Int. J. Anal. Appl. (2025), 23:93

International Journal of Analysis and Applications

# A Modern Stability Analysis of Mixed Duodecic-Tridecic Functional Equations in Neutrosophic Normed Spaces: A Hyers-Ulam Perspective

**P. Agilan[1,*], K. Julietraja[2], Ahmad Aloqaily[3,4], Nabil Mlaiki[4,*]**

[1]*Department of Mathematics, St.Joseph's College of Engineering, OMR, Chennai - 600 119, TamilNadu, India*

[2]*Department of Mathematics, School of Engineering, Presidency University, Bengaluru-560 064, Karnataka, India*

[3]*School of Computer, Data and Mathematical Sciences, Western Sydney University, Sydney, 2150, Australia*

[4]*Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia*

[*]*Corresponding authors:* agilram@gmail.com; nmlaiki@psu.edu.sa, nmlaiki2012@gmail.com

**Abstract.** This study delves into the stability of mixed duodecic-tridecic functional equations within the framework of neutrosophic normed spaces, employing the Hyers-Ulam stability perspective. The paper extends classical stability concepts to this enriched mathematical setting, incorporating uncertainty and imprecision inherent in neutrosophic spaces. By utilizing advanced analytical techniques, it establishes sufficient conditions for stability, providing a significant contribution to functional equation theory and its applications in areas with uncertainty modeling.

## 1. Introduction

The concept of Ulam-Hyers stability has a rich history, starting with Stanislaw Ulam's question (1940) about the stability of group homomorphisms and its formal resolution by Donald Hyers (1941), who proved stability for the Cauchy functional equation in Banach spaces. This seminal result, now known as the Hyers Stability Theorem, became the foundation for further studies in the field see [1, 2]. In 1950, Th. Aoki extended Hyers' work to accommodate unbounded perturbations, leading to the Hyers-Ulam-Aoki stability framework. The theory underwent significant development with the contributions of S.M. Rassias (1978), who introduced a generalization where the perturbation bounds depend on a function rather than a constant, known as Hyers-Ulam-Rassias stability ( [3–7]).

Received: Dec. 10, 2024.

2020 *Mathematics Subject Classification.* 39B52, 39B82.

*Key words and phrases.* Ulam-Hyers stability; Neutrosophic normed spaces; duodecic-tridecic functional equations.

https://doi.org/10.28924/2291-8639-23-2025-93

ISSN: 2291-8639

A neutrosophic normed space is a mathematical extension of normed spaces designed to address uncertain, indeterminate, or inconsistent information. This framework is particularly valuable in fields such as decision-making, artificial intelligence, and fuzzy logic, where imprecise or incomplete data is common. Introduced by Florentin Smarandache in the 1990s [8–10], neutrosophy studies indeterminacy, ambiguity, and incompleteness across various domains of knowledge. Central to this philosophy is the concept of a "neutrosophic set," where elements can simultaneously possess degrees of truth, falsity, and indeterminacy. This approach proves useful in scenarios where classical logic and set theory are inadequate for handling vagueness or uncertainty. The concept has been applied across diverse areas ( [11–20]).

In essence, neutrosophic normed spaces provide a robust mathematical framework for addressing uncertainty and ambiguity in normed spaces, broadening their applicability to scenarios where classical mathematical methods may fall short. They serve as a powerful tool for modeling and analyzing real-world phenomena characterized by imprecision or incomplete information. Agilan et al. have introduced novel functional equations and established the Hyers-Ulam stability of these equations across various normed spaces ( [21–27]).

This article introduces a novel mixed duodecic-tridecic functional equation and investigates its Ulam-Hyers stability within neutrosophic normed linear spaces (NNLS). Classical approaches are employed for the stability analysis in the newly proposed equation. Given the unique properties of neutrosophic normed spaces and their broad potential applications, the stability analysis of this equation is of considerable importance. Notably, this study marks the first instance in the literature where the stability of a functional equation is examined within the framework of neutrosophic normed spaces, underscoring the distinctiveness and significance of the research.

$$\mathfrak{F}(4\mathfrak{z}) = 41,943,040\mathfrak{F}(\mathfrak{z}) - 25,165,824\mathfrak{F}(-\mathfrak{z}) \tag{1.1}$$

The above equation having solution $\mathfrak{F}(\mathfrak{z}) = \mathcal{A}_1\, \mathfrak{z}^{13} + \mathcal{A}_2\, \mathfrak{z}^{12}$.

**Remark 1.1.** *Let us take an odd mapping $\mathfrak{F} : \mathscr{X}_M \to \mathscr{Y}_M$ which satisfies the FE (1.1) then it is tridecic*

$$\mathfrak{F}(4\mathfrak{z}) = 67,108,864\mathfrak{F}(\mathfrak{z}) = 4^{13}\mathfrak{F}(\mathfrak{z})$$

*for all $\mathfrak{z} \in \mathscr{X}_M$*

**Remark 1.2.** *Let us take an even mapping $\mathfrak{F} : \mathscr{A}_\varphi \to \mathscr{Y}_M$ which satisfies the FE (1.1) then it is duodecic*

$$\mathfrak{F}(4\mathfrak{z}) = 16,777,216\mathfrak{F}(\mathfrak{z}) = 4^{12}\mathfrak{F}(\mathfrak{z})$$

*for all $\mathfrak{z} \in \mathscr{X}_M$*

## 2. Definition of neutrosophic normed spaces

**Definition 2.1.** *The Seven-tuple $(\mathbb{A}, \mathscr{E}_{NS}, \mathscr{F}_{NS}, \mathscr{G}_{NS}*, \diamond, \oslash)$ is said to be a neutrosophic normed space (for short, NNS) if $\mathbb{A}$ is a vector space, $*$ is a continuous $\kappa$-norm, $\diamond$ and $\oslash$ is a continuous $\kappa-$ conorm, and $\mathscr{E}_{NS}, \mathscr{F}_{NS}, \mathscr{G}_{NS}$ are fuzzy sets on $\mathbb{A} \times (0, \infty)$ satisfying the following conditions. For every $p, q \in \mathbb{A}$ and*

$s, \kappa > 0$,
$\mathbb{C}_1 : \mathscr{E}_{NS}(p,\kappa) + \mathscr{F}_{NS}(p,\kappa) + \mathscr{G}_{NS}(p,\kappa) \leq 3$,
$\mathbb{C}_2 : 0 \leq \mathscr{E}_{NS}(p,\kappa) \leq 1, 0 \leq \mathscr{F}_{NS}(p,\kappa) \leq 1, 0 \leq \mathscr{G}_{NS}(p,\kappa) \leq 1$,
$\mathbb{C}_3 : \mathscr{E}_{NS}(p,\kappa) > 0$,
$\mathbb{C}_4 : \mathscr{E}_{NS}(p,\kappa) = 1$, *if and only if* $p = 0$.
$\mathbb{C}_5 : \mathscr{E}_{NS}(\alpha p,\kappa) = \mathscr{E}_{NS}\left(p, \frac{\kappa}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,
$\mathbb{C}_6 : \mathscr{E}_{NS}(p,\kappa) * \mathscr{E}_{NS}(q,s) \leq \mathscr{E}_{NS}(p+q,\kappa+s)$,
$\mathbb{C}_7 : \mathscr{E}_{NS}(p,\cdot) : (0,\infty) \to [0,1]$ *is continuous*,
$\mathbb{C}_8 : \lim_{\kappa\to\infty} \mathscr{E}_{NS}(p,\kappa) = 1$ *and* $\lim_{\kappa\to 0} \mathscr{E}_{NS}(p,\kappa) = 0$,
$\mathbb{C}_9 : \mathscr{F}_{NS}(p,\kappa) < 1$,
$\mathbb{C}_{10} : \mathscr{F}_{NS}(p,\kappa) = 0$, *if and only if* $p = 0$.
$\mathbb{C}_{11} : \mathscr{F}_{NS}(\alpha p,\kappa) = \mathscr{F}_{NS}\left(p, \frac{\kappa}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,
$\mathbb{C}_{12} : \mathscr{F}_{NS}(p,\kappa) \diamond \mathscr{F}_{NS}(q,s) \geq \mathscr{F}_{NS}(p+q,\kappa+s)$,
$\mathbb{C}_{13} : \mathscr{F}_{NS}(p,\cdot) : (0,\infty) \to [0,1]$ *is continuous*,
$\mathbb{C}_{14} : \lim_{\kappa\to\infty} \mathscr{F}_{NS}(p,\kappa) = 0$ *and* $\lim_{\kappa\to 0} \mathscr{F}_{NS}(p,\kappa) = 1$
$\mathbb{C}_{15} : \mathscr{G}_{NS}(p,\kappa) < 1$,
$\mathbb{C}_{16} : \mathscr{G}_{NS}(p,\kappa) = 0$, *if and only if* $p = 0$.
$\mathbb{C}_{17} : \mathscr{G}_{NS}(\alpha p,\kappa) = \mathscr{G}_{NS}\left(p, \frac{\kappa}{|\alpha|}\right)$ *for each* $\alpha \neq 0$,
$\mathbb{C}_{18} : \mathscr{G}_{NS}(p,\kappa) \oslash \mathscr{G}_{NS}(q,s) \geq \mathscr{G}_{NS}(p+q,\kappa+s)$,
$\mathbb{C}_{19} : \mathscr{G}_{NS}(p,\cdot) : (0,\infty) \to [0,1]$ *is continuous*,
$\mathbb{C}_{20} : \lim_{\kappa\to\infty} \mathscr{G}_{NS}(p,\kappa) = 0$ *and* $\lim_{\kappa\to 0} \mathscr{G}_{NS}(p,\kappa) = 1$.

## 3. Stability analysis in neutrosophic normed spaces

**Theorem 3.1.** *Assume that* $\mathscr{X}_M$ *is a LS,* $\left(\mathscr{Z}_m, \mathscr{E}'_{NS}, \mathscr{F}'_{NS}, \mathscr{G}'_{NS}\right)$ *is a NNS and* $(\mathscr{Y}_M, \mathscr{E}_{NS}, \mathscr{F}_{NS}, \mathscr{G}_{NS})$ *an NBS . Let* $\eth : \mathscr{X}_M \longrightarrow \mathscr{Z}_m$ *be a function such that for some* $0 < \left(\frac{\mathscr{Q}}{4^{13}}\right)^F < 1$ *with* $F \in \{1,-1\}$.

$$\left.\begin{aligned} \mathscr{E}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) &\geq \mathscr{E}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{F}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) &\leq \mathscr{F}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{G}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) &\leq \mathscr{G}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{aligned}\right\} \tag{3.1}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$ *and*

$$\left.\begin{aligned} \lim_{n\to\infty} \mathscr{E}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{13Fn}\mathscr{K}\right) &= 1 \\ \lim_{n\to\infty} \mathscr{F}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{13Fn}\mathscr{K}\right) &= 0 \\ \lim_{n\to\infty} \mathscr{G}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{13Fn}\mathscr{K}\right) &= 0 \end{aligned}\right\} \tag{3.2}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$ *. Let an odd function* $\mathfrak{F} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ *satisfying*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{array}\right\} \quad (3.3)$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$*. Then there exists a unique tridecic mapping* $\mathscr{T} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ *satisfying (1.1) and*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}|4^{13} - \mathscr{Q}|\right) \\ \\ \mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}|4^{13} - \mathscr{Q}|\right) \\ \\ \mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}|4^{13} - \mathscr{Q}|\right) \end{array}\right\} \quad (3.4)$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$*.*

*Proof.* For the first case $F = 1$. Using oddness of $\mathfrak{F}$ in in (3.3), we obtain

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{array}\right\} \quad (3.5)$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Using $\mathbb{C}_5$, $\mathbb{C}_11$ and $\mathbb{C}_17$ in (3.5), we have

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4\mathfrak{Z})}{4^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4\mathfrak{Z})}{4^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4\mathfrak{Z})}{4^{13}} - \mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \end{array}\right\} \quad (3.6)$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Let us take $\mathfrak{Z}$ by $4^n\mathfrak{Z}$ in (3.6), we arrive

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{(n+1)}\mathfrak{Z})}{4^{13}} - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \geq \mathscr{E}'_{NS}\left(\eth(4^n\mathfrak{Z}), \mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{(n+1)}\mathfrak{Z})}{4^{13}} - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \leq \mathscr{F}'_{NS}\left(\eth(4^n\mathfrak{Z}), \mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{(n+1)}\mathfrak{Z})}{4^{13}} - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}}{4^{13}}\right) \leq \mathscr{G}'_{NS}\left(\eth(4^n\mathfrak{Z}), \mathscr{K}\right) \end{array}\right\} \quad (3.7)$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. It is simple to confirm that (3.7) and using (3.1), $\mathbb{C}_5$, $\mathbb{C}_1 1$ and $\mathbb{C}_1 7$ that

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{(n+1)}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}}{4^{13}\cdot 4^{13n}}\right) \geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \frac{\mathscr{K}}{\mathscr{Q}^n}\right) \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n+1}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}}{4^{13}\cdot 4^{13n}}\right) \leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \frac{\mathscr{K}}{\mathscr{Q}^n}\right) \\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{(n+1)}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}}{4^{13}\cdot 4^{13n}}\right) \leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \frac{\mathscr{K}}{\mathscr{Q}^n}\right) \end{aligned}\right\} \tag{3.8}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Swapping $\mathscr{K}$ into $\mathscr{Q}^n\mathscr{K}$ in (3.8), we have

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{n+1}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}\cdot\mathscr{Q}^n}{4^{13}\cdot 4^{13n}}\right) \geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n+1}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}\cdot\mathscr{Q}^n}{4^{13}\cdot 4^{13n}}\right) \leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{n+1}\mathfrak{Z})}{4^{13(n+1)}} - \frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}, \frac{\mathscr{K}\cdot\mathscr{Q}^n}{4^{13}\cdot 4^{13n}}\right) \leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \end{aligned}\right\} \tag{3.9}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. It is simple to observe that

$$\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}) = \sum_{i=0}^{n-1} \frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}} \tag{3.10}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$. It follows from (3.9) and (3.10), we get

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) = \mathscr{E}_{NS}\left(\sum_{i=0}^{n-1}\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}}, \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) = \mathscr{F}_{NS}\left(\sum_{i=0}^{n-1}\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{i}}, \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \\ *\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) = \mathscr{G}_{NS}\left(\sum_{i=0}^{n-1}\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}}, \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \end{aligned}\right\} \tag{3.11}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Using $\mathbb{C}_5$, $\mathbb{C}_1 1$ and $\mathbb{C}_1 7$in (3.11), we have

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \geq \prod_{i=0}^{n-1}\mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}}, \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \leq \prod_{i=0}^{n-1}\mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}}, \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1}\frac{\mathscr{Q}^i\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \leq \coprod_{i=0}^{n-1}\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{i+1}\mathfrak{Z})}{4^{13(i+1)}} - \frac{\mathfrak{F}(4^i\mathfrak{Z})}{4^{13i}}, \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \end{aligned}\right\} \tag{3.12}$$

where

$$\prod_{\mathcal{J}=0}^{n-1} Q_j = Q_1 * Q_2 * \cdots * Q_n \quad and \quad \coprod_{\mathcal{J}=0}^{n-1} R_j = R_1 \diamond R_2 \diamond \cdots \diamond R_n and \quad \coprod_{\mathcal{J}=0}^{n-1} S_j = S_1 \oslash S_2 \oslash \cdots \oslash S_n$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Hence, from (3.12) and (3.9), we arrive

$$\left.\begin{aligned} &\mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \geq \prod_{i=0}^{n-1} \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) = \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ &\mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \leq \prod_{i=0}^{n-1} \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) = \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ &\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}} - \mathfrak{F}(\mathfrak{Z}), \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13i}}\right) \leq \prod_{i=0}^{n-1} \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) = \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \end{aligned}\right\} \tag{3.13}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Replacing $\mathfrak{Z}$ by $4^m\mathfrak{Z}$ in (3.13) and using (3.1), $\mathbb{C}_5$, $\mathbb{C}_11$ and $\mathbb{C}_17$, we obtain

$$\left.\begin{aligned} &\mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \geq \mathscr{E}'_{NS}\left(\eth(4^m\mathfrak{Z}), \mathscr{K}\right) = \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{\mathscr{K}}{\mathscr{Q}^m}\right) \\ &\mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \leq \mathscr{F}'_{NS}\left(\eth(4^m\mathfrak{Z}), \mathscr{K}\right) = \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{\mathscr{K}}{\mathscr{Q}^m}\right) \\ &\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^i\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \leq \mathscr{G}'_{NS}\left(\eth(4^m\mathfrak{Z}), \mathscr{K}\right) = \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{\mathscr{K}}{\mathscr{Q}^m}\right) \end{aligned}\right\} \tag{3.14}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ also $m, n$ are positive numbers. Changing $\mathscr{K}$ by $\mathscr{Q}^m\mathscr{K}$ in (3.14), we get

$$\left.\begin{aligned} &\mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^{i+m}\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ &\mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^{i+m}\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \\ &\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}} - \frac{\mathfrak{F}(4^m\mathfrak{Z})}{4^{13m}}, \sum_{i=0}^{n-1} \frac{\mathscr{Q}^{i+m}\,\mathscr{K}}{4^{13}\cdot 4^{13(i+m)}}\right) \leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \mathscr{K}\right) \end{aligned}\right\} \tag{3.15}$$

which implies

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}}-\frac{\mathfrak{F}(4^{m}\mathfrak{Z})}{4^{13m}},\mathscr{K}\right) &\geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=m}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right)\\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}}-\frac{\mathfrak{F}(4^{m}\mathfrak{Z})}{4^{13m}},\mathscr{K}\right) &\leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=m}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right)\\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{n+m}\mathfrak{Z})}{4^{13(n+m)}}-\frac{\mathfrak{F}(4^{m}\mathfrak{Z})}{4^{13m}},\mathscr{K}\right) &\leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=m}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right) \end{aligned}\right\} \tag{3.16}$$

Here $\left\{\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}\right\}$ is a Cauchy sequence in $(\mathscr{Y}_M,\mathscr{E}_{NS},\mathscr{F}_{NS},\mathscr{G}_{NS})$ also a complete NNS-space is $(\mathscr{Y}_M,\mathscr{E}_{NS},\mathscr{F}_{NS},\mathscr{G}_{NS})$ then this sequence converges to a particular point $\mathscr{T}(\mathfrak{Z}) \in Y$.

$$\lim_{n\to\infty}\mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}-\mathscr{T}(\mathfrak{Z}),\mathscr{K}\right)=1,$$

$$\lim_{n\to\infty}\mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}-\mathscr{T}(\mathfrak{Z}),\mathscr{K}\right)=0$$

$$\lim_{n\to\infty}\mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}-\mathscr{T}(\mathfrak{Z}),\mathscr{K}\right)=0$$

and

$$\frac{\mathfrak{F}(4^n\mathfrak{Z})}{4^{13n}}\xrightarrow{NNS}\mathscr{T}(\mathfrak{Z}),\quad as\quad n\to\infty.$$

Taking $m=0$ in (3.15), we reach

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\frac{\mathfrak{F}(4^{n}\mathfrak{Z})}{4^{13n}}-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\geq \mathscr{E}'_{NS}\left(\Psi(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=0}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right)\\ \mathscr{F}_{NS}\left(\frac{\mathfrak{F}(4^{n}\mathfrak{Z})}{4^{13n}}-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\leq \mathscr{F}'_{NS}\left(\Psi(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=0}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right)\\ \mathscr{G}_{NS}\left(\frac{\mathfrak{F}(4^{n}\mathfrak{Z})}{4^{13n}}-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\leq \mathscr{G}'_{NS}\left(\Psi(\mathfrak{Z}),\frac{\mathscr{K}}{\sum_{i=0}^{n-1}\frac{\mathscr{Q}^i}{4^{13}.4^{13i}}}\right) \end{aligned}\right\} \tag{3.17}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Considering $n\to\infty$ in (3.17), we arrive

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\mathscr{T}(\mathfrak{Z})-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}),\mathscr{K}|4^{13}-\mathscr{Q}|\right)\\ \mathscr{F}_{NS}\left(\mathscr{T}(\mathfrak{Z})-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}),\mathscr{K}|4^{13}-\mathscr{Q}|\right)\\ \mathscr{G}_{NS}\left(\mathscr{T}(\mathfrak{Z})-\mathfrak{F}(\mathfrak{Z}),\mathscr{K}\right) &\leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}),\mathscr{K}|4^{13}-\mathscr{Q}|\right) \end{aligned}\right\} \tag{3.18}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ . Next, we have to show $\mathfrak{F}$ satisfies (1.1), letting $\mathfrak{Z}$ by $4^n\mathfrak{Z}$ in (3.3) respectively, we have

$$\left.\begin{aligned}
&\mathscr{E}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \mathscr{K}\right)\\
&\qquad\qquad\qquad\qquad \geq \mathscr{E}'_{NS}\left(\eth(4^n\mathfrak{Z}), 4^{13n}\mathscr{K}\right)\\
&\mathscr{F}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \mathscr{K}\right)\\
&\qquad\qquad\qquad\qquad \leq \mathscr{F}'_{NS}\left(\eth(4^n\mathfrak{Z}), 4^{13n}\mathscr{K}\right)\\
&\mathscr{G}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \mathscr{K}\right)\\
&\qquad\qquad\qquad\qquad \leq \mathscr{G}'_{NS}\left(\eth(4^n\mathfrak{Z}), 4^{13n}\mathscr{K}\right)
\end{aligned}\right\} \tag{3.19}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ . Now,

$$\begin{aligned}
&\mathscr{E}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - 41,943,040\mathscr{T}(\mathfrak{Z}) + 25,165,824\mathscr{T}(-\mathfrak{Z}), \mathscr{K}\Big)\\
&\geq \mathscr{E}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - \frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad * \mathscr{E}_{NS}\Big(-41,943,040\mathscr{T}(\mathfrak{Z}) + 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad * \mathscr{E}_{NS}\Big(25,165,824\mathscr{T}(-\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad * \mathscr{E}_{NS}\Big(\frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\mathscr{F}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - 41,943,040\mathscr{T}(\mathfrak{Z}) + 25,165,824\mathscr{T}(-\mathfrak{Z}), \mathscr{K}\Big)\\
&\geq \mathscr{E}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - \frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \diamond \mathscr{F}_{NS}\Big(-41,943,040\mathscr{T}(\mathfrak{Z}) + 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \diamond \mathscr{F}_{NS}\Big(-25,165,824\mathscr{T}(-\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \diamond \mathscr{F}_{NS}\Big(\frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)
\end{aligned} \tag{3.21}$$

and

$$\begin{aligned}
&\mathscr{G}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - 41,943,040\mathscr{T}(\mathfrak{Z}) + 25,165,824\mathscr{T}(-\mathfrak{Z}), \mathscr{K}\Big)\\
&\geq \mathscr{G}_{NS}\Big(\mathscr{T}(4\mathfrak{Z}) - \frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \oslash \mathscr{G}_{NS}\Big(-41,943,040\mathscr{T}(\mathfrak{Z}) + 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \oslash \mathscr{G}_{NS}\Big(25,165,824\mathscr{T}(-\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)\\
&\quad \oslash \mathscr{E}_{NS}\Big(\frac{1}{4^{13n}}\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\frac{1}{4^{13n}}\mathfrak{F}(\mathfrak{Z}) + 25,165,824\frac{1}{4^{13n}}\mathfrak{F}(-\mathfrak{Z}), \frac{\mathscr{K}}{4}\Big)
\end{aligned} \tag{3.22}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ . Also,

$$\left.\begin{aligned}\lim_{n\to\infty} \mathscr{E}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \tfrac{\mathscr{K}}{4}\right) = 1\\ \lim_{n\to\infty} \mathscr{F}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \tfrac{\mathscr{K}}{4}\right) = 0\\ \lim_{n\to\infty} \mathscr{G}_{NS}\left(\tfrac{1}{4^{13n}}\left[\mathfrak{F}(3\cdot 4^n\mathfrak{Z}) - 41,943,040\mathfrak{F}(4^n\mathfrak{Z}) + 25,165,824\mathfrak{F}(-4^n\mathfrak{Z})\right], \tfrac{\mathscr{K}}{4}\right) = 0\end{aligned}\right\} \tag{3.23}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ .

By Taking $n \to \infty$ in (3.22), (3.21) and using (3.23), we proved that $\mathscr{T}$ satisfies (1.1). Therefore, $\mathscr{T}$ is a tridecic mapping.

Next, we need to prove $\mathscr{T}(\mathfrak{Z})$ is unique,

let $\mathscr{T}'(\mathfrak{Z})$ be another tridecic FE satisfying (1.1) and (3.4), Then

$$\begin{aligned}&\mathscr{E}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K})\\ &\geq \mathscr{E}_{NS}\left(\mathscr{T}(4^n\mathfrak{Z}) - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right) * \mathscr{E}_{NS}\left(\mathfrak{F}(4^n\mathfrak{Z}) - \mathscr{T}'(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right)\\ &\geq \mathscr{E}'_{NS}\left(\eth(4^n\mathfrak{Z}), \frac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2}\right) \geq \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \frac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2\cdot \mathscr{Q}^n}\right)\\ &\mathscr{F}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K})\\ &\leq \mathscr{F}_{NS}\left(\mathscr{T}(4^n\mathfrak{Z}) - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right) \diamond \mathscr{F}_{NS}\left(\mathfrak{F}(4^n\mathfrak{Z}) - \mathscr{T}'(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right)\\ &\leq \mathscr{F}'_{NS}\left(\eth(4^n\mathfrak{Z}), \frac{4^{13n}\mathscr{K}\ |4^{13} - \mathscr{Q}|}{2}\right) \leq \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \frac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2\cdot \mathscr{Q}^n}\right)\\ &\mathscr{G}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K})\\ &\leq \mathscr{G}_{NS}\left(\mathscr{T}(4^n\mathfrak{Z}) - \mathfrak{F}(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right) * \mathscr{G}_{NS}\left(\mathfrak{F}(4^n\mathfrak{Z}) - \mathscr{T}'(4^n\mathfrak{Z}), \frac{\mathscr{K}.4^{13n}}{2}\right)\\ &\leq \mathscr{G}'_{NS}\left(\eth(4^n\mathfrak{Z}), \frac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2}\right) \leq \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \frac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2\cdot \mathscr{Q}^n}\right)\end{aligned}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ . Since $\lim_{n\to\infty} \dfrac{4^{13n}\ \mathscr{K}\ |4^{13} - \mathscr{Q}|}{2\ \mathscr{Q}^n} = \infty$, we obtain

$$\left.\begin{aligned}\lim_{n\to\infty} \mathscr{E}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{4^{13n}\ \mathscr{K}\ |4^{13}-\mathscr{Q}|}{2\cdot\mathscr{Q}^n}\right) = 1\\ \lim_{n\to\infty} \mathscr{F}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{4^{13n}\ \mathscr{K}\ |4^{13}-\mathscr{Q}|}{2\cdot\mathscr{Q}^n}\right) = 0\\ \lim_{n\to\infty} \mathscr{G}'_{NS}\left(\eth(\mathfrak{Z}), \tfrac{4^{13n}\ \mathscr{K}\ |4^{13}-\mathscr{Q}|}{2\cdot\mathscr{Q}^n}\right) = 0\end{aligned}\right\}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ . Thus

$$\left.\begin{aligned}\mathscr{E}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K}) = 1\\ \mathscr{F}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K}) = 0\\ \mathscr{G}_{NS}(\mathscr{T}(\mathfrak{Z}) - \mathscr{T}'(\mathfrak{Z}), \mathscr{K}) = 0\end{aligned}\right\}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ .
Hence, $\mathscr{T}(\mathfrak{Z}) = \mathscr{T}'(\mathfrak{Z})$. Therefore, $\mathscr{T}(\mathfrak{Z})$ is unique.
For second case, we have to take $F = -1$. Considering $\mathfrak{Z}$ by $\frac{\mathfrak{Z}}{13}$ in (3.5), we have

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - 4^{13}\mathfrak{F}\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\frac{\mathfrak{Z}}{13}\right), \mathscr{K}\right) \end{array}\right\} \tag{3.24}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. □

**Corollary 3.1.** *Let $\aleph, r$ are constants, with $\aleph > 0$ and $r \neq 12$ and let an odd function $\mathfrak{F} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ satisfies*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \geq \left\{\begin{array}{l} \mathscr{E}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{E}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \left\{\begin{array}{l} \mathscr{F}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{F}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \left\{\begin{array}{l} \mathscr{G}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{G}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \end{array}\right\} \tag{3.25}$$

*for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$, Then there exists a unique tridecic function $\mathscr{T} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ such that*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \geq \left\{\begin{array}{l} \mathscr{E}'_{NS}\left(\aleph, |4^{13} - 1|\mathscr{K}\right), \\ \mathscr{E}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{13} - 4^r|\mathscr{K}\right), \end{array}\right. \\ \mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \left\{\begin{array}{l} \mathscr{F}'_{NS}\left(\aleph, |4^{13} - 1|\mathscr{K}\right), \\ \mathscr{F}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{13} - 4^r|\mathscr{K}\right), \end{array}\right. \\ \mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \left\{\begin{array}{l} \mathscr{G}'_{NS}\left(\aleph, |4^{13} - 1|\mathscr{K}\right), \\ \mathscr{G}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{13} - 4^r|\mathscr{K}\right), \end{array}\right. \end{array}\right\} \tag{3.26}$$

*for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ .*

**Theorem 3.2.** *Assume that $\mathscr{X}_M$ is a LS, $\left(\mathscr{Z}_m, \mathscr{E}'_{NS}, \mathscr{F}'_{NS}, \mathscr{G}'_{NS}\right)$ is a NNS and $(\mathscr{Y}_M, \mathscr{E}_{NS}, \mathscr{F}_{NS}, \mathscr{G}_{NS})$ an NBS . Let $\eth : \mathscr{X}_M \longrightarrow \mathscr{Z}_m$ be a function such that for some $0 < \left(\frac{\mathscr{Q}}{4^{12}}\right)^F < 1$ with $F \in \{1, -1\}$.*

$$\left.\begin{array}{l} \mathscr{E}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{F}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{G}'_{NS}\left(\eth\left(4^{nF}\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\mathscr{Q}^{nF}\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{array}\right\} \tag{3.27}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$ *and*

$$\left.\begin{aligned} \lim_{n\to\infty} \mathscr{E}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{12Fn}\mathscr{K}\right) = 1 \\ \lim_{n\to\infty} \mathscr{F}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{12Fn}\mathscr{K}\right) = 0 \\ \lim_{n\to\infty} \mathscr{G}'_{NS}\left(\eth\left(4^{Fn}\mathfrak{Z}\right), 4^{12Fn}\mathscr{K}\right) = 0 \end{aligned}\right\} \tag{3.28}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$ *. Let an even function* $\mathfrak{F} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ *satisfying*

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{aligned}\right\} \tag{3.29}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$*. Then there exists a unique duodecic mapping* $\mathscr{D} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ *satisfying (1.1) and*

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \end{aligned}\right\} \tag{3.30}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$*.*

*Proof.* For the first case $F = 1$. By applying the evenness condition of $\mathfrak{F}$ in in (3.29), we arrive

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 25,165,824\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 25,165,824\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 25,165,824\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{aligned}\right\} \tag{3.31}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. From (3.31) we have

$$\left.\begin{aligned} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{12}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{12}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 4^{12}\mathfrak{F}\left(\mathfrak{Z}\right), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{aligned}\right\} \tag{3.32}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. □

**Corollary 3.2.** *Let $\aleph, r$ are constants, with $\aleph > 0$ and $r \neq 12$ and let an even mapping $\mathfrak{F} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ satisfies*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \geq \left\{ \begin{array}{l} \mathscr{E}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{E}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \left\{ \begin{array}{l} \mathscr{F}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{F}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \left\{ \begin{array}{l} \mathscr{G}'_{NS}\left(\aleph, \mathscr{K}\right), \\ \mathscr{G}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right), \mathscr{K}\right), \end{array}\right. \end{array}\right\} \tag{3.33}$$

*for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$, Then there exists a unique duodecic function $\mathscr{D} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ such that*

$$\left.\begin{array}{r} \mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \geq \left\{ \begin{array}{l} \mathscr{E}'_{NS}\left(\aleph, |4^{12} - 1|\mathscr{K}\right), \\ \mathscr{E}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{12} - 4^r|\mathscr{K}\right), \end{array}\right. \\ \mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \leq \left\{ \begin{array}{l} \mathscr{F}'_{NS}\left(\aleph, |4^{12} - 1|\mathscr{K}\right), \\ \mathscr{F}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{12} - 4^r|\mathscr{K}\right), \end{array}\right. \\ \mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \leq \left\{ \begin{array}{l} \mathscr{G}'_{NS}\left(\aleph, |4^{12} - 1|\mathscr{K}\right), \\ \mathscr{G}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r, |4^{12} - 4^r|\mathscr{K}\right), \end{array}\right. \end{array}\right\} \tag{3.34}$$

*for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$ .*

**Theorem 3.3.** *Assume that $\mathscr{X}_M$ is a LS, $\left(\mathscr{Z}_m, \mathscr{E}'_{NS}, \mathscr{F}'_{NS}, \mathscr{G}'_{NS}\right)$ is a NNS and $\left(\mathscr{Y}_M, \mathscr{E}_{NS}, \mathscr{F}_{NS}, \mathscr{G}_{NS}\right)$ an NBS. Let $\eth : \mathscr{X}_M \longrightarrow \mathscr{Z}_m$ be a function such that for some $0 < \left(\frac{\mathscr{Q}}{4^{12}}\right)^F < 1, 0 < \left(\frac{\mathscr{Q}}{4^{13}}\right)^F < 1$ with $F \in \{1, -1\}$. Let $\mathfrak{F} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ be a function satisfying the inequality with conditions (3.1), (3.27), (3.2) and (3.28). Then*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z}) - 41,943,040\mathfrak{F}(\mathfrak{Z}) + 25,165,824\mathfrak{F}(-\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), \mathscr{K}\right) \end{array}\right\} \tag{3.35}$$

*for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Then there exists a unique tridecic function $\mathscr{T} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ and a unique duodecic function $\mathscr{D} : \mathscr{X}_M \longrightarrow \mathscr{Y}_M$ satisfying (1.1) and*

$$\left.\begin{array}{l} \mathscr{E}_{NS}\left(g(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \\ \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \\ \quad * \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \\ \\ \mathscr{F}_{NS}\left(g(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \\ \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \\ \quad \diamond \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \\ \\ \mathscr{G}_{NS}\left(g(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \\ \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \oslash \mathscr{G}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \\ \quad \oslash \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \oslash \mathscr{G}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \end{array}\right\} \tag{3.36}$$

*for all* $\mathfrak{Z} \in \mathscr{X}_M$ *and all* $\mathscr{K} > 0$.

*Proof.* Let $\mathfrak{F}_o(\mathfrak{Z}) = \dfrac{\mathfrak{F}(\mathfrak{Z}) - \mathfrak{F}(-\mathfrak{Z})}{2}$ for all $\mathfrak{Z} \in \mathscr{X}_M$. Then $\mathfrak{F}_o(0) = 0$ and $\mathfrak{F}_o(-\mathfrak{Z}) = -\mathfrak{F}_o(\mathfrak{Z})$ for all $\mathfrak{Z} \in \mathscr{X}_M$. Hence by Theorem 3.1, we have

$$\left.\begin{aligned}
&\mathscr{E}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right)\\
&\mathscr{F}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right)\\
&\mathscr{G}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \oslash \mathscr{G}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right)
\end{aligned}\right\} \tag{3.37}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Also, let $\mathfrak{F}_e(\mathfrak{Z}) = \dfrac{\mathfrak{F}(\mathfrak{Z}) + \mathfrak{F}(-\mathfrak{Z})}{2}$ for all $\mathfrak{Z} \in \mathscr{X}_M$. Then $\mathfrak{F}_e(0) = 0$ and $\mathfrak{F}_e(-\mathfrak{Z}) = \mathfrak{F}_e(\mathfrak{Z})$ for all $\mathfrak{Z} \in \mathscr{X}_M$. Hence by Theorem 3.2, we have

$$\left.\begin{aligned}
&\mathscr{E}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right)\\
&\mathscr{F}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right)\\
&\mathscr{G}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right) \leq \mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \oslash \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right)
\end{aligned}\right\} \tag{3.38}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$ and all $\mathscr{K} > 0$. Define

$$g(\mathfrak{Z}) = \mathfrak{F}_o(\mathfrak{Z}) + \mathfrak{F}_e(\mathfrak{Z}) \tag{3.39}$$

for all $\mathfrak{Z} \in \mathscr{X}_M$. From (3.37),(3.38) and (3.39), we arrive

$$\begin{aligned}
\mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), 2\mathscr{K}\right) &= \mathscr{E}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) + \mathfrak{F}_e(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), 2\mathscr{K}\right)\\
&\geq \mathscr{E}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) * \mathscr{E}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right)\\
&\geq \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right)\\
&\quad * \mathscr{E}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) * \mathscr{E}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right)
\end{aligned}$$

and

$$\begin{aligned}
\mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), 2\mathscr{K}\right) &= \mathscr{F}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) + \mathfrak{F}_e(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), 2\mathscr{K}\right)\\
&\leq \mathscr{F}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z}) - \mathscr{T}(\mathfrak{Z}), \mathscr{K}\right) \diamond \mathscr{F}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z}) - \mathscr{D}(\mathfrak{Z}), \mathscr{K}\right)\\
&\leq \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{13} - \mathscr{Q}|\mathscr{K}\right)\\
&\quad \diamond \mathscr{F}'_{NS}\left(\eth\left(\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right) \diamond \mathscr{F}'_{NS}\left(\eth\left(-\mathfrak{Z}\right), |4^{12} - \mathscr{Q}|\mathscr{K}\right)
\end{aligned}$$

and

$$\begin{aligned}\mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),2\mathscr{K}\right)&=\mathscr{G}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z})+\mathfrak{F}_e(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),2\mathscr{K}\right)\\&\le\mathscr{F}_{NS}\left(\mathfrak{F}_o(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z}),\mathscr{K}\right)\oslash\mathscr{G}_{NS}\left(\mathfrak{F}_e(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),\mathscr{K}\right)\\&\le\mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right),|4^{13}-\mathscr{Q}|\mathscr{K}\right)\oslash\mathscr{G}'_{NS}\left(\eth\left(-\mathfrak{Z}\right),|4^{13}-\mathscr{Q}|\mathscr{K}\right)\\&\oslash\mathscr{G}'_{NS}\left(\eth\left(\mathfrak{Z}\right),|4^{12}-\mathscr{Q}|\mathscr{K}\right)\oslash\mathscr{G}'_{NS}\left(\eth\left(-\mathfrak{Z}\right),|4^{12}-\mathscr{Q}|\mathscr{K}\right)\end{aligned}$$

for all $\mathfrak{Z}\in\mathscr{X}_M$ and all $\mathscr{K}>0$. □

**Corollary 3.3.** *Let $\aleph, r$ are constants with $\aleph>0$ and $r\neq 13,12$ and let the function $\mathfrak{F}:\mathscr{X}_M\longrightarrow\mathscr{Y}_M$ satisfies*

$$\left.\begin{array}{l}\mathscr{E}_{NS}\left(\mathfrak{F}(4\mathfrak{Z})-41,943,040\mathfrak{F}(\mathfrak{Z})+25,165,824\mathfrak{F}(-\mathfrak{Z}),\mathscr{K}\right)\ge\begin{cases}\mathscr{E}'_{NS}\left(\aleph,\mathscr{K}\right),\\\mathscr{E}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right),\mathscr{K}\right),\end{cases}\\\mathscr{F}_{NS}\left(\mathfrak{F}(4\mathfrak{Z})-41,943,040\mathfrak{F}(\mathfrak{Z})+25,165,824\mathfrak{F}(-\mathfrak{Z}),\mathscr{K}\right)\le\begin{cases}\mathscr{F}'_{NS}\left(\aleph,\mathscr{K}\right),\\\mathscr{F}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right),\mathscr{K}\right),\end{cases}\\\mathscr{G}_{NS}\left(\mathfrak{F}(4\mathfrak{Z})-41,943,040\mathfrak{F}(\mathfrak{Z})+25,165,824\mathfrak{F}(-\mathfrak{Z}),\mathscr{K}\right)\le\begin{cases}\mathscr{G}'_{NS}\left(\aleph,\mathscr{K}\right),\\\mathscr{G}'_{NS}\left(\aleph\left(\|\mathfrak{Z}\|^r\right),\mathscr{K}\right),\end{cases}\end{array}\right\}\tag{3.40}$$

*for all $\mathfrak{Z}\in\mathscr{X}_M$ and all $\mathscr{K}>0$. Then there exists a unique tridecic function $\mathscr{T}:\mathscr{X}_M\longrightarrow\mathscr{Y}_M$ and a unique duodecic function $\mathscr{D}:\mathscr{X}_M\longrightarrow\mathscr{Y}_M$ such that*

$$\left.\begin{array}{l}\mathscr{E}_{NS}\left(\mathfrak{F}(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),\mathscr{K}\right)\\\ge\begin{cases}\mathscr{E}'_{NS}\left(\aleph,|4^{13}-1|\mathscr{K}\right)*\mathscr{E}'_{NS}\left(\aleph,|4^{12}-1|\mathscr{K}\right),\\\mathscr{E}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{13}-4^r|\mathscr{K}\right)*\mathscr{E}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{12}-4^r|\mathscr{K}\right),\end{cases}\\\\\mathscr{F}_{NS}\left(\mathfrak{F}(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),\mathscr{K}\right)\\\le\begin{cases}\mathscr{F}'_{NS}\left(\aleph,|4^{13}-1|\mathscr{K}\right)\diamond\mathscr{F}'_{NS}\left(\aleph,|4^{12}-1|\mathscr{K}\right),\\\mathscr{F}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{13}-4^r|\mathscr{K}\right)\diamond\mathscr{F}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{12}-4^r|\mathscr{K}\right),\end{cases}\\\\\mathscr{G}_{NS}\left(\mathfrak{F}(\mathfrak{Z})-\mathscr{T}(\mathfrak{Z})-\mathscr{D}(\mathfrak{Z}),\mathscr{K}\right)\\\le\begin{cases}\mathscr{G}'_{NS}\left(\aleph,|4^{13}-1|\mathscr{K}\right)\oslash\mathscr{G}'_{NS}\left(\aleph,|4^{12}-1|\mathscr{K}\right),\\\mathscr{G}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{13}-4^r|\mathscr{K}\right)\oslash\mathscr{G}'_{NS}\left(\aleph\|\mathfrak{Z}\|^r,|4^{12}-4^r|\mathscr{K}\right),\end{cases}\end{array}\right\}\tag{3.41}$$

*for all $\mathfrak{Z}\in\mathscr{X}_M$ and all $\mathscr{K}>0$ .*

## 4. Conclusion

The study successfully extends the Hyers-Ulam stability analysis to mixed duodecic-tridecic functional equations in neutrosophic normed spaces, demonstrating the robustness of these equations under perturbations. The findings highlight the adaptability of neutrosophic spaces in handling uncertainty, offering a deeper understanding of functional stability in complex systems. This work paves the way for future research into stability phenomena in diverse mathematical and applied contexts involving neutrosophic frameworks.

**Acknowledgment:** The authors A. Aloqaily and N. Mlaiki would like to thank Prince Sultan University for the support through the TAS research lab and for paying the APC.

**Conflicts of Interest:** The authors declare that there are no conflicts of interest regarding the publication of this paper.

## References


[1] S.M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.

[2] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Nat. Acad. Sci. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.

[3] P. Kannappan, Functional Equations and Inequalities with Applications, Springer, New York, 2009.

[4] J.M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, J. Funct. Anal. 46 (1982), 126–130. https://doi.org/10.1016/0022-1236(82)90048-9.

[5] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300. https://doi.org/10.1090/S0002-9939-1978-0507327-1.

[6] T.M. Rassias, ed., Functional Equations, Inequalities and Applications, Springer, Dordrecht, 2003. https://doi.org/10.1007/978-94-017-0225-6.

[7] P. Gavruta, A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings, J. Math. Anal. Appl. 184 (1994), 431–436. https://doi.org/10.1006/jmaa.1994.1211.

[8] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.

[9] F. Smarandache, Neutrosophy: Neutrosophic Probability, Set, and Logic: Analytic Synthesis & Synthetic Analysis, American Research Press, 1998.

[10] F. Smarandache, A Unifying Field In Logics: Neutrosophic Logic, American Research Press, 1999.

[11] T. Bera, N.K. Mahapatra, On Neutrosophic Soft Linear Spaces, Fuzzy Inf. Eng. 9 (2017), 299–324. https://doi.org/10.1016/j.fiae.2017.09.004.

[12] T. Bera, N.K. Mahapatra, Neutrosophic Soft Normed Linear Spaces, Neutrosophic Sets Syst. 23 (2018), 52–71.

[13] A.A.A. Agboola, A.D. Akwu, Y.T. Oyebo, Neutrosophic Groups and Subgroups, Int. J. Math. Combin. 3 (2012), 1-9.

[14] A.A.A. Agboola, S.A. Akinleye, Neutrosophic Vector Spaces. Neutrosophic Sets Syst. 4 (2014), 9-18.

[15] W.B. Vasantha, F. Smarandache, Neutrosophic Rings, Hexis, Phoenix, 2006.

[16] M. Abobala, Classical Homomorphisms Between n-Refined Neutrosophic Rings, Int. J. Neutrosophic Sci. 7 (2020), 74–78. https://doi.org/10.54216/IJNS.070204.

[17] M. Abobala, On the Representation of Neutrosophic Matrices by Neutrosophic Linear Transformations, J. Math. 2021 (2021), 5591576. https://doi.org/10.1155/2021/5591576.

[18] M. Abobala, Partial Foundation of Neutrosophic Number Theory, Neutrosophic Sets Syst. 39 (2021), 120-132.

[19] W.B.V. Kandasamy, K. Ilanthenral, F. Smarandache, Neutrosophic Graphs: A New Dimension To Graph Theory, Europa Nova, 2015. https://doi.org/10.5281/ZENODO.31820.

[20] F. Smarandache, Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability, Sitech, 2013.

[21] A. Pasupathi, J. Konsalraj, N. Fatima, et al. Direct and Fixed-Point Stability–Instability of Additive Functional Equation in Banach and Quasi-Beta Normed Spaces, Symmetry 14 (2022), 1700. https://doi.org/10.3390/sym14081700.

[22] P. Agilan, K. Julietraja, N. Mlaiki, A. Mukheimer, Intuitionistic Fuzzy Stability of an Euler–Lagrange Symmetry Additive Functional Equation via Direct and Fixed Point Technique (FPT), Symmetry 14 (2022), 2454. https://doi.org/10.3390/sym14112454.

[23] P. Agilan, M.M.A. Almazah, K. Julietraja, A. Alsinai, Classical and Fixed Point Approach to the Stability Analysis of a Bilateral Symmetric Additive Functional Equation in Fuzzy and Random Normed Spaces, Mathematics 11 (2023), 681. https://doi.org/10.3390/math11030681.

[24] P. Agilan, K. Julietraja, M.M.A. Almazah, A. Alsinai, Stability Analysis of a New Class of Series Type Additive Functional Equation in Banach Spaces: Direct and Fixed Point Techniques, Mathematics 11 (2023), 887. https://doi.org/10.3390/math11040887.

[25] A. Aloqaily, P. Agilan, K. Julietraja, S. Annadurai, N. Mlaiki, A Novel Stability Analysis of Functional Equation in Neutrosophic Normed Spaces, Bound. Value Probl. 2024 (2024), 47. https://doi.org/10.1186/s13661-024-01854-2.

[26] P. Agilan, K. Julietraja, B. Kanimozhi, A. Alsinai, Hyers Stability of AQC Functional Equation, Dyn. Contin. Discr. Impuls. Syst. Ser. B: Appl. Algorithms 31 (2024), 63–75.

[27] P. Agilan, K. Julietraja, S. Aljohani, N. Mlaiki, Generalised Ulam-Hyers Stability Analysis for System of Additive Functional Equation in Fuzzy and Random Normed Spaces: Direct and Fixed Point Approach, Int. J. Anal. Appl. 22 (2024), 201. https://doi.org/10.28924/2291-8639-22-2024-201.