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Extended q - Difference Operator of the Second Kind: Analytical Insights and Diverse Applications

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Abstract. The q-difference operator is a basic tool in q-calculus, widely used in many mathematical and scientific fields, such as statistical physics, fractal geometry, quantum mechanics, number theory, combinatorics, and orthogonal polynomials. Its applications are also found in advanced sciences, like quantum theory, mechanics, and the theory of relativity. In this paper, we define the extended q-difference operator of the second kind and its inverse. We further derive the Leibniz theorem, Montmorte's theorem, and several properties associated with the extended q-difference operator of the second kind. Additionally, a formula for the sum of partial sums of higher powers of real numbers in an arithmetic progression is constructed using its inverse. Numerical examples are given to illustrate the results.

1. Introduction

The foundation for a theory of difference equations based on the difference operator Δ is defined as

$$\Delta u(k) = u(k+1) - u(k), k \in N = \{0, 1, 2, \cdots\}. \tag{1.1}$$

Moreover, a number of authors ([1], [8], [10], [18]), have proposed that the difference operator Δ_ℓ can be defined as

$$\Delta_\ell u(k) = u(k+\ell) - u(k), k \in [0, \infty), \ell \in (0, \infty), \tag{1.2}$$

with no appreciable findings in the field of numerical techniques. An alternative approach to the theory of difference equations was taken in 2006, yielding many interesting findings in the area of Numerical Methods ([12]- [16]), by using the definition of Δ as stated in (1.2).

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Moreover, applications of q-calculus have emerged in the last thirty years in approximation theory. The well-known Bernstein polynomials were used in the first q-analogue by Saw Lupas in 1987. Further, in 1997, Phillips explored an addition option in the traditional Bernstein polynomial q-analogue. Subsequently, some scholars proposed the q-extension of the highly exponential type operators, such as Picard, Weierstrass, Bleiman, Szasz-Mirakyan, Meyer-Konig-Zeller, and Baskakov. Also, a discussion on the approximation features of $q-$analogues for some standard integral operators of Durrmeyer and Kantorovich type was conducted ([3]).

In literature ([2], [4], [11], [17], [19], [20]), mathematicians have introduced q-numbers, including q-discrete distribution, q-difference equations, q-series, and q-calculus.

In ([9]), the $q-$derivative operator, Δ_q is defined as

$$\Delta_q u(k) = \frac{u(kq) - u(k)}{(q-1)k}, q \in (0, \infty).$$

They also did not produce any noteworthy results in the numerical analysis. Recently, Chandrasekar and Suresh ([5]), defined Δ_q by replacing $\Delta_{q(\ell)}$ as

$$\Delta_{q(\ell)} u(k) = \frac{u(kq) - \ell u(k)}{(q-\ell)k},$$

for the real-valued function $u(k), \ell \in (0, \infty)$ and derived numerous forms of arithmetic-geometric progressions in the field of Numerical techniques.

In ([6], [7]), we defined the first-kind extended q-difference operator for real-valued functions $u(k)$ as

$$\Delta_{q(\ell)} u(k) = u((k+\ell)q) - u(k) \tag{1.3}$$

and developed the formula for the sum of real numbers of arithmetic progression in Numerical Methods.

Stated on this background, we describe the extended q-difference operator of the second kind and make use of its inverse operator to obtain the formula for fractional series in Numerical Analysis.

2. Basic Definitions

In this section, we provide basic definitions and preliminary results for future discussion.

Definition 2.1. *Let $u(k)$ be a real-valued function. Then the extended second-kind $q-$difference operator of $\Delta_{q(\ell_1,\ell_2)}$ is defined as*

$$\Delta_{q(\ell_1,\ell_2)} u(k) = u\left(((k+\ell_1)q + \ell_2)q\right) - \left[u\left((k+\ell_1)q\right) + u\left((k+\ell_2)q\right)\right] + u(k),$$
$$\text{where } q, \ell_1, \ell_2 \in (0, \infty). \tag{2.1}$$

Lemma 2.1. *The relation between $\Delta_{q(\ell_1,\ell_2)}$ and $E^{q(\ell_1,\ell_2)}$ is*

$$\Delta_{q(\ell_1,\ell_2)} = E^{q(\ell_1,\ell_2)} - \left[E^{q(\ell_1)} + E^{q(\ell_2)}\right] + 1. \tag{2.2}$$

Proof. The shift operator $E^{q(\ell_1,\ell_2)}$ is defined as

$$E^{q(\ell_1,\ell_2)}u(k) = u\left(((k+\ell_1)q+\ell_2)q\right), k \in [0,\infty). \tag{2.3}$$

The proof follows from (2.1) and (2.3). □

Theorem 2.1. *For real numbers q, ℓ_1 and ℓ_2, we have*

$$\Delta_{q(\ell_1,\ell_2)} = \Delta_{q(\ell_1)}\Delta_{q(\ell_2)}. \tag{2.4}$$

Proof. Equation (2.4) can be obtained using (2.1) and the relation $E^{q(\ell)} = \Delta_{q(\ell)} + 1$. □

Remark 2.1. *For $\ell_1, \ell_2 \in N(1)$,*

$$\Delta_{q(\ell_1,\ell_2)} = \left(\sum_{m=1}^{q(\ell_1)} q(\ell_1)C_m\Delta^m\right)\left(\sum_{n=1}^{q(\ell_2)} q(\ell_2)C_n\Delta^n\right). \tag{2.5}$$

Proof. Using (2.2) and $1+\Delta_{q(\ell)} = (1+\Delta)^{q(\ell)}$, Equation (2.5) can be obtained. □

Lemma 2.2. *If $u(k)$ and $v(k)$ are any two real-valued functions, then*

$$\Delta_{q(\ell_1,\ell_2)}\left[au(k)+bv(k)\right] = a\Delta_{q(\ell_1,\ell_2)}u(k) + b\Delta_{q(\ell_1,\ell_2)}v(k),$$

where a and b are any two non-zero scalars.

Lemma 2.3. *Given any two real valued functions $u(k)$ and $v(k) \neq 0$, we have*

$$\begin{aligned}\Delta_{q(\ell_1,\ell_2)}[u(k)v(k)] &= v(k)\Delta_{q(\ell_1,\ell_2)}u(k) \\ &\quad + \Big[u\left(((k+\ell_1)q+\ell_2)q\right)\left(E^{q(\ell_1,\ell_2)}-1\right) \\ &\quad - u\left((k+\ell_1)q\right)\Delta_{q(\ell_1)} - u\left((k+\ell_2)q\right)\Delta_{q(\ell_2)}\Big]v(k). \end{aligned} \tag{2.6}$$

Lemma 2.4. *For any two real valued functions, $u(k)$ and $v(k) \neq 0$,*

$$\begin{aligned}\Delta_{q(\ell_1,\ell_2)}\left[\frac{u(k)}{v(k)}\right] &= \frac{\left[v(k)\left(E^{q(\ell_1,\ell_2)}-1\right)u(k) - u(k)\left(E^{q(\ell_1,\ell_2)}-1\right)v(k)\right]}{v(k)\,v\left((k+\ell_1)q\right)v\left((k+\ell_2)q\right)v\left(((k+\ell_1)q+\ell_2)q\right)} \\ &\quad - \Delta_{q(\ell_1)}\left[\frac{u(k)}{v(k)}\right] - \Delta_{q(\ell_2)}\left[\frac{u(k)}{v(k)}\right].\end{aligned}$$

3. Superior Grade for Extended q-difference Operator of the Second Kind

This section establishes the generalized Leibniz theorem according to $\Delta_{q(\ell_1,\ell_2)}$ and defines the higher order of $\Delta_{q(\ell_1,\ell_2)}$.

Definition 3.1. *The second order of $\Delta_{q(\ell_1,\ell_2)}$ is defined as*

$$\Delta^2_{q(\ell_1,\ell_2)} = \left[\Delta_{q(\ell_1,\ell_2)}\left(\Delta_{q(\ell_1,\ell_2)}\right)\right].$$

In general, the n^{th} order of $\Delta_{q(\ell_1,\ell_2)}$ is defined as

$$\Delta^n_{q(\ell_1,\ell_2)} = \Delta_{q(\ell_1,\ell_2)}\left[\Delta^{n-1}_{q(\ell_1,\ell_2)}\right].$$

Remark 3.1. *For $q, \ell_1, \ell_2 \in [0, \infty)$ and for any two positive integers m and n,*

$$\Delta^m_{q(\ell_1,\ell_2)}\Delta^n_{q(\ell_1,\ell_2)} = \Delta^n_{q(\ell_1,\ell_2)}\Delta^m_{q(\ell_1,\ell_2)}.$$

Remark 3.2. *Suppose $u(k)$ is any real valued function then*

$$\Delta^m_{q(\ell_1,\ell_2)}cu(k) = c\Delta^m_{q(\ell_1,\ell_2)}u(k),$$

where c is a constant.

Theorem 3.1. *Given n is a positive integer and q, ℓ_1, and ℓ_2 are positive reals. Then*

$$\Delta^n_{q(\ell_1,\ell_2)} = \prod_{m=1}^{2}\left[\sum_{i=0}^{n}(-1)^i\binom{n}{i}E^{q(\ell_m)(n-i)}\right], \tag{3.1}$$

and also

$$\Delta^n_{q(\ell_1,\ell_2)}u(k) = \sum_{s=0}^{n}\sum_{r=0}^{n}(-1)^{s+r}\binom{n}{s}\binom{n}{r}\left[u\left(kq^{2n-s-r} + \sum_{t=1}^{n-s}\ell_1 q^{t+n-r} + \sum_{p=1}^{n-r}\ell_2 q^p\right)\right]. \tag{3.2}$$

Proof. By Binomial theorem, we find

$$\Delta^n_{q(\ell_1,\ell_2)} = \left[\left(E^{q(\ell_1)}-1\right)\left(E^{q(\ell_2)}-1\right)\right]^n. \tag{3.3}$$

Equation (3.1) follows from (3.3). Further on operating both sides with $u(k)$ and simplifying (3.3), we get (3.2). □

Corollary 3.1. *For any two positive integers m and n,*

$$\Delta^m_{q(\ell_1,\ell_2)}k^n = \sum_{s=0}^{m}\sum_{r=0}^{m}(-1)^{s+r}\binom{m}{s}\binom{m}{r}\left[kq^{2m-s-r} + \sum_{t=1}^{m-s}\ell_1 q^{t+m-r} + \sum_{p=1}^{m-r}\ell_2 q^p\right]^n.$$

Proof. The proof follows by considering $u(k) = k^n$ in (3.2). □

Lemma 3.1. *If ℓ_{1i} and ℓ_{2m} are positive integers with $q(\ell_1) = \sum_{i=1}^{n} q(\ell_{1i})$ and $q(\ell_2) = \sum_{m=1}^{n} q(\ell_{2m})$ then*

$$\Delta_{q(\ell_1,\ell_2)} = \left[\prod_{i=1}^{n}\left(\Delta_{q(\ell_{1i})}+1\right)-1\right]\left[\prod_{m=1}^{n}\left(\Delta_{q(\ell_{2m})}+1\right)-1\right]. \tag{3.4}$$

Proof. Equation (3.4) follows by (3.3). □

Remark 3.3. *The results are easily deduced from $\Delta_{q(\ell_1,\ell_2)}$ and $E^{q(\ell_1,\ell_2)}$ as follows:*

(i) $\Delta_{nq(\ell_1,\ell_2)} = \left(E^{nq(\ell_1)}-1\right)\left(E^{nq(\ell_2)}-1\right) = \left[\left(1+\Delta_{q(\ell_1)}\right)^n-1\right]\left[\left(1+\Delta_{q(\ell_2)}\right)^n-1\right]$

(ii) $\Delta_{nq(\ell_1,\ell_2)} = \prod_{m=1}^{2}\left(\sum_{i_{m=1}}^{n}\binom{n}{i_m}\Delta^{i_m}_{q(\ell_m)}\right)$

(iii) $\Delta^n_{q(\ell_1,\ell_2)} = \prod_{m=1}^{2}\left(\sum_{i_m=0}^{n-1}(-1)^{i_m}nC_{i_m}\Delta_{q(\ell_m)(n-i_m)}\right)$

Corollary 3.2. *For the shift operator* $E^{q(\ell_1,\ell_2)}$

$$\left[E^{q(\ell_1,\ell_2)} - E^{q(\ell_1)} - E^{q(\ell_2)}\right]^n u(k) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \Delta^{(n-r)}_{q(\ell_1,\ell_2)} u(k). \tag{3.5}$$

Proof. Equation (3.5) is followed by (2.2). □

Example 3.1. *When ϕ_1 and ϕ_2 are expressed in degrees, the integer values should be taken only anticlockwise.*

$$\left[E^{q(\phi_1,\phi_2)} - E^{q(\phi_1)} - E^{q(\phi_2)}\right]^n sin(k) = \left[\Delta_{q(\phi_1,\phi_2)} - 1\right]^n sin(k). \tag{3.6}$$

Proof. Taking $\phi_1 = \ell_1$, $\phi_2 = \ell_2$ and operating $u(k) = sin(k)$ in (3.5). □

The discrete version of the Leibniz theorem in accordance with $\Delta_{q(\ell_1,\ell_2)}$ is as follows.

Theorem 3.2. *If $u(k)$ and $v(k)$ are any two real functions, then*

$$\Delta^n_{q(\ell_1,\ell_2)}\left[u(k)v(k)\right] = \sum_{r=0}^{n} \binom{n}{r} \Delta^n_{q(\ell_2)} \left[\Delta^r_{q(\ell_1)} u(k) \Delta^{n-r}_{q(\ell_1)} v\left(kq^r + \ell_1 \sum_{t=1}^{r} q^t\right)\right]. \tag{3.7}$$

Proof. Define the operator $E_1^{q(\ell_2)}$ and $E_2^{q(\ell_2)}$ as

$$E_1^{q(\ell_2)}\left[u(k)v(k)\right] = u\left((k+\ell_2)\,q\right)v(k)$$

and

$$E_2^{q(\ell_2)}\left[u(k)v(k)\right] = u(k)v\left((k+\ell_2)\,q\right). \tag{3.8}$$

Hence, we get

$$E^{q(\ell_2)} = E_1^{q(\ell_2)} E_2^{q(\ell_2)}. \tag{3.9}$$

Also, we define

$$\left[\Delta_{q(\ell_2)}\right]_1 = E_1^{q(\ell_2)} - 1 \quad \text{and} \quad \left[\Delta_{q(\ell_2)}\right]_2 = E_2^{q(\ell_2)} - 1. \tag{3.10}$$

This implies

$$\Delta_{q(\ell_2)} = E_1^{q(\ell_2)} E_2^{q(\ell_2)} - 1.$$

From (3.10), we get

$$\Delta_{q(\ell_2)} = \left[\Delta_{q(\ell_2)}\right]_2 + \left[\Delta_{q(\ell_2)}\right]_1 E_2^{q(\ell_2)}. \tag{3.11}$$

Again operating $\Delta_{q(\ell_1)}$ in equation (3.11) and using Binomial theorem we get equation (3.7). □

Lemma 3.2. *When an integer x is positive and a real valued function $a(k)$ exists, then*

$$\sum_{k=0}^{\infty}\left[\frac{x^{kq(\ell_1,\ell_2)}}{k!\left[q(\ell_1,\ell_2)\right]^k} a_{[kq(\ell_1,\ell_2)]}\right] = \left[e^{\frac{x^{q(\ell_1,\ell_2)}E^{q(\ell_1,\ell_2)}}{q(\ell_1,\ell_2)}}\right] a(0) = \left[e^{\frac{x^{q(\ell_1,\ell_2)}\prod_{m=1}^{2}\left(\Delta_{q(\ell_m)}+1\right)}{q(\ell_1,\ell_2)}}\right] a(0).$$

Proof. The proof follows from the relation $E^{q(\ell)}u(k) = u\left((k+\ell)q\right)$, $a(k) = E^k a(0)$ and (2.3). □

The following theorem is generalized version of Montmorte's theorem for $\Delta_{q(\ell_1,\ell_2)}$.

Theorem 3.3. *If the series* $\sum_{k=0}^{\infty} x^{kq(\ell_1,\ell_2)} a\left(kq(\ell_1,\ell_2)\right)$ *converges, then it can be expressed as*

$$\sum_{k=0}^{\infty} x^{kq(\ell_1,\ell_2)} a\left(kq(\ell_1,\ell_2)\right) = \left[\sum_{k=0}^{\infty} \frac{x^{kq(\ell_1,\ell_2)}\left(\Delta_{q(\ell_1,\ell_2)} + \Delta_{q(\ell_1)} + \Delta_{q(\ell_2)}\right)^k}{\left(1 - x^{q(\ell_1,\ell_2)}\right)^{k+1}}\right] a(0).$$

Proof. The proof follows from the relation $E^{q(\ell_1,\ell_2)} = \left(\Delta_{q(\ell_1)} + 1\right)\left(\Delta_{q(\ell_2)} + 1\right)$. □

Corollary 3.3. *If the series* $\sum_{k=0}^{\infty} x^{kq(\ell,\ell)} a\left(kq(\ell,\ell)\right)$ *converges, then*

$$\sum_{k=0}^{\infty} x^{kq(\ell,\ell)} a\left(kq(\ell,\ell)\right) = \left[\sum_{k=0}^{\infty} \frac{x^{kq(\ell,\ell)}\left(\Delta^2_{q(\ell)} + 2\Delta_{q(\ell)}\right)^k}{\left(1 - x^{q(\ell,\ell)}\right)^{k+1}}\right] a(0).$$

4. Extended q-polynomial Factorial of the Second Kind

This section presents the relationship between the polynomial factorial and the q-difference operator, denoted by $\Delta_{q(\ell_1,\ell_2)}$.

Definition 4.1. *If n is a positive integer, then the extended q-polynomial factorial of second kind denoted by $k^{(n)}_{q(\ell_1,\ell_2)}$ is defined as*

$$k^{(n)}_{q(\ell_1,\ell_2)} = \left[(k+\ell_1)\,q\right]^{(n)}_{q(\ell_2)} + \left[(k+\ell_2)\,q\right]^{(n)}_{q(\ell_1)} - \left[k^{(n)}_{q(\ell_2)} + k^{(n)}_{q(\ell_1)}\right]. \tag{4.1}$$

Lemma 4.1. *If m and n are positive integers, then*

$$\Delta^m_{q(\ell)} k^{(n)}_{q(\ell)} = \left[\frac{q^n-1}{q^{n-1}} \Delta^{m-1}_{q(\ell)} \left(k^{(1)}_{q(\ell)}\right) + \frac{(q^n + (n-1))\,\ell}{q^{n-1}} \Delta^{m-1}_{q(\ell)}\right]\left(k^{(n-1)}_{q(\ell)}\right). \tag{4.2}$$

Proof. The proof is obtained using the induction approach on m and n. □

Lemma 4.2. *For the positive integers ℓ_1, ℓ_2, t and n,*

$$\Delta_{q(\ell_1,\ell_2)} k^{(n)}_t = \begin{cases} \left[\frac{q^n-1}{q^{n-1}}\left(((k+\ell_2)\,q)^{(1)}_t - k^{(1)}_t\right) + \frac{(q^n+(n-1))\ell_1}{q^{n-1}}\right] \\ \left(((k+\ell_2)\,q)^{(n-1)}_t - k^{(n-1)}_t\right), \text{if } t = q(\ell_1) \\ \\ \left[\frac{q^n-1}{q^{n-1}}\left(((k+\ell_1)\,q)^{(1)}_t - k^{(1)}_t\right) + \frac{(q^n+(n-1))\ell_1}{q^{n-1}}\right] \\ \left(((k+\ell_1)\,q)^{(n-1)}_t - k^{(n-1)}_t\right), \text{if } t = q(\ell_2). \end{cases} \tag{4.3}$$

Proof. The proof is derived from (2.2) and (4.1). □

Lemma 4.3. *For the real numbers q, ℓ_1, ℓ_2 and any positive integer n,*

$$\begin{aligned} \Delta_{q(\ell_1,\ell_2)} k^{(n)}_{q(\ell_1,\ell_2)} &= \Delta_{q(\ell_1,\ell_1)} \left[\frac{q^n-1}{q^{n-1}}\left(k^{(1)}_{q(\ell_2)}\right) + \frac{(q^n+(n-1))\,\ell_1}{q^{n-1}}\right]\left(k^{(n-1)}_{q(\ell_2)}\right) \\ &\quad + \Delta_{q(\ell_2,\ell_2)} \left[\frac{q^n-1}{q^{n-1}}\left(k^{(1)}_{q(\ell_1)}\right) + \frac{(q^n+(n-1))\,\ell_1}{q^{n-1}}\right]\left(k^{(n-1)}_{q(\ell_1)}\right). \end{aligned}$$

Also, if $\ell_1 = \ell_2 = \ell$, then

$$\Delta_{q(\ell,\ell)} k^{(n)}_{q(\ell,\ell)} = 2\Delta_{q(\ell,\ell)} \left[\frac{q^n - 1}{q^{n-1}} \left(k^{(1)}_{q(\ell)} \right) + \frac{(q^n + (n-1))\,\ell}{q^{n-1}} \right] \left(k^{(n-1)}_{q(\ell)} \right).$$

Proof. The proof follows from (4.1). □

5. Main Results on Inverse of the Extended q-difference Operator of the Second Kind

In this section, we define the inverse operator and derive some interesting results using its inverse.

Definition 5.1. *The inverse of the extended q-difference operator of the second kind denoted by $\Delta^{-1}_{q(\ell_1,\ell_2)}$ is defined as,*

$$if\ \Delta_{q(\ell_1,\ell_2)} v(k) = u(k)\ then\ v(k) = \Delta^{-1}_{q(\ell_1,\ell_2)} u(k) + c_j, \tag{5.1}$$

and the n^{th} order inverse operator denoted by $\Delta^{-n}_{q(\ell_1,\ell_2)}$ is defined as

$$if\ \Delta^{n}_{q(\ell_1,\ell_2)} v(k) = u(k)\ then\ v(k) = \Delta^{-n}_{q(\ell_1,\ell_2)} u(k) + c_j,$$

where c_j is constant, depending upon $k \in N_\ell(j)$, $j = k - \left[\frac{k}{\ell}\right]\ell$.

Remark 5.1. *Let $u(k)$ be a real-valued function. Then*

$$\Delta_{q(\ell_1,\ell_2)} \left[\Delta^{-1}_{q(\ell_1,\ell_2)} u(k) \right] \neq \Delta^{-1}_{q(\ell_1,\ell_2)} \left[\Delta_{q(\ell_1,\ell_2)} u(k) \right].$$

Theorem 5.1. *If k, ℓ and q are positive real values, then*

$$\begin{aligned} &\sum_{r=2}^{j^\star+1} (r-1)\, u\left(\frac{k - \ell \sum\limits_{t=1}^{r} q^t}{q^r} \right) + \sum_{r=1}^{j^\star-1} (j^\star - r)\, u\left(\frac{k - \ell \sum\limits_{t=1}^{j^\star+r+1} q^t}{q^{j^\star+r+1}} \right) \\ &= \Delta^{-2}_{q(\ell)} u(k)\Big|^{k}_{j_{1q(\ell)}} - \Delta^{-2}_{q(\ell)} u(k)\Big|_{j_{1q(\ell)}} + \Delta^{-2}_{q(\ell)} u(k)\Big|_{j_{2q(\ell)}}), \end{aligned} \tag{5.2}$$

where $j_{1q(\ell)} = \left(\frac{k - \ell \sum\limits_{t=1}^{j^\star} q^t}{q^{j^\star}} \right)$, $j_{2q(\ell)} = \left(\frac{k - \ell \sum\limits_{t=1}^{2j^\star} q^t}{q^{2j^\star}} \right)$ and $j^\star = \left[\frac{k}{\ell}\right]$ is the integral part of $\frac{k}{l}$.

Proof. The proof follows from (5.1) and the relation

$$\Delta^2_{q(\ell)} \left[\sum_{r=2}^{j^\star+1} (r-1)\, u\left(\frac{k - \ell \sum\limits_{t=1}^{r} q^t}{q^r} \right) + \sum_{r=1}^{j^\star-1} (j^\star - r)\, u\left(\frac{k - \ell \sum\limits_{t=1}^{j^\star+r+1} q^t}{q^{j^\star+r+1}} \right) \right] = u(k).$$

□

Theorem 5.2. *If k, ℓ and q are positive real values, then*

$$\left[\sum_{r=2}^{j^\star+1}(r-1)\,u\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\frac{\ell q}{q-1}\right)+\sum_{r=1}^{j^\star-1}(j^\star-r)\,u\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\frac{\ell q}{q-1}\right)\right]$$
$$=\left(\frac{k}{(q-1)^2}\right)-2\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star}q^t}{(q-1)^2q^{j^\star}}\right)+\left(\frac{k-\ell\sum\limits_{t=1}^{2j^\star}q^t}{(q-1)^2q^{2j^\star}}\right). \tag{5.3}$$

Proof. Let $u(k)=k$. From (5.1), we have

$$\Delta^{-1}_{q(\ell,\ell)}\left(k+\frac{\ell q}{q-1}\right)=\frac{k}{(q-1)^2}+c_j \tag{5.4}$$

The proof follows by (5.2) and (5.4). □

Lemma 5.1. *For $\lambda\neq1$, $k\geq 2q\ell$ and $P(k)$ is any function of k, then*

$$\sum_{r=2}^{j^\star+1}(r-1)\lambda^{\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)}P\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)+\sum_{r=1}^{j^\star-1}(j^\star-r)\,\lambda^{\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}\right)}P\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}\right)$$
$$=\frac{\lambda^k}{\left(\lambda^{\Delta_{q(\ell)}(k)}-1\right)^2}\left[1-\frac{\lambda^{\Delta_{q(\ell)}(k)}\Delta_{q(\ell)}}{\left(\lambda^{\Delta_{q(\ell)}(k)}-1\right)}+\frac{\lambda^{2\Delta_{q(\ell)}(k)}\Delta^2_{q(\ell)}}{\left(\lambda^{\Delta_{q(\ell)}(k)}-1\right)^2}+\cdots+\right]^2P(k)+c_j.$$

Proof. Let $\Delta_{q(\ell,\ell)}\lambda^kF(k)=\lambda^kP(k)$, where $P(k)=\left(\lambda^{\Delta_{q(\ell)}(k)}E^{q(\ell)}-1\right)^2F(k)$. Operating $\Delta^{-1}_{q(\ell,\ell)}$ on both sides of $\Delta_{q(\ell,\ell)}\lambda^kF(k)=\lambda^kP(k)$, we obtain

$$\Delta^{-1}_{q(\ell,\ell)}\lambda^kP(k)=\lambda^kF(k)+c_j=\left(\lambda^{\Delta_{q(\ell(k)}E^{q(\ell)}}-1\right)^{-2}P(k)+c_j.$$

Now the proof follows from $\Delta_{q(\ell)}=E^{q(\ell)}+1$ and the Binomial theorem. □

Theorem 5.3. *Let $k\in[0,\infty)$ and $j^\star=\left[\frac{k}{\ell}\right]$. Then*

$$\sum_{r=2}^{j^\star+1}(r-1)\left[\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q+\ell\right)q\right)^{-1}-2\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q\right)^{-1}+\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}\right)^{-1}\right]$$
$$+\sum_{r=1}^{j^\star-1}(j^\star-r)\left[\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+\ell\right)q\right)^{-1}-2\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q\right)^{-1}\right. \tag{5.5}$$

$$+\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}\right)^{-1}\Bigg] = \frac{1}{k}-2\left(\frac{q^{j^\star}}{k-\ell\sum\limits_{t=1}^{j^\star}q^t}\right)+\left(\frac{q^{2j^\star}}{k-\ell\sum\limits_{t=1}^{2j^\star}q^t}\right)$$

Proof. In (5.1) by taking $u(k)=\frac{1}{k}$, we have

$$\Delta_{q(\ell,\ell)}\left(\frac{1}{k}\right)=\left[\left(\left(\left(k+\ell\right)q+\ell\right)q\right)^{-1}-2\left(\left(k+\ell\right)q\right)^{-1}+\left(k\right)^{-1}\right]. \tag{5.6}$$

From equation (5.1), we obtain

$$\Delta^{-1}_{q(\ell,\ell)}\left[\left(\left(\left(k+\ell\right)q+\ell\right)q\right)^{-1}-2\left(\left(k+\ell\right)q\right)^{-1}+\left(k\right)^{-1}\right]=\left(\frac{1}{k}\right)+c_j$$

The proof follows from (5.2) and (5.6). ◻

Theorem 5.4. *If $k\in[0,\infty)$ and n is an integer, then*

$$\sum_{r=2}^{j^\star+1}(r-1)\left[\frac{\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q+\ell\right)q+n\ell\right)}{\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q+\ell\right)q+(n-1)\ell\right)}-2\frac{\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q+n\ell\right)}{\left(\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+\ell\right)q+(n-1)\ell\right)}\right.$$
$$\left.+\frac{\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+n\ell\right)}{\left(\frac{k-\ell\sum\limits_{t=1}^{r}q^t}{q^r}+(n-1)\ell\right)}\right]$$

$$+\sum_{r=1}^{j^\star-1}(j^\star-r)\left[\frac{\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+\ell\right)q+n\ell\right)}{\left(\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+\ell\right)q+(n-1)\ell\right)}-2\frac{\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+n\ell\right)}{\left(\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+(n-1)\ell\right)}\right.$$
$$\left.+\frac{\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+n\ell\right)}{\left(\frac{k-\ell\sum\limits_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+(n-1)\ell\right)}\right]$$

$$=\left[\frac{k+n\ell}{k+(n-1)\ell}\right]-2\left[\frac{\left(\frac{k-\ell\sum_{t=1}^{j^\star}q^t}{q^{j^\star}}\right)+n\ell}{\left(\frac{k-\ell\sum_{t=1}^{j^\star}q^t}{q^{j^\star}}\right)+(n-1)\ell}\right]+\left[\frac{\left(\frac{k-\ell\sum_{t=1}^{2j^\star}q^t}{q^{2j^\star}}\right)+n\ell}{\left(\frac{k-\ell\sum_{t=1}^{2j^\star}q^t}{q^{2j^\star}}\right)+(n-1)\ell}\right]. \tag{5.7}$$

Proof. Taking $u(k)=k+n\ell$ and $v(k)=k+(n-1)\,\ell$ in Lemma 2.7, we have

$$\begin{aligned}&\Delta_{q(\ell,\ell)}\left[\frac{k+n\ell}{k+(n-1)\ell}\right]\\&=\left[\frac{(((k+\ell)\,q+\ell)\,q+n\ell)}{(((k+\ell)\,q+\ell)\,q+(n-1)\ell)}-2\frac{((k+\ell)\,q+n\ell)}{((k+\ell)\,q+(n-1)\ell)}+\frac{k+n\ell}{k+(n-1)\ell}\right]\end{aligned} \tag{5.8}$$

Apply (5.1) in (5.8), we get

$$\begin{aligned}&\Delta^{-1}_{q(\ell,\ell)}\left[\frac{(((k+\ell)\,q+\ell)\,q+n\ell)}{(((k+\ell)\,q+\ell)\,q+(n-1)\ell)}-2\frac{((k+\ell)\,q+n\ell)}{((k+\ell)\,q+(n-1)\ell)}+\frac{k+n\ell}{k+(n-1)\ell}\right]\\&=\left[\frac{k+n\ell}{k+(n-1)\ell}\right]+c_j.\end{aligned}$$

The proof follows from (5.2) and (5.8). □

Theorem 5.5. *If $k\in[0,\infty)$ and $n\in N(1)$, then*

$$\begin{aligned}&\sum_{r=2}^{j^\star+1}(r-1)\left[\frac{\left(\left(\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+\ell\right)q+\ell\right)q+n\ell\right)}{\left(\left(\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+\ell\right)q+\ell\right)q+(n+1)\ell\right)}-2\frac{\left(\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+\ell\right)q+n\ell\right)}{\left(\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+\ell\right)q+(n+1)\ell\right)}\right.\\&\qquad\left.+\frac{\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+n\ell\right)}{\left(\frac{k-\ell\sum_{t=1}^{r}q^t}{q^r}+(n+1)\ell\right)}\right]\\&+\sum_{r=1}^{j^*-1}(j^\star-r)\left[\frac{\left(\left(\left(\frac{k-\ell\sum_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+\ell\right)q+n\ell\right)}{\left(\left(\left(\frac{k-\ell\sum_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+\ell\right)q+(n+1)\ell\right)}-2\frac{\left(\left(\frac{k-\ell\sum_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+n\ell\right)}{\left(\left(\frac{k-\ell\sum_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+\ell\right)q+(n+1)\ell\right)}\right.\end{aligned}$$

$$+\frac{\left(\frac{k-\ell\sum_{t=1}^{j^\star+r+1}q^t}{q^{j^\star+r+1}}+n\ell\right)}{\left(\frac{k-\ell\sum_{t=1}^{j^*+r+1}q^t}{q^{j^*+r+1}}+(n+1)\ell\right)}\Bigg]$$

$$=\left[\frac{k+n\ell}{k+(n+1)\ell}\right]-2\left[\frac{\left(\frac{k-\ell\sum_{t=1}^{j^\star}q^t}{q^{j^\star}}\right)+n\ell}{\left(\frac{k-\ell\sum_{t=1}^{j^\star}q^t}{q^{j^\star}}\right)+(n+1)\ell)}\right]+\left[\frac{\left(\frac{k-\ell\sum_{t=1}^{2j^\star}q^t}{q^{2j^\star}}\right)+n\ell}{\left(\frac{k-\ell\sum_{t=1}^{2j^\star}q^t}{q^{2j^\star}}\right)+(n+1)\ell}\right]. \tag{5.9}$$

Proof. Taking $u(k)=k+n\ell$ and $v(k)=k+(n+1)\,\ell$ in Lemma 2.7, we have

$$\begin{aligned}&\Delta_{q(\ell,\ell)}\left[\frac{k+n\ell}{k+(n+1)\ell}\right]\\&=\left[\frac{(((k+\ell)\,q+\ell)\,q+n\ell)}{(((k+\ell)\,q+\ell)\,q+(n+1)\ell)}-2\frac{((k+\ell)\,q+n\ell)}{((k+\ell)\,q+(n-1)\ell)}+\frac{k+n\ell}{k+(n+1)\ell}\right]\end{aligned} \tag{5.10}$$

Equation (5.1) using in (5.10), we have

$$\begin{aligned}&\Delta^{-1}_{q(\ell,\ell)}\left[\frac{(((k+\ell)\,q+\ell)\,q+n\ell)}{(((k+\ell)\,q+\ell)\,q+(n+1)\ell)}-2\frac{((k+\ell)\,q+n\ell)}{((k+\ell)\,q+(n-1)\ell)}+\frac{k+n\ell}{k+(n+1)\ell}\right]\\&=\left[\frac{k+n\ell}{k+(n+1)\ell}\right]+c_j.\end{aligned}$$

Equation (5.9) follows from (5.2) and (5.10). □

6. Numerical Evaluation of Results

In this section, we obtain several fractional series using the inverse of extended q-difference operators, along with appropriate examples.

The following example is an illustration of Theorem 5.2.

Example 6.1. *Consider the series*

$$\begin{aligned}S=&1\left(\frac{85}{2^2}\right)+2\left(\frac{85}{2^3}\right)+3\left(\frac{85}{2^4}\right)+\cdots+26\left(\frac{85}{2^{27}}\right)\\&+25\left(\frac{85}{2^{28}}\right)+24\left(\frac{85}{2^{29}}\right)+23\left(\frac{85}{2^{30}}\right)+\cdots+1\left(\frac{85}{2^{52}}\right)\end{aligned}$$

In (5.3), by taking $k = 79$ and $\ell = 3$, we get

$$\left[\sum_{r=2}^{27}(r-1)\left(\frac{79-3\sum\limits_{t=1}^{r}q^t}{q^r}+\frac{3q}{q-1}\right)+\sum_{r=1}^{25}(26-r)\left(\frac{79-3\sum\limits_{t=1}^{27+r}q^t}{q^{27+r}}+\frac{3q}{q-1}\right)\right]$$
$$=\left(\frac{79}{(q-1)^2}\right)-2\left(\frac{79-3\sum\limits_{t=1}^{26}q^t}{(q-1)^2q^{26}}\right)+\left(\frac{79-3\sum\limits_{t=1}^{2(26)}q^t}{(q-1)^2q^{2(26)}}\right). \tag{6.1}$$

In particular, when $q = 2$ we have

$$\begin{aligned} S &= 79-2\left(\frac{79-3\sum\limits_{t=1}^{26}2^t}{2^{26}}\right)+\left(\frac{79-3\sum\limits_{t=1}^{52}2^t}{2^{52}}\right) \\ &= 79+11.999997466802597-5.999999999999981 \\ &= 84.999997466802616. \end{aligned}$$

The following example illustrates Theorem 5.3.

Example 6.2. *Consider the series*

$$\begin{aligned} S = {} & 1\left(\frac{7081}{74725}\right)-2\left(\frac{17666}{28175}\right)+3\left(\frac{49348}{68425}\right)-\cdots-10\left(\frac{462004736}{1817075464295}\right)-9\left(\frac{1842562048}{14689548132455}\right) \\ & -8\left(\frac{7359334400}{118130904838247}\right)-7\left(\frac{29415510016}{947510810202215}\right)-\cdots-1\left(\frac{120397920206848}{249020931804471365735}\right) \end{aligned}$$

On substituting $k = 61$, $\ell = 6$ and $q = 2$ in (5.5), we get

$$\begin{aligned} & \sum_{r=2}^{11}(r-1)\left[\left(\left(\left(\frac{61-6\sum\limits_{t=1}^{r}2^t}{2^r}+6\right)2+6\right)2\right)^{-1}-2\left(\left(\frac{61-6\sum\limits_{t=1}^{r}2^t}{2^r}+6\right)2\right)^{-1}+\left(\frac{61-6\sum\limits_{t=1}^{r}2^t}{2^r}\right)^{-1}\right] \\ & +\sum_{r=1}^{9}(10-r)\left[\left(\left(\left(\frac{61-6\sum\limits_{t=1}^{11+r}2^t}{2^{11+r}}+6\right)2+6\right)2\right)^{-1}-2\left(\left(\frac{61-6\sum\limits_{t=1}^{11+r}2^t}{2^{11+r}}+6\right)2\right)^{-1}+\left(\frac{61-6\sum\limits_{t=1}^{11+r}2^t}{2^{11+r}}\right)^{-1}\right] \end{aligned}$$

$$S = \frac{1}{61} - 2\left(\frac{2^{10}}{61 - 6\sum\limits_{t=1}^{10} 2^t}\right) + \left(\frac{2^{20}}{61 - 6\sum\limits_{t=1}^{20} 2^t}\right)$$

$$= 0.01639344262295082 + 0.16766270978305362 - 0.08333381679603466$$

$$= 0.10072233560996978.$$

This example serves as an illustration of Theorem 5.4.

Example 6.3. *Consider the fractional series*

$$F = 1\left(\frac{3696}{7163}\right) - 2\left(\frac{1464}{13949}\right) - 3\left(\frac{636}{43993}\right) - \cdots - 19\left(\frac{4529860608}{50096022907760639}\right)$$
$$- 18\left(\frac{18119417856}{400770085790638079}\right) - 17\left(\frac{72477622272}{3206168296454799359}\right) - 16\left(\frac{289910390784}{25649376812188139519}\right)$$
$$- \cdots - 1\left(\frac{311288806247069908992}{902458288606356895746195676200959}\right).$$

Here $n = -12$, $k = 58$, $\ell = 3$, and $q = 2$ can be substituted in (5.7) to get obtain

$$\sum_{r=2}^{20} (r-1)\left[\frac{\left(\left(\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} + 3\right)2 + 3\right)2 - 36\right)}{\left(\left(\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} + 3\right)2 + 3\right)2 - 39\right)} - 2\frac{\left(\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} + 3\right)2 - 36\right)}{\left(\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} + 3\right)2 - 39\right)} + \frac{\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} - 36\right)}{\left(\frac{58-3\sum\limits_{t=1}^{r} 2^t}{2^r} - 39\right)}\right]$$

$$+ \sum_{r=1}^{18} (19-r)\left[\frac{\left(\left(\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} + 3\right)2 + 3\right)2 - 36\right)}{\left(\left(\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} + 3\right)2 + 3\right)2 - 39\right)} - 2\frac{\left(\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} + 3\right)2 - 36\right)}{\left(\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} + 3\right)2 - 39\right)} + \frac{\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} - 36\right)}{\left(\frac{58-3\sum\limits_{t=1}^{20+r} 2^t}{2^{20+r}} - 39\right)}\right]$$

$$F = \left[\frac{22}{19}\right] - 2\left[\frac{\left(\frac{58-3\sum\limits_{t=1}^{19} 2^t}{2^{19}}\right) - 36}{\left(\frac{58-3\sum\limits_{t=1}^{19} 2^t}{2^{19}}\right) - 39}\right] + \left[\frac{\left(\frac{58-3\sum\limits_{t=1}^{38} 2^t}{2^{38}}\right) - 36}{\left(\frac{58-3\sum\limits_{t=1}^{38} 2^2}{2^{38}}\right) - 39}\right]$$

$$= 1.1578947368421053 - 1.8666663049758707 + 0.9333333333329884$$

$$= 0.224561765199223.$$

The subsequent example clarifies Theorem 5.5.

Example 6.4. *Consider the fractional series*

$$\begin{aligned}F &= 1\left(\frac{175152}{20924365}\right)+2\left(\frac{2544624}{244017235}\right)+3\left(\frac{25806384}{4532162005}\right)+\cdots\\&\quad+12\left(\frac{10560576640823280}{2779680185412620837178 7}\right)+11\left(\frac{95045361291055152}{7505077809838689262526 05}\right)\\&\quad+10\left(\frac{855408766190433264}{20263657264993711846847395}\right)+9\left(\frac{7698680439426710064}{547118270761074774586047685}\right)\\&\quad+\cdots+1\left(\frac{3314029821649821711035151 84}{1545222924889698643654681100606969048 05}\right).\end{aligned}$$

Take $n=10$, $k=73$, $\ell=6$, and $q=3$, in (5.9), we get

$$\sum_{r=2}^{13}(r-1)\left[\frac{\left(\left(\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+6\right)3+6\right)3+60\right)}{\left(\left(\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+6\right)3+6\right)3+66\right)}-2\frac{\left(\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+6\right)3+60\right)}{\left(\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+6\right)3+66\right)}+\frac{\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+60\right)}{\left(\frac{73-6\sum_{t=1}^{r}3^t}{3^r}+66\right)}\right]$$

$$+\sum_{r=1}^{11}(12-r)\left[\frac{\left(\left(\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+6\right)3+6\right)3+60\right)}{\left(\left(\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+6\right)3+6\right)3+66\right)}-2\frac{\left(\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+6\right)3+60\right)}{\left(\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+6\right)3+66\right)}+\frac{\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+60\right)}{\left(\frac{73-6\sum_{t=1}^{13+r}3^t}{3^{13+r}}+66\right)}\right]$$

$$\begin{aligned}F &= \left[\frac{133}{139}\right]-2\left[\frac{\left(\frac{73-6\sum_{t=1}^{12}3^t}{3^{12}}\right)+60}{\left(\frac{73-6\sum_{t=1}^{12}3^t}{3^{12}}\right)+66}\right]+\left[\frac{\left(\frac{73-6\sum_{t=1}^{24}3^t}{3^{24}}\right)+60}{\left(\frac{73-6\sum_{t=1}^{24}3^t}{3^{24}}\right)+66}\right]\\&= 0.9568345323741008-1.7894742540980573+0.8947368421057993\\&= 0.062097120381842785.\end{aligned}$$

7. Conclusion

This paper develops an advancement in Numerical Analysis regarding some conclusions on the numerical solutions of extended q-difference equations governed by (5.2) in conjunction with the function $u(k)$ examination. Additionally, the theorem mentioned in this paper can be used to compute the sum of many fractional series quickly by choosing a big value for k and a tiny positive value for q and ℓ.

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