WK}\left(h_1,h_2\right)\right\|_{L^{\alpha}(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)} \leq C'\left\|h_1\right\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}\left\|h_2\right\|_{L^{\alpha}(\mathbb{R}_+,d\mu)}, \alpha<2, 2\leq\beta\leq\infty.$$

Therefore, we finish the proof. □

4. Weyl Operator Associated with the Windowed Kontorovich-Lebedev-Clifford Transform

In this section, we examine the Weyl transform in the context of the windowed Kontorovich-Lebedev-Clifford transform. Specifically, we establish the boundedness and compactness of the Weyl transform when the symbol ϕ belongs to the space $L^{\alpha}\left(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv\right)$, where $\alpha\in[1,2]$. The results presented here provide a deeper understanding of the operator-theoretic properties of the Weyl transform in this framework.

Definition 4.1. *Let the symbol $\phi\in C_c^{\infty}\left(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv\right)$, and $h_1,h_2\in C_c^{\infty}(\mathbb{R}_+)$. The Weyl transform $\mathcal{A}_\phi$ is defined by*

$$\left\langle \mathcal{A}_\phi h_1, \overline{h_2}\right\rangle = \int_0^{\infty}\int_0^{\infty}\phi(a_1,\lambda)\mathcal{WK}\left(h_1,h_2\right)(a_1,\lambda)\,d\mu(a_1)\,dv(\lambda). \tag{4.1}$$

Theorem 4.1. *Let $\phi\in C_c^{\infty}\left(\mathbb{R}_+\times\mathbb{R}_+, d\mu\otimes dv\right)$, and $h_1\in C_c^{\infty}\mathbb{R}_+$. Then, we have*

$$\left(\mathcal{A}_\phi h_1\right)(a_1) = \int_0^{\infty}\Upsilon(a_1,a_2)h_1(a_2)\,d\mu(a_2),$$

where

$$\Upsilon(a_1,a_2) = \int_0^{\infty}\left(\mathcal{T}_{h_2}\phi\right)(a_1,\lambda)K_{2i\sqrt{\lambda}}(2\sqrt{a_2})\,dv(\lambda).$$

Proof. Invoking (2.7) and Fubini's theorem, we have

$$\begin{aligned}\left\langle \mathcal{A}_\phi h_1, \overline{h_2}\right\rangle &= \int_0^{\infty}\int_0^{\infty}\phi(a_1,\lambda)\left[\int_0^{\infty}h_1(a_2)\left(\mathcal{T}_{a_2}h_2\right)(a_1)K_{2i\sqrt{\lambda}}(2\sqrt{a_2})\,d\mu(a_2)\right]d\mu(a_1)dv(\lambda)\\ &= \int_0^{\infty}\int_0^{\infty}\left[\int_0^{\infty}\left(\mathcal{T}_{a_2}\phi\right)(a_1,\lambda)K_{2i\sqrt{\lambda}}(2\sqrt{a_2})\,dv(\lambda)\right]h_1(a_2)h_2(a_1)\,d\mu(a_1)d\mu(a_2)\end{aligned}$$

Thus,

$$\left(\mathcal{A}_\phi h_1\right)(a_1) = \int_0^\infty \Upsilon(a_1, a_2) h_1(a_2)\, d\mu(a_2),$$

where

$$\Upsilon(a_1, a_2) = \int_0^\infty \left(\mathcal{T}_{h_2}\phi\right)(a_1, \lambda) K_{2i\sqrt{\lambda}}(2\sqrt{a_2})\, dv(\lambda).$$

This completes the proof. □

Theorem 4.2. *Let $\Upsilon \in C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$. Then, for all $h_1 \in C_c^\infty(\mathbb{R}_+)$,*

$$\mathcal{A}_\phi : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}_+, d\mu)$$

is a Hilbert-Schmidt operator and, consequently, it is compact. Moreover, for $\phi \in L^\alpha\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right), \alpha \in [1, 2]$, the operator $\mathcal{A}_\phi$ can be extended to a bounded operator from $L^2(\mathbb{R}_+, d\mu)$ to itself.

Proof. From Theorem (4.1) and by the L^2-boundedness of the translation operator in (2.2), and Theorem (2.1), it follows that $\Upsilon(a_1, a_2) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\mu\, d\mu)$. Hence, we have

$$\begin{aligned}\left\|\mathcal{A}_\phi\right\|_2 &\leq \left(\int_0^\infty \int_0^\infty |\Upsilon(a_1, a_2)|^2\, d\mu(a_1)\, d\mu(a_2) \int_0^\infty \left|h_1(a_1)\right|^2\, d\mu(a_1)\right)^{\frac{1}{2}} \\ &= \|\Upsilon\|_2 \|h_1\|_2.\end{aligned}$$

Since $C_c^\infty(\mathbb{R}_+)$ is dense in $L^2(\mathbb{R}_+, d\mu)$, it follows that $\mathcal{A}_\phi$ can be extended to a bounded operator from $L^2(\mathbb{R}_+, d\mu)$ to $L^2(\mathbb{R}_+, d\mu)$.

In particular, $\Upsilon(a_1, a_2) \in L^2(\mathbb{R}_+ \times \mathbb{R}_+, d\mu\, d\mu)$ implies that $\mathcal{A}_\phi : L^2(\mathbb{R}_+, d\mu) \to L^2(\mathbb{R}_+, d\mu)$ is a Hilbert-Schmidt operator. Consequently, it is compact. By the density of $C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$ in $L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$, the proof is complete. □

We denote by $\mathcal{Z}(E, F)$ (or $\mathcal{Z}(E)$ if $E = F$) the space of bounded linear operators from a Banach space E into a Banach space F.

Theorem 4.3. *Let $\phi \in L^\alpha\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$, $\alpha \in [1, 2]$. Then the operator $\mathcal{A}_\phi$ is a bounded operator on $L^2(\mathbb{R}_+ d\mu)$, and hence*

$$\left\|\mathcal{A}_\phi\right\|_{\mathcal{Z}(L^2)} \leq C_1 \|\phi|_{L^\alpha(\mathbb{R}_+ \times \mathbb{R}_+, dx \otimes dv)}$$

Proof. Using Theorem (3.1) , Definition (4.1), and the density of $C_c^\infty\left(\mathbb{R}_+ \times \mathbb{R}_+\right)$ in $L^2\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$, as well as the density of $C_c^\infty \mathbb{R}_+$ in $L^2(\mathbb{R}_+ d\mu)$, the proof follows. □

Theorem 4.4. *The Weyl operator $\mathcal{A}_\phi$ with symbol $\phi \in L^\beta\left(\mathbb{R}_+ \times \mathbb{R}_+, d\mu \otimes dv\right)$ is bounded on $L^\alpha(\mathbb{R}_+ d\mu)$ if $\beta \leq 2$ and $\beta \leq \alpha \leq \beta'$, with the corresponding norm estimate*

$$\left\|\mathcal{A}_\phi\right\|_{\mathcal{Z}(L^\alpha)} \leq C' \|\phi|_{L^\beta(I \times \mathbb{R}_+, dx \otimes dv)}$$

Proof. (i) Let us assume $\alpha \neq 1$ and $\alpha \neq \infty$. In this case, the boundedness follows from Definition (4.1) and Theorem (3.5). We have

$$\begin{aligned}\left\|\mathcal{A}_\phi\right\| &= \sup_{\|h_1\|_{L^\alpha(\mathbb{R}_+,d\mu)}=\|h_2\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}=1} \left|\int_0^\infty \int_0^\infty \phi(a_1,\lambda)\mathcal{WK}(h_1,h_2)(a_1,\lambda)\,d\mu(a_1)\,dv(\lambda)\right| \\ &\leq \sup_{\|h_1\|_{L^\alpha(\mathbb{R}_+,d\mu)}=\|h_2\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}=1} \left(\int_0^\infty \int_0^\infty |\phi(a_1,\lambda)|^\beta\,d\mu(a_1)\,dv(\lambda)\right)^{\frac{1}{\beta}} \\ &\quad \times\left(\int_0^\infty \int_0^\infty \left|\mathcal{WK}(h_1,h_2)(a_1,\lambda)\right|^{\beta'}\,d\mu(a_1)\,dv(\lambda)\right)^{\frac{1}{\beta'}} \\ &\leq C'\|\phi\|_{L^\beta(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)}\,\|h_1\|_{L^\alpha(\mathbb{R}_+,d\mu)}\,\|h_2\|_{L^{\alpha'}(\mathbb{R}_+,d\mu)}\end{aligned}$$

(ii) If $\alpha = 1$, then $\beta = 1$, and by using Theorem (3.5), we readily obtain

$$\begin{aligned}\left\|\mathcal{A}_\phi\right\| &= \sup_{\|h_1\|_{L^\infty(\mathbb{R}_+,d\mu)}=\|h_2\|_{L^1(\mathbb{R}_+,d\mu)}=1} \left|\int_0^\infty \int_0^\infty \phi(a_1,\lambda)\mathcal{WK}(h_1,h_2)(a_1,\lambda)\,d\mu(a_1)\,dv(\lambda)\right| \\ &\leq \sup_{\|h_1\|_{L^\infty(\mathbb{R}_+,d\mu)}=\|h_2\|_{L^1(\mathbb{R}_+,d\mu)}=1} \left|\int_0^\infty \int_0^\infty \phi(a_1,\lambda)\left\|\mathcal{WK}(h_1,h_2)\right\|_\infty\,d\mu(a_1)\,dv(\lambda)\right| \\ &\leq C'\|\phi\|_{L^1(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)}\end{aligned}$$

(iii) If $\alpha = \infty$, then we must have $\beta = 1$ too. Using the same arguments, we can obtain that

$$\left\|\mathcal{A}_\phi\right\| \leq C'\|\phi\|_{L^1(\mathbb{R}_+\times\mathbb{R}_+,d\mu\otimes dv)}.$$

Using the boundedness of the Weyl operator, we can easily deduce that the bounded operator is also compact. □

Theorem 4.5. *In the same hypotheses as in Theorem* (4.4) *, the Weyl operator is compact on* $L^\alpha(\mathbb{R}_+,d\mu)$.

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