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Optimization of the Approximate Integration Formula Using the Quadrature Formula

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Abstract. The article explores Sard's problem of constructing optimal quadrature formulas in the space $W_2^{(4,0)}(0,1)$ by using Sobolev's method. This problem involves two steps: first, calculating the norm of the error functional, and then finding the minimum of this norm by the coefficients of the quadrature formulas. The norm of the error functional is computed using the extremal function. Then, by using the method of Lagrange multipliers, a system of linear equations for the coefficients of the optimal quadrature formulas in the space $W_2^{(4,0)}(0,1)$ is derived, and the existence and uniqueness of the solution to this system are discussed. The paper then proceeds to construct the optimal quadrature formula using the discrete analogue $D_4(h\beta)$ of the high-order differential operator. Finally, the optimal quadrature formulas that are exact for exponential-trigonometric functions are obtained.

1. Introduction and problem statement

Quadrature formulas are widely used in various sections of mathematics and its applications. In obtaining a discrete approximation, a crucial requirement is that the quadrature formula closely approximates the given definite integrals. Such formulas can be obtained using variational principles. Therefore, constructing lattice optimal quadrature formulas on classes of differentiable functions using the variational method is one of the pressing problems of computational mathematics. The problem of optimizing numerical integration formulas in the variational approach is the problem of finding the minimum of the norm of the error functional on a given space of functions. The variational approach to optimizing numerical integration formulas involves finding the

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minimum error functional norm within a given function space. The problem introduced by S.M. Nikol'skii [13], [14] involves minimizing the error functional's norm using coefficients and nodes. On the other hand, the problem by A. Sard [12], [15], [16] focuses on minimizing the norm of the error functional by modifying coefficients while keeping the nodes fixed. The solutions of Nikolskii's and Sard's problems are called the optimal quadrature formula in the sense of Nikol'skii and the sense of Sard, respectively.

In this paper, we study Sard's problem of constructing optimal quadrature formulas in Hilbert space.

We indicate $W_2^{(4,0)}(0,1)$ the class of functions φ defined on the $[0,1]$, which possesses a continuous third derivative on $[0,1]$ and fourth derivative is in $L_2(0,1)$.

The class $W_2^{(4,0)}(0,1)$ under the pseudo-inner product

$$\langle\varphi,\psi\rangle_{W_2^{(4,0)}}=\int_0^1\left(\varphi^{(4)}(x)+\varphi(x)\right)\left(\psi^{(4)}(x)+\psi(x)\right)dx \tag{1.1}$$

is a Hilbert space if we can find functions that are different from the equation's solution $f^{(4)}(x)+f(x)=0$ (see, [1]). Thus, $W_2^{(4,0)}$ is the Hilbert space equipped with the norm

$$\|\varphi\|_{W_2^{(4,0)}}=\left(\int_0^1\left(\varphi^{(4)}(x)+\varphi(x)\right)^2dx\right)^{1/2} \tag{1.2}$$

corresponding to the inner product (1.1).

For a function φ from the space $W_2^{(4,0)}$ we consider a quadrature formula of the form

$$\int_0^1\varphi(x)dx\cong\sum_{\beta=0}^{N}C_\beta\varphi\left(x_\beta\right), \tag{1.3}$$

where C_β and x_β are called the coefficients and nodes of formula (1.1), φ is an element of the Hilbert space $W_2^{(4,0)}(0,1)$.

The following difference between integral and quadrature sum

$$(\ell,\varphi)=\int_0^1\varphi(x)dx-\sum_{\beta=0}^{N}C_\beta\varphi\left(x_\beta\right) \tag{1.4}$$

is termed the error of the quadrature formula (1.3) and $(\ell,\varphi)=\int_{-\infty}^{\infty}\ell(x)\varphi(x)dx$. This difference corresponds to the error functional ℓ, which has the form

$$\ell(x)=\varepsilon_{[0,1]}(x)-\sum_{\beta=0}^{N}C_\beta\delta(x-x_\beta), \tag{1.5}$$

here $\varepsilon_{[0,1]}$ is the characteristic function of the interval $[0,1]$, δ is Dirac's delta-function.

According to the Cauchy-Schwarz inequality, the absolute value of the error (1.4) is appraised by the norm

$$||\ell||_{W_2^{(4,0)*}} = \sup_{||\varphi||_{W_2^{(4,0)}}=1} \left|(\ell,\varphi)\right| \tag{1.6}$$

of the error functional ℓ as follows

$$\left|(\ell,\varphi)\right| \le \left\|\varphi\right\|_{W_2^{(4,0)}} ||\ell||_{W_2^{(4,0)*}}$$

where $W_2^{(4,0)*}$ is the conjugate space to the space $W_2^{(4,0)}$.

Sard's problem on the construction of optimal quadrature formulas in the space $W_2^{(4,0)}(0,1)$ which satisfy the equality

$$\left\|\overset{\circ}{\ell}\right\|_{W_2^{(4,0)*}} = \inf_{C_\beta} ||\ell||_{W_2^{(4,0)*}}, \tag{1.7}$$

i.e., to find the minimum of the norm (1.6) of the error functional ℓ by coefficients C_β for fixed nodes x_β.

This problem consists of two parts: first calculating the norm (1.6) of the error functional ℓ in the space $W_2^{(4,0)*}$ and then finding the minimum of the norm (1.6) by coefficients C_β for fixed nodes x_β.

There are several methods for constructing optimal quadrature formulas in the sense of Sard, for example, the spline method [9] [17], [20], [21], [22], [30], the φ-function method (see, for example, [2], [8], [10], [11], [18], [19]) and the Sobolev method [26], [27], [28], [29]. In different spaces, based on these methods, Sard's problem has been studied by many authors (see, for example, [1], [5], [6], [7], [23], [24], [25] and references therein).

One of the main goals of this paper is to study Sard's problem of constructing optimal quadrature formulas of the form (1.3) in the space $W_2^{(4,0)}$ by the Sobolev method. As a result, we obtain an optimal quadrature formula that is exact for the basis functions of the norm kernel (1.2). Here, the basis functions consist of exponential-trigonometric functions.

The paper is organized as follows: Section 2 presents the extremal function, which corresponds to the error functional ℓ. Section 3 calculates the norm of the error functional using this extremal function, thereby solving the first part of Sard's problem. In Section 4, the system of linear equations for the coefficients of optimal quadrature formulas in the space $W_2^{(4,0)}$ is discussed, along with the existence and uniqueness of the solution to this system. Section 5 details the obtained optimal quadrature formula exact on exponential-trigonometric functions.

2. Extremal function of the error functional

To find the norm of the error functional (1.5) of the quadrature formula (1.3), we will use the extremal function of this functional

The function ψ_ℓ satisfying the equation

$$(\ell,\psi_\ell) = ||\ell||_{W_2^{(4,0)*}} \cdot \left\|\psi_\ell\right\|_{W_2^{(4,0)}} \tag{2.1}$$

is called the **extremal function** for the functional ℓ [26], [27], [28], [29].

Since $W_2^{(4,0)}$ is the Hilbert space then, by the Riesz theorem on the general form of a linear continuous functional on Hilbert spaces, for the error functional $\ell \in W_2^{(4,0)*}$ there exists a unique function $\psi_\ell \in W_2^{(4,0)}$ such that for any $\varphi \in W_2^{(4,0)}$ following equality is fulfilled

$$(\ell, \varphi) = < \psi_\ell, \varphi >_{W_2^{(4,0)}} \tag{2.2}$$

and $\|\ell\|_{W_2^{(4,0)*}} = \left\|\psi_\ell\right\|_{W_2^{(4,0)}}$, where $< \psi_\ell, \varphi >_{W_2^{(4,0)}}$ is the inner product of two functions ψ_ℓ and φ from $W_2^{(4,0)}$ space. Recall that the inner product $< \psi_\ell, \varphi >_{W_2^{(4,0)}}$ is defined by (1.1). In particular, from (2.2) when $\varphi = \psi_\ell$ we have

$$(\ell, \psi_\ell) = < \psi_\ell, \psi_\ell >_{W_2^{(4,0)}}$$

$$= \left\|\psi_\ell\right\|^2_{W_2^{(4,0)}} = \left\|\psi_\ell\right\|_{W_2^{(4,0)}} \cdot \|\ell\|_{W_2^{(4,0)*}} = \|\ell\|^2_{W_2^{(4,0)*}}$$

From this, it is clear that the solution ψ_ℓ of equation (2.2) satisfies equation (2.1) and is an extremal function. Thus, to calculate the norm of the error functional ℓ, we must first find the extremal function ψ_ℓ from equation (2.2), and then calculate the square of the norm of the error functional ℓ as follows

$$\|\ell\|^2_{W_2^{(4,0)*}} = (\ell, \psi_\ell)\,. \tag{2.3}$$

Let us solve equation (2.2). Integrating by parts the right side of equation (2.2), we have

$$(\ell, \varphi) = \int_0^1 \left(\psi_\ell^{(8)}(x) + 2\psi_\ell^4(x) + \psi_\ell(x)\right)\varphi(x)\,dx$$

$$+ \sum_{s=0}^{3} (-1)^s \left(\psi_\ell^{(4+s)}(x) + \psi_\ell^{(s)}(x)\right)\varphi^{(3-s)}(x)\Big|_0^1 . \tag{2.4}$$

From (2.4), taking into account the uniqueness of the function ψ_ℓ, we have the following equation

$$\psi_\ell^{(8)}(x) + 2\psi_\ell^{(4)}(x) + \psi_\ell(x) = \ell(x) \tag{2.5}$$

with the boundary conditions

$$\left[\left(\psi_\ell^{(4+s)}(x) + \psi_\ell^{(s)}(x)\right)\right]\Big|_{x=0}^{x=1} = 0, \quad s = \overline{0,3}. \tag{2.6}$$

It should be noted that in work [3], the following result was obtained for solving the boundary value problem (2.5) - (2.6).

Theorem 2.1. *The solution of equation* (2.5) *with boundary conditions* (2.6) *is an extremal function* ψ_ℓ *of the error functional* ℓ *of the quadrature formula* (1.3) *and has the form*

$$\psi_\ell(x) = \ell(x) * G_4(x) + Y_4(x)$$

, where $G_4(x)$*is Green's function, i.e., the fundamental solution of the equation*

$$G_4^{(8)}(x) + 2G_4^{(4)}(x) + G_4(x) = \delta(x) \tag{2.7}$$

and is expressed as follows

$$G_4(x) = \frac{sign\,(x)}{8}\cdot\left[\frac{3\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}x\right)ch\left(\frac{\sqrt{2}}{2}x\right)-\frac{3\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}x\right)sh\left(\frac{\sqrt{2}}{2}x\right)\right.$$

$$\left.-x\sin\left(\frac{\sqrt{2}}{2}x\right)sh\left(\frac{\sqrt{2}}{2}x\right)\right], \tag{2.8}$$

$$Y_4(x) = r_1e^{\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)+r_2e^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)+r_3e^{-\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)+r_4e^{-\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right), \tag{2.9}$$

r_1, r_2, r_3 *and* r_4 *are constants.*

Since the error functional ℓ is defined in the space $W_2^{(4,0)}$, the following conditions must be satisfied

$$\left(\ell, \mathrm{e}^{\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)\right)=0, \tag{2.10}$$

$$\left(\ell, \mathrm{e}^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)\right)=0, \tag{2.11}$$

$$\left(\ell, \mathrm{e}^{-\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)\right)=0, \tag{2.12}$$

$$\left(\ell, \mathrm{e}^{-\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)\right)=0. \tag{2.13}$$

This means that our quadrature formula will be exact on functions:

$$\mathrm{e}^{\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right),\quad \mathrm{e}^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right),\quad \mathrm{e}^{-\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right),\quad \mathrm{e}^{-\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right).$$

3. The norm of the error functional

The square of the error norm for the functional (1.5) can be calculated by determining the value (ℓ,ψ_ℓ) of the error functional ℓ on the function ψ_ℓ. First, we need to use equalities (2.10) - (2.13) to derive this value. To do this, first, using equalities (2.10) - (2.13), we obtain

$$(\ell, Y_4\,(x))=0,$$

where $Y_4\,(x)$ is the function defined by (2.9). Then, using (2.9) we have

$$\|\ell\|^2=(\ell,\psi_\ell)=\int\limits_{-\infty}^{\infty}\ell\,(x)\,[\ell\,(x)*G_4\,(x)+Y_4\,(x)]\,dx=\int\limits_{-\infty}^{\infty}\ell\,(x)\,[\ell\,(x)*G_4\,(x)]dx, \tag{3.1}$$

where $G_4\,(x)$ function defined by (2.8).

Now for the convolution in (3.1) taking into account (1.5) we obtain

$$\ell\,(x)*G_4\,(x)=\int\limits_{-\infty}^{\infty}\ell\,(y)\,G_4\,(x-y)\,dy=\int\limits_{0}^{1}G_4\,(x-y)\,dy-\sum_{\beta=0}^{N}C_\beta G_4\left(x-x_\beta\right).$$

Then the square of the norm of the error functional ℓ is reduced to the form

$$||\ell||^2 = \sum_{\beta=0}^{N} C_\beta \left(\int_0^1 G_4(x-x_\beta) + G_4(x_\beta - x) \right) dx - \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G_4(x_\beta - x_\gamma) - \int_0^1 \int_0^1 G_4(x-y)dxdy. \tag{3.2}$$

Since $G_4\,(x)$ is an even function, i.e.

$$G_4(x_\beta - x) = G_4(x - x_\beta)$$

then, taking into account the last equality, from (3.2) we obtain

$$\left\| \ell |\, W_2^{(4,0)*} \right\|^2 = \sum_{\beta=0}^{N}\sum_{\gamma=0}^{N} C_\beta C_\gamma G_4(x_\beta - x_\gamma) + \int_0^1 \int_0^1 G_4(x-y)dxdy - 2\sum_{\beta=0}^{N} C_\beta \int_0^1 G_4(x-x_\beta)dx\,. \tag{3.3}$$

Thus, the first part of Sard's problem on the construction of optimal quadrature formulas in the space $W_2^{(4,0)}$ is solved. Next, we consider the second part of the problem.

4. Wiener-Hopf type system for finding optimal coefficients

Let's minimise the square of the error functional's norm (3.3). It is known that the error functional ℓ meets conditions (2.10) - (2.13). The norm's square (3.3) of the error functional ℓ depends on many variable coefficients $C_\beta\ (\beta = \overline{0,N})$ from the quadrature formula (1.3).

To find the conditional minimum point of the square of the norm of the error functional (1.5) under conditions (2.10) - (2.13), we use the method of undetermined Lagrange multipliers. Let us denote $\mathbb{C} = (C_0, C_1, ..., C_N)$ and $r = (r_1, r_2, r_3, r_4)$.

Let's consider the following function

$$\Psi(\mathbb{C}, r) = ||\ell||^2 - 2r_1\left(\ell, e^{\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)\right) - 2r_2\left(\ell, e^{\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)\right)$$
$$-2r_3\left(\ell, e^{-\frac{\sqrt{2}}{2}x}\cos\left(\frac{\sqrt{2}}{2}x\right)\right) - 2r_4\left(\ell, e^{-\frac{\sqrt{2}}{2}x}\sin\left(\frac{\sqrt{2}}{2}x\right)\right).$$

Equating to zero the partial derivatives of the function $\Psi(\mathbb{C}, r)$ by coefficients $C_\beta(\beta = \overline{0,N})$ and r_1, r_2, r_3 and r_4, we obtain the following system of linear equations

$$\sum_{\gamma=0}^{N} C_\gamma G_4(x_\beta - x_\gamma) + Y_4(x_\beta) = f_4(x_\beta), \;\; \beta = 0, 1, \ldots, N, \tag{4.1}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{\frac{\sqrt{2}}{2}x_\gamma}\cos\left(\frac{\sqrt{2}}{2}x_\gamma\right) = \frac{\sqrt{2}}{2}e^{\frac{\sqrt{2}}{2}}\left(\sin\frac{\sqrt{2}}{2} + \cos\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}, \tag{4.2}$$

$$\sum_{\gamma=0}^{N} C_\gamma e^{\frac{\sqrt{2}}{2}x_\gamma}\sin\left(\frac{\sqrt{2}}{2}x_\gamma\right) = \frac{\sqrt{2}}{2}e^{\frac{\sqrt{2}}{2}}\left(\sin\frac{\sqrt{2}}{2} - \cos\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}, \tag{4.3}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{-\frac{\sqrt{2}}{2}x_{\gamma}} \cos\left(\frac{\sqrt{2}}{2}x_{\gamma}\right) = \frac{\sqrt{2}}{2} e^{-\frac{\sqrt{2}}{2}} \left(\sin\frac{\sqrt{2}}{2} - \cos\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}, \tag{4.4}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{-\frac{\sqrt{2}}{2}x_{\gamma}} \sin\left(\frac{\sqrt{2}}{2}x_{\gamma}\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} e^{-\frac{\sqrt{2}}{2}} \left(\sin\frac{\sqrt{2}}{2} + \cos\frac{\sqrt{2}}{2}\right), \tag{4.5}$$

where

$$f_4\left(x_\beta\right) = 1 - \frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}x_\beta\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}x_\beta\right) - \frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right)$$
$$-\frac{\sqrt{2}\left(1-x_\beta\right)}{16}\sin\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right) + \frac{\sqrt{2}\left(1-x_\beta\right)}{16}\cos\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}\left(1-x_\beta\right)\right)$$
$$+\frac{\sqrt{2}x_\beta}{16}\sin\left(\frac{\sqrt{2}}{2}x_\beta\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}x_\beta\right) - \frac{\sqrt{2}x_\beta}{16}\cos\left(\frac{\sqrt{2}}{2}x_\beta\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}x_\beta\right), \tag{4.6}$$

$G_4(x)$ and $Y_4(x)$ defined by Theorem 1.

We notice that the system (4.1) - (4.5) is called the **discrete system of Wiener-Hopf** type [26], [27], [28], [29].

It is worth noting that the study in [26], [27], [28], [29] investigated the existence and uniqueness of an optimal quadrature formula of the form (1.3) in the sense of Sard in Hilbert spaces. The difference system (4.1) - (4.5) has a unique solution for any set of different nodes x_β, $\beta = 0, 1, ..., N$, when $N \geq 3$. This solution provides the minimum $\|\ell\|^2$ as determined by equality (3.3) under conditions (2.10) - (2.13). The solution for this type of difference systems' existence and uniqueness was also studied in [26], [27], [28], [29].

Further, we consider the case of equally spaced nodes. Suppose $x_\beta = h\beta$, $\beta = 0, 1, ...N$, $h = \frac{1}{N}$, $N = 1, 2, ...$

We suppose that $C_\beta = 0$ for $\beta < 0$ and $\beta > N$. Then, using the convolution of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ (see, [26], [29])

$$\varphi(h\beta) * \psi(h\beta) = \sum_{\gamma=-\infty}^{\infty} \varphi(h\gamma)\cdot\psi(h\beta - h\gamma),$$

we will rewrite the system (4.1) - (4.5) in the following convolution form:

$$C_\beta * G_4(h\beta) + Y_4(h\beta) = f_4(h\beta), \quad \beta = \overline{0, N}, \tag{4.7}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{\frac{\sqrt{2}}{2}x_{\gamma}} \cos\left(\frac{\sqrt{2}}{2}x_{\gamma}\right) = \frac{\sqrt{2}}{2} e^{\frac{\sqrt{2}}{2}} \left(\sin\frac{\sqrt{2}}{2} + \cos\frac{\sqrt{2}}{2}\right) - \frac{\sqrt{2}}{2}, \tag{4.8}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{\frac{\sqrt{2}}{2}x_{\gamma}} \sin\left(\frac{\sqrt{2}}{2}x_{\gamma}\right) = \frac{\sqrt{2}}{2} e^{\frac{\sqrt{2}}{2}} \left(\sin\frac{\sqrt{2}}{2} - \cos\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}, \tag{4.9}$$

$$\sum_{\gamma=0}^{N} C_{\gamma} e^{-\frac{\sqrt{2}}{2}x_{\gamma}} \cos\left(\frac{\sqrt{2}}{2}x_{\gamma}\right) = \frac{\sqrt{2}}{2} e^{-\frac{\sqrt{2}}{2}} \left(\sin\frac{\sqrt{2}}{2} - \cos\frac{\sqrt{2}}{2}\right) + \frac{\sqrt{2}}{2}, \tag{4.10}$$

$$\sum_{\gamma=0}^{N} C_\gamma \mathrm{e}^{-\frac{\sqrt{2}}{2}x_\gamma} \sin\left(\frac{\sqrt{2}}{2}x_\gamma\right) = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\mathrm{e}^{-\frac{\sqrt{2}}{2}}\left(\sin\frac{\sqrt{2}}{2} + \cos\frac{\sqrt{2}}{2}\right), \tag{4.11}$$

where $G_4(h\beta)$, $Y_4(h\beta)$ and $f_4(h\beta)$ are defined by (2.8), (2.9) and (4.6), respectively. There are $N+5$ unknowns C_β $(\beta = 0, 1, ..., N)$, r_1, r_2, r_3, r_4 and $N+5$ linear equations in the system (4.7) - (4.11).

By using the Sobolev method to solve system (4.7) - (4.11), we require $D_4(h\beta)$ a discrete analogue of the differential operator $\frac{\mathrm{d}^8}{\mathrm{d}x^8} + 2\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 1$, which should satisfy the equality

$$D_4(h\beta) * G_4(h\beta) = \delta_d(h\beta), \tag{4.12}$$

where $G_4(h\beta)$ is determined by formula (2.8) and

$$\delta_d(h\beta) = \begin{cases} 1, & \beta = 0, \\ 0, & \beta \neq 0, \end{cases} \quad h = \frac{1}{N}, \quad N = 1, 2, ...$$

In this regard, in the work [4] a discrete analogue $D_4(h\beta)$ of the differential operator $\frac{\mathrm{d}^8}{\mathrm{d}x^8} + 2\frac{\mathrm{d}^4}{\mathrm{d}x^4} + 1$ satisfying (4.12) is constructed and some of its properties are investigated.

Further from [4] we have:

Theorem 4.1. *The discrete analogue of $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8} + 2\frac{d^4}{dx^4} + 1$ has the form:*

$$D_4(h\beta) = \frac{8}{K} \cdot \begin{cases} \sum\limits_{k=1}^{3} A_k \cdot \lambda_k^{|\beta|-1}, & |\beta| \geq 2, \\ 1 + \sum\limits_{k=1}^{3} A_k, & |\beta| = 1, \\ F_1 + \sum\limits_{k=1}^{3} \frac{A_k}{\lambda_k}, & \beta = 0, \end{cases} \tag{4.13}$$

where K, F_1, A_k $(k = 1, 2, 3)$ known and h is small parameter, λ_k $(|\lambda_k| < 1,\ k = 1, 2, 3)$ are roots of the polynomial $P_6(\lambda)$, which are given in the work [4].

Theorem 4.2. *The discrete analogue $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8} + 2\frac{d^4}{dx^4} + 1$ satisfies the following equalities*

$$D_4[\beta] * e^{\frac{\sqrt{2}}{2}[\beta]} \cos\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0, \quad D_4[\beta] * e^{\frac{\sqrt{2}}{2}[\beta]} \sin\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0,$$

$$D_4[\beta] * e^{-\frac{\sqrt{2}}{2}[\beta]} \cos\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0, \quad D_4[\beta] * e^{-\frac{\sqrt{2}}{2}[\beta]} \sin\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0,$$

$$D_4[\beta] * [\beta]\, e^{\frac{\sqrt{2}}{2}[\beta]} \cos\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0, \quad D_4[\beta] * [\beta]\, e^{\frac{\sqrt{2}}{2}[\beta]} \sin\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0,$$

$$D_4[\beta] * [\beta]\, e^{-\frac{\sqrt{2}}{2}[\beta]} \cos\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0, \quad D_4[\beta] * [\beta]\, e^{-\frac{\sqrt{2}}{2}[\beta]} \sin\left(\frac{\sqrt{2}}{2}[\beta]\right) = 0.$$

Here $[\beta] = h\beta$.

5. Solution of the discrete Wiener-Hopf system

In this section, we present an algorithm for determining the exact solution of system (4.7) - (4.11) using the discrete analogue $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8}+2\frac{d^4}{dx^4}+1$.

We introduce the following functions

$$\vartheta(h\beta)=C_\beta * G_4(h\beta) \tag{5.1}$$

and

$$u(h\beta)=\vartheta(h\beta)+Y_4(h\beta) \tag{5.2}$$

Then, considering equation (4.12), we can determine the optimal coefficients C_β as follows:

$$C_\beta=D_4(h\beta)*u(h\beta)\,. \tag{5.3}$$

If we find the function $u(h\beta)$, then the optimal coefficients are determined using the formula (5.3). To calculate the convolution (5.3), we need to find the representation of the function $u(h\beta)$ for all integer values of the variable β. From equation (4.7), it is evident that $u(h\beta)=f_4(h\beta)$ corresponds to $h\beta\in[0,1]$.

Now we need to find the representation of the function $u(h\beta)$ for $\beta<0$ and $\beta>N$. Using the formula (2.8), we calculate the convolution

$$\vartheta(h\beta)=C_\beta * G_4(h\beta)$$

for $h\beta\notin[0,1]$.

Let $\beta<0$, then taking into account equalities (4.8) - (4.11) we have

$$\begin{aligned}\vartheta(h\beta)=G_4(h\beta)*C_\beta=-\Bigg\{&b_1e^{\frac{\sqrt{2}}{2}h\beta}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)+b_2e^{\frac{\sqrt{2}}{2}h\beta}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)\\&+b_3e^{-\frac{\sqrt{2}}{2}h\beta}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)+b_4e^{-\frac{\sqrt{2}}{2}h\beta}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)+\frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}h\beta\right)\\&-\frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}h\beta\right)+\frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\\&-\frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\Bigg\}\end{aligned} \tag{5.4}$$

and for $\beta>N$

$$\begin{aligned}\vartheta(h\beta)=\,&b_1e^{\frac{\sqrt{2}}{2}h\beta}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)+b_2e^{\frac{\sqrt{2}}{2}h\beta}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)\\&+b_3e^{-\frac{\sqrt{2}}{2}h\beta}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)+b_4e^{-\frac{\sqrt{2}}{2}h\beta}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)+\frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}h\beta\right)\\&-\frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)\mathrm{ch}\left(\frac{\sqrt{2}}{2}h\beta\right)+\frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\mathrm{sh}\left(\frac{\sqrt{2}}{2}(1-h\beta)\right)\end{aligned}$$

$$-\frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right)\text{ch}\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right), \tag{5.5}$$

here b_1, b_2, b_3 and b_4 are unknowns.

Now we introduce the following notations

$$r_1^- = r_1 - b_1,\ r_2^- = r_2 - b_2,\ r_3^- = r_3 - b_3,\ r_4^- = r_4 - b_4,$$

$$r_1^+ = r_1 + b_1,\ r_2^+ = r_2 + b_2,\ r_3^+ = r_3 + b_3,\ r_4^+ = r_4 + b_4.$$

Then, taking into account (5.2), for $u\left(h\beta\right)$ we have

$$u\left(h\beta\right) = \begin{cases} \sin\left(\frac{\sqrt{2}h\beta}{2}\right)\left(r_2^- e^{\frac{\sqrt{2}}{2}h\beta} + r_4^- e^{-\frac{\sqrt{2}}{2}h\beta}\right) \\ +\cos\left(\frac{\sqrt{2}}{2}h\beta\right)\left(r_3^- e^{-\frac{\sqrt{2}}{2}h\beta} + r_1^- e^{\frac{\sqrt{2}}{2}h\beta}\right) \\ -Q_4\left(h\beta\right), & \beta < 0, \\ f_4\left(h\beta\right), & \beta = 0,1,...,N, \\ \sin\left(\frac{\sqrt{2}}{2}h\beta\right)\left(r_2^+ e^{\frac{\sqrt{2}}{2}h\beta} + r_4^+ e^{-\frac{\sqrt{2}}{2}h\beta}\right) \\ \cos\left(\frac{\sqrt{2}}{2}h\beta\right)\left(+r_3^+ e^{-\frac{\sqrt{2}}{2}h\beta} + r_1^+ e^{\frac{\sqrt{2}}{2}h\beta}\right) \\ +Q_4\left(h\beta\right), & \beta > N, \end{cases} \tag{5.6}$$

where

$$Q_4\left(h\beta\right) = \frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}h\beta\right)\text{sh}\left(\frac{\sqrt{2}}{2}h\beta\right) - \frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}h\beta\right)\text{ch}\left(\frac{\sqrt{2}}{2}h\beta\right)$$

$$+\frac{\sqrt{2}h\beta}{16}\cos\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right)\text{sh}\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right) - \frac{\sqrt{2}h\beta}{16}\sin\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right)\text{ch}\left(\frac{\sqrt{2}}{2}\left(1-h\beta\right)\right)$$

and r_1^-, r_2^-, r_3^-, r_4^-, r_1^+, r_2^+, r_3^+ and r_4^+ are unknowns. From here:

$$r_1 = \frac{1}{2}\left(r_1^- + r_1^+\right),\ r_2 = \frac{1}{2}\left(r_2^- + r_2^+\right),$$

$$r_3 = \frac{1}{2}\left(r_3^- + r_3^+\right),\ r_4 = \frac{1}{2}\left(r_4^- + r_4^+\right),$$

$$b_1 = \frac{1}{2}\left(r_1^+ - r_1^-\right),\ b_2 = \frac{1}{2}\left(r_2^+ - r_2^-\right),$$

$$b_3 = \frac{1}{2}\left(r_3^+ - r_3^-\right),\ b_4 = \frac{1}{2}\left(r_4^+ - r_4^-\right).$$

Since for $h\beta \notin [0,1]$, $C_\beta = 0$, then

$$C_\beta = D_4\left(h\beta\right) * u\left(h\beta\right) = 0, \quad \left(h\beta\right) \notin [0,1]. \tag{5.7}$$

From here we get a system of linear equations for finding unknowns r_1^-, r_2^-, r_3^-, r_4^-, r_1^+, r_2^+, r_3^+ and r_4^+.

From (5.6) for $\beta = 0$ and $\beta = N$ we obtain

$$\begin{cases} r_1^- = F_2 - r_3^-, \\ r_1^+ = F_3 - \text{tg}\frac{\sqrt{2}}{2}\left(r_2^+ + r_4^+ e^{-\sqrt{2}}\right) - r_3^+ e^{-\sqrt{2}} \end{cases} \tag{5.8}$$

$$F_2=\frac{1}{2}-\frac{1}{2}\cos\frac{\sqrt{2}}{2}\text{ch}\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{16}\left(\sin\frac{\sqrt{2}}{2}\text{ch}\frac{\sqrt{2}}{2}-\cos\frac{\sqrt{2}}{2}\text{sh}\frac{\sqrt{2}}{2}\right),$$

$$F_3=\frac{\frac{1}{2}-\frac{1}{2}\cos\frac{\sqrt{2}}{2}\text{ch}\frac{\sqrt{2}}{2}}{e^{\frac{\sqrt{2}}{2}}\cos\frac{\sqrt{2}}{2}}.$$

Now we have six unknowns $r_2^-, r_3^-, r_4^-, r_2^+, r_3^+$ and r_4^+. From equation (5.7) for $\beta=-1,-2,-3$ and $\beta=N+1$, $N+2$, $N+3$, and considering (5.6), we can derive a system of six linear equations for $r_2^-, r_3^-, r_4^-, r_2^+, r_3^+$ and r_4^+. Since this problem has a unique solution, the main matrix of this system is non-degenerate

Solving system (5.6) for $\beta=-1,-2,-3$ and $\beta=N+1$, $N+2$, $N+3$, we find $r_2^-, r_3^-, r_4^-, r_2^+, r_3^+$ and r_4^+. Then using (5.2) we find r_1^- and r_1^+. From (5.3) for $\beta=0,1,...,N$

$$\begin{aligned}
C_\beta &= T+\sum_{\gamma=1}^{\infty}D_4\left(h\beta+h\gamma\right)\left[r_1^-\text{e}^{-\frac{\sqrt{2}}{2}h\gamma}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)-r_2^-\text{e}^{-\frac{\sqrt{2}}{2}h\gamma}\sin\left(\frac{\sqrt{2}}{2}h\gamma\right)\right.\\
&+r_3^-\text{e}^{\frac{\sqrt{2}}{2}h\gamma}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)-r_4^-\text{e}^{\frac{\sqrt{2}}{2}h\gamma}\sin\left(\frac{\sqrt{2}}{2}h\gamma\right)+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\text{ch}\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)-1\\
&+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{ch}\left(\frac{\sqrt{2}}{2}h\gamma\right)+\frac{\sqrt{2}h\gamma}{8}\sin\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{ch}\left(\frac{\sqrt{2}}{2}h\gamma\right)\\
&-\frac{\sqrt{2}h\gamma}{8}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{sh}\left(\frac{\sqrt{2}}{2}h\gamma\right)+\frac{\sqrt{2}}{16}\sin\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\text{ch}\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\\
&\left.-\frac{\sqrt{2}}{16}\cos\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\text{sh}\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\right]+\sum_{\gamma=1}^{\infty}D_4\left(h\left(N+\gamma-\beta\right)\right)\times\\
&\times\left[r_1^+\text{e}^{\frac{\sqrt{2}}{2}(1+h\gamma)}\cos\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)+r_2^+\text{e}^{\frac{\sqrt{2}}{2}(1+h\gamma)}\sin\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\right.\\
&+r_3^+\text{e}^{-\frac{\sqrt{2}}{2}(1+h\gamma)}\cos\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)+r_4^+\text{e}^{-\frac{\sqrt{2}}{2}(1+h\gamma)}\sin\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\\
&+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{ch}\left(\frac{\sqrt{2}}{2}h\gamma\right)+\frac{\sqrt{2}\left(1+2h\gamma\right)}{16}\sin\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{ch}\left(\frac{\sqrt{2}}{2}h\gamma\right)\\
&\left.-\frac{\sqrt{2}\left(1+2h\gamma\right)}{16}\cos\left(\frac{\sqrt{2}}{2}h\gamma\right)\text{sh}\left(\frac{\sqrt{2}}{2}h\gamma\right)-1+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\text{ch}\left(\frac{\sqrt{2}}{2}\left(1+h\gamma\right)\right)\right],
\end{aligned}\tag{5.9}$$

where

$$T=\frac{8}{K}\left[F_1+2+\sum_{k=1}^{3}\frac{A_k\left(1+\lambda_k\right)}{\lambda_k\left(1-\lambda_k\right)}\right].$$

Thus the following is true.

Theorem 5.1. *The coefficients of optimal quadrature formulas of the form* (1.3) *with equally spaced nodes in the space* $W_2^{(4,0)}(0,1)$ *are expressed by formulas*

$$C_0 = T + \frac{8}{K}\left[Q_1(h) + \sum_{k=1}^{3}\frac{A_k}{\lambda_k}\left[M_k + \lambda_k^N \cdot N_k\right]\right], \quad C_\beta = T + \frac{8}{K}\sum_{k=1}^{3}\frac{A_k}{\lambda_k}\left[\lambda_k^\beta \cdot M_k + \lambda_k^{N-\beta}\cdot N_k\right],$$

$$\beta = 1,2,\ldots,N-1, \; C_N = T + \frac{8}{K}\left[Q_2(h) + \sum_{k=1}^{3}\frac{A_k}{\lambda_k}\left[\lambda_k^N \cdot M_k + N_k\right]\right],$$

where

$$Q_1(h) = r_1^- e^{-\frac{\sqrt{2}}{2}h}\cos\left(\frac{\sqrt{2}}{2}h\right) - r_2^- e^{-\frac{\sqrt{2}}{2}h}\sin\left(\frac{\sqrt{2}}{2}h\right) + r_3^- e^{\frac{\sqrt{2}}{2}h}\cos\left(\frac{\sqrt{2}}{2}h\right) - r_4^- e^{\frac{\sqrt{2}}{2}h}\sin\left(\frac{\sqrt{2}}{2}h\right)$$
$$+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}(1+h)\right)ch\left(\frac{\sqrt{2}}{2}(1+h)\right) - 1 + \frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}h\right)ch\left(\frac{\sqrt{2}}{2}h\right)$$
$$+\frac{\sqrt{2}h}{8}\sin\left(\frac{\sqrt{2}}{2}h\right)ch\left(\frac{\sqrt{2}}{2}h\right) - \frac{\sqrt{2}h}{8}\cos\left(\frac{\sqrt{2}}{2}h\right)sh\left(\frac{\sqrt{2}}{2}h\right)$$
$$+\frac{\sqrt{2}}{16}\sin\left(\frac{\sqrt{2}}{2}(1+h)\right)ch\left(\frac{\sqrt{2}}{2}(1+h)\right) - \frac{\sqrt{2}}{16}\cos\left(\frac{\sqrt{2}}{2}(1+h)\right)sh\left(\frac{\sqrt{2}}{2}(1+h)\right),$$

$$Q_2(h) = r_1^+ e^{\frac{\sqrt{2}}{2}(1+h)}\cos\left(\frac{\sqrt{2}}{2}(1+h)\right) + r_2^+ e^{\frac{\sqrt{2}}{2}(1+h)}\sin\left(\frac{\sqrt{2}}{2}(1+h)\right)$$
$$+r_3^+ e^{-\frac{\sqrt{2}}{2}(1+h)}\cos\left(\frac{\sqrt{2}}{2}(1+h)\right) + r_4^+ e^{-\frac{\sqrt{2}}{2}(1+h)}\sin\left(\frac{\sqrt{2}}{2}(1+h)\right)$$
$$+\frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}h\right)ch\left(\frac{\sqrt{2}}{2}h\right) + \frac{\sqrt{2}(1+2h)}{16}\sin\left(\frac{\sqrt{2}}{2}h\right)ch\left(\frac{\sqrt{2}}{2}h\right)$$
$$-\frac{\sqrt{2}(1+2h)}{16}\cos\left(\frac{\sqrt{2}}{2}h\right)sh\left(\frac{\sqrt{2}}{2}h\right) - 1 + \frac{1}{2}\cos\left(\frac{\sqrt{2}}{2}(1+h)\right)ch\left(\frac{\sqrt{2}}{2}(1+h)\right),$$

$$Q_3(h) = \frac{\sqrt{2}h}{16}\left[\frac{e^{\frac{\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h - \sin\frac{\sqrt{2}}{2}h\right)}{\left(e^{\sqrt{2}h}\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + 1\right)^2} + \frac{\lambda_k^2 e^{\frac{\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h + \sin\frac{\sqrt{2}}{2}h\right) - 2\lambda_k e^{\sqrt{2}h}}{\left(e^{\sqrt{2}h}\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + 1\right)^2}\right.$$
$$\left.-\frac{e^{\frac{3\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h + \sin\frac{\sqrt{2}}{2}h\right) - 2\lambda_k e^{\sqrt{2}h}}{\left(\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + e^{\sqrt{2}h}\right)^2} + \frac{\lambda_k^2 e^{\frac{\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h - \sin\frac{\sqrt{2}}{2}h\right)}{\left(\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + e^{\sqrt{2}h}\right)^2}\right],$$

$$M_k = \lambda_k\left[\frac{\left(r_1^- + \frac{1}{4}\right)\left(e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h - \lambda_k\right)}{\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + e^{\sqrt{2}h}} - \frac{r_2^- e^{\frac{\sqrt{2}}{2}h}\sin\frac{\sqrt{2}}{2}h}{\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + e^{\sqrt{2}h}}\right.$$
$$+\frac{e^{\frac{\sqrt{2}}{2}(h-1)}\cos\frac{\sqrt{2}}{2}(h+1) - \lambda_k e^{-\frac{\sqrt{2}}{2}}\cos\frac{\sqrt{2}}{2}}{4\left(\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + e^{\sqrt{2}h}\right)} + \frac{\left(r_3^- + \frac{1}{4}\right)\left(e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h - \lambda_k e^{\sqrt{2}h}\right)}{e^{\sqrt{2}h}\lambda_k^2 - 2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h + 1}$$

$$
-\frac{r_4^- e^{\frac{\sqrt{2}}{2}h}\sin\frac{\sqrt{2}}{2}h}{e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}+\frac{e^{\frac{\sqrt{2}}{2}(1+h)}\cos\frac{\sqrt{2}}{2}(1+h)-\lambda_k e^{\frac{\sqrt{2}}{2}(1+2h)}\cos\frac{\sqrt{2}}{2}}{4\left(e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1\right)}
$$

$$
-\frac{\sqrt{2}}{32}\left(\frac{e^{\frac{\sqrt{2}}{2}(1+h)}\left(\cos\frac{\sqrt{2}}{2}(1+h)-\sin\frac{\sqrt{2}}{2}(1+h)\right)}{\lambda_k^2 e^{\sqrt{2}h}-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}-\frac{\lambda_k e^{\frac{\sqrt{2}}{2}(1+2h)}\left(\cos\frac{\sqrt{2}}{2}-\sin\frac{\sqrt{2}}{2}\right)}{\lambda_k^2 e^{\sqrt{2}h}-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}\right.
$$

$$
\left.-\frac{e^{\frac{\sqrt{2}}{2}(h-1)}\left(\sin\frac{\sqrt{2}}{2}(1+h)+\cos\frac{\sqrt{2}}{2}(1+h)\right)}{\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}}+\frac{\lambda_k e^{-\frac{\sqrt{2}}{2}}\left(\sin\frac{\sqrt{2}}{2}+\cos\frac{\sqrt{2}}{2}\right)}{\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}}\right)
$$

$$
\left.-\frac{1}{1-\lambda_k}-Q_3(h)\right], \qquad k=1,2,3,
$$

$$
N_k=\lambda_k\left[\frac{e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h-\lambda_k e^{\sqrt{2}h}}{4\left(e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1\right)}+\frac{\left(r_1^++\frac{1}{4}\right)\left(e^{\frac{\sqrt{2}}{2}(1+h)}\cos\frac{\sqrt{2}}{2}(1+h)\right)}{e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}\right.
$$

$$
-\frac{\left(r_1^++\frac{1}{4}\right)\left(\lambda_k e^{\frac{\sqrt{2}}{2}(1+2h)}\cos\frac{\sqrt{2}}{2}\right)}{e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}+\frac{r_2^+\left(e^{\frac{\sqrt{2}}{2}(1+h)}\sin\frac{\sqrt{2}}{2}(1+h)-\lambda_k e^{\frac{\sqrt{2}}{2}(1+2h)}\sin\frac{\sqrt{2}}{2}\right)}{e^{\sqrt{2}h}\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}
$$

$$
+\frac{r_4^+\left(e^{\frac{\sqrt{2}}{2}(h-1)}\sin\frac{\sqrt{2}}{2}(1+h)-\lambda_k e^{-\frac{\sqrt{2}}{2}}\sin\frac{\sqrt{2}}{2}\right)}{\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}}-\frac{\sqrt{2}}{32}\frac{e^{\frac{\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h-\sin\frac{\sqrt{2}}{2}h\right)-\lambda_k e^{\sqrt{2}h}}{\lambda_k^2 e^{\sqrt{2}h}-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+1}
$$

$$
+\frac{\sqrt{2}}{32}\frac{e^{\frac{\sqrt{2}}{2}h}\left(\cos\frac{\sqrt{2}}{2}h+\sin\frac{\sqrt{2}}{2}h\right)-\lambda_k}{\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}}+\frac{\left(r_3^-+\frac{1}{4}\right)\left(e^{\frac{\sqrt{2}}{2}(h-1)}\cos\frac{\sqrt{2}}{2}(h+1)-\lambda_k e^{-\frac{\sqrt{2}}{2}}\cos\frac{\sqrt{2}}{2}\right)}{\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}}
$$

$$
\left.+\frac{e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h-\lambda_k}{4\left(\lambda_k^2-2\lambda_k e^{\frac{\sqrt{2}}{2}h}\cos\frac{\sqrt{2}}{2}h+e^{\sqrt{2}h}\right)}-\frac{1}{1-\lambda_k}-Q_3(h)\right] \quad k=1,2,3.
$$

6. Conclusion

Thus, in this paper, we used the Sobolev method to develop an algorithm for solving a system of algebraic equations that determines the coefficients of quadrature formulas of the form (1.3). To achieve this, we used the discrete analogue $D_4(h\beta)$ of the differential operator $\frac{d^8}{dx^8}+2\frac{d^4}{dx^4}+1$ to solve the system (4.7) - (4.11). We then obtained explicit expressions for the optimal coefficients C_β and used them to construct an optimal quadrature formula of the form (1.3) in the space $W_2^{(4,0)}(0,1)$. It is important to note that the optimal quadrature formula of the form (1.3) in the space $W_2^{(4,0)}(0,1)$ is exact for the exponential-trigonometric functions.

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