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K-Frames in Super Hilbert C^*-Modules

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Abstract. In this paper, we study the theory of K-frames in super Hilbert C^*-modules. We introduce the concept of super Hilbert modules as direct sums of Hilbert C^*-modules and explore how frames and K-frames can be defined and characterized within this framework. Our main results provide new characterizations of K-frames in super Hilbert C^*-modules, as well as necessary and sufficient conditions under which sequences in super Hilbert C^*-modules form K-frames. Additionally, we investigate the relationships between K-frames, minimal frames, and orthonormal bases, offering several propositions and illustrative examples. These findings extend the existing frame theory in Hilbert spaces to the richer structure of Hilbert C^*-modules, thereby contributing to a deeper understanding of operator theory and functional analysis in the context of C^*-algebras.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been a subject of extensive research since its introduction by Duffin and Schaeffer [5]. Frames generalize the notion of bases and allow for redundant yet stable representations of elements in Hilbert spaces, which is particularly useful in signal processing, harmonic analysis, and other applied fields. The development of frame theory has led to numerous generalizations and extensions, including frames in Hilbert C^*-modules, K-frames, and operator frames.

Hilbert C^*-modules, as introduced by Paschke, are generalizations of Hilbert spaces where the inner product takes values in a C^*-algebra rather than in the field of complex numbers. These modules provide a natural setting for extending operator theory and functional analysis to more general contexts, particularly in relation to C^*-algebras and non-commutative geometry.

In recent years, the study of frames in Hilbert C^*-modules has gained significant attention. Frank and Larson [8] investigated frames in Hilbert C^*-modules and established foundational

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results analogous to those in Hilbert spaces. Furthermore, the concept of K-frames was introduced by Găvruţa [6] to address the need for frames that are compatible with a given bounded operator K, allowing for more flexibility in applications such as sampling theory and reconstruction. For more detailed information on frame theory, readers are recommended to consult: [2,7,11,12,15–18].

Super Hilbert modules [14], which can be viewed as the direct sum of Hilbert C^*-modules, provide a richer structure for exploring frames and K-frames. They allow for the incorporation of additional algebraic or topological properties that may not be present in standard Hilbert modules.

The primary objective of this paper is to investigate K-frames in the context of super Hilbert C^*-modules. We aim to extend the existing theory of frames and K-frames to this setting, providing new insights and generalizations. We begin by reviewing the necessary background on Hilbert C^*-modules, super Hilbert modules, and frames. We then introduce the concept of K-frames in super Hilbert modules and establish various characterizations and properties.

Our main contributions include:

- Providing necessary and sufficient conditions for sequences in super Hilbert modules to be K-frames.
- Exploring the relationships between K-frames, minimal frames, and orthonormal bases in super Hilbert modules.
- Presenting new propositions and examples that illustrate the behavior of K-frames in this setting.

The paper is organized as follows. In Section 1, we review the necessary preliminaries on Hilbert C^*-modules, frames, and K-frames. In Section 2, we present our main results on K-frames in super Hilbert modules, including various propositions and their proofs. Finally, we conclude with remarks on potential future research directions in this area.

We believe that this work contributes to the broader understanding of frame theory in the context of Hilbert C^*-modules and provides a foundation for further exploration of operator theory and functional analysis in settings involving C^*-algebras.

We start by defining the key structures and notations that will be used throughout the paper.

Definition 1.1 ([4]). *A C^*-algebra A is a complex Banach algebra equipped with an involution * satisfying the C^*-identity:*

$$\|a^*a\| = \|a\|^2 \quad \text{for all } a \in A.$$

Definition 1.2 ([10]). *Let A be a unital C^*-algebra and $\mathcal{H}$ a left A-module such that the linear structures of A and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert A-module if $\mathcal{H}$ is equipped with an A-valued inner product*

$$\langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \to A$$

satisfying the following properties:

(i) *For all $x \in \mathcal{H}$, $\langle x, x\rangle_A \geq 0$, and $\langle x, x\rangle_A = 0$ if and only if $x = 0$.*
(ii) *$\langle ax + y, z\rangle_A = a\langle x, z\rangle_A + \langle y, z\rangle_A$, for all $a \in A$ and $x, y, z \in \mathcal{H}$.*

(iii) $\langle x, y\rangle_A = \langle y, x\rangle_A^*$, *for all* $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, *we define* $\|x\| = \|\langle x, x\rangle_A\|_A^{\frac{1}{2}}$. *If* $\mathcal{H}$ *is complete with respect to* $\|\cdot\|$, *it is called a Hilbert* A*-module or a Hilbert* C^**-module over* A.

Definition 1.3 ([14]). *Let* $\mathcal{H}_0$ *and* $\mathcal{H}_1$ *be two Hilbert* $\mathcal{A}$*-modules with inner products* $\langle\cdot,\cdot\rangle_0$ *and* $\langle\cdot,\cdot\rangle_1$, *respectively. The super Hilbert module space* $\mathcal{H}$ *is the direct sum* $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, *equipped with the inner product*

$$\langle x, y\rangle = \langle x_0, y_0\rangle_0 + \langle x_1, y_1\rangle_1,$$

where $x = x_0 + x_1$ *and* $y = y_0 + y_1$, *with* $x_i, y_i \in \mathcal{H}_i$ *for* $i = 0, 1$.

Definition 1.4 ([3]). *Let* $\mathcal{H}$ *and* $\mathcal{K}$ *be Hilbert* A*-modules over a* C^**-algebra* A. *An operator* $T : \mathcal{H} \to \mathcal{K}$ *is called adjointable if there exists an operator* $T^* : \mathcal{K} \to \mathcal{H}$ *such that*

$$\langle Tx, y\rangle_A = \langle x, T^*y\rangle_A \quad \text{for all } x \in \mathcal{H},\ y \in \mathcal{K}.$$

The set of all adjointable operators from $\mathcal{H}$ *to* $\mathcal{K}$ *is denoted by* $\mathrm{End}_A^*(\mathcal{H}, \mathcal{K})$. *When* $\mathcal{H} = \mathcal{K}$, *we abbreviate* $\mathrm{End}_A^*(\mathcal{H}, \mathcal{H})$ *to* $\mathrm{End}_A^*(\mathcal{H})$.

Definition 1.5 ([3]). *Let* $\{x_n\}_{n\geq 1}$ *be a Bessel sequence in* $\mathcal{H}$.

i. *The analysis operator of* $\{x_n\}_{n\geq 1}$, *denoted by* T, *is the bounded adjointable operator defined by*

$$T : \begin{array}{ccc} \mathcal{H} & \longrightarrow & \ell^2(A) \\ x & \mapsto & \{\langle x, x_n\rangle_A\}_{n\geq 1}. \end{array}$$

ii. *The synthesis operator of* $\{x_n\}_{n\geq 1}$ *is the adjoint of its analysis operator, denoted by* T^*. *It is defined explicitly by:*

$$T^* : \begin{array}{ccc} \ell^2(A) & \longrightarrow & \mathcal{H} \\ \{a_n\}_{n\geq 1} & \mapsto & \displaystyle\sum_{n=1}^{+\infty} x_n a_n. \end{array}$$

iii. *The frame operator, denoted by* S, *is the composition of* T^* *and* T. *It is defined explicitly by:*

$$S : \begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathcal{H} \\ x & \mapsto & T^*Tx = \displaystyle\sum_{n=1}^{+\infty} x_n\langle x_n, x\rangle_A. \end{array}$$

Definition 1.6 ([9]). *A sequence* $\{x_n\}_{n\geq 1}$ *in* $\mathcal{H}$ *is said to be a frame for* $\mathcal{H}$ *if there exist constants* $A, B > 0$ *such that for all* $x \in \mathcal{H}$,

$$A\langle x, x\rangle_A \leq \sum_{n=1}^{+\infty} \langle x, x_n\rangle_A \langle x_n, x\rangle_A \leq B\langle x, x\rangle_A.$$

If $A = B$, $\{x_n\}_{n\geq 1}$ *is called a tight frame; if* $A = B = 1$, *it is called a Parseval frame. If only the upper inequality holds,* $\{x_n\}_{n\geq 1}$ *is called a Bessel sequence.*

Definition 1.7 ([13]). *Let $K \in \mathrm{End}_A^*(\mathcal{H})$. A sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{H}$ is said to be a K-frame for $\mathcal{H}$ if there exist constants $A, B > 0$ such that for all $x \in \mathcal{H}$,*

$$A\langle K^*x, K^*x\rangle_A \leq \sum_{n=1}^{+\infty} \langle x, x_n\rangle_A \langle x_n, x\rangle_A \leq B\langle x, x\rangle_A.$$

The numbers A and B are called the lower and upper bounds of the K-frame, respectively.

If there exists $A > 0$ such that $\{x_n\}_{n\geq 1}$ satisfies

$$A\langle K^*x, K^*x\rangle_A = \sum_{n=1}^{+\infty} \langle x, x_n\rangle_A \langle x_n, x\rangle_A \quad \text{for all } x \in \mathcal{H},$$

it is called a tight K-frame. If $A = 1$, it is called a Parseval K-frame.

Example 1.1. *Let $\{e_n\}_{n\geq 1}$ be an orthonormal basis for $\mathcal{H}$. Define $K \in \mathrm{End}_A^*(\mathcal{H})$ such that for all $n \geq 1$, $K(e_n) := e_{n+1}$. The sequence $\{K(e_n)\}_{n\geq 1} = \{e_{n+1}\}_{n\geq 1}$ is not a frame (it is not a complete sequence).*

For all $x \in \mathcal{H}$, we have

$$K^*(x) = \sum_{n=1}^{+\infty} \langle x, K(e_n)\rangle_A e_n,$$

so

$$\|K^*(x)\|^2 = \sum_{n=1}^{+\infty} \|\langle x, K(e_n)\rangle_A\|^2.$$

Hence, $\{K(e_n)\}_{n\geq 1} = \{e_{n+1}\}_{n\geq 1}$ is a Parseval K-frame for $\mathcal{H}$.

Remark 1.1.

i. *If $K = I$, then a K-frame is just an ordinary frame.*
ii. *An ordinary frame for $\mathcal{H}$ is a K-frame for any $K \in \mathrm{End}_A^*(\mathcal{H})$.*
iii. *Let $K \in \mathrm{End}_A^*(\mathcal{H})$. Every K-frame is, in particular, a Bessel sequence, so the synthesis, analysis, and frame operators are well-defined and bounded. Unlike the frame case, the synthesis operator is not surjective, the analysis operator is not injective, and the frame operator is not invertible.*

The following results provide characterizations of K-frames in Hilbert C^*-modules.

Proposition 1.1 ([19]). *Let $\{x_n\}_{n\geq 1}$ be a Bessel sequence in $\mathcal{H}$ with frame operator S and synthesis operator T^*. Let $K \in \mathrm{End}_A^*(\mathcal{H})$. The following statements are equivalent:*

i. *$\{x_n\}$ is a K-frame.*
ii. *There exists a constant $A > 0$ such that $AKK^* \leq S$.*

Proposition 1.2 ([6]). *Let $\{x_n\}_{n\geq 1}$ be a Bessel sequence in $\mathcal{H}$ with frame operator S and synthesis operator T^*. Let $K \in \mathrm{End}_A^*(\mathcal{H})$. The following statements are equivalent:*

i. *$\{x_n\}$ is a K-frame.*
ii. $\mathrm{Range}(K) \subseteq \mathrm{Range}(T^*)$.

Proposition 1.3 ([6]). *Let $\{x_n\}_{n\geq 1}$ be a Bessel sequence in $\mathcal{H}$ with frame operator S and synthesis operator T^*. Let $K \in \mathrm{End}_A^*(\mathcal{H})$. The following statements are equivalent:*

i. $\{x_n\}$ *is a K-frame.*

ii. *There exists a Bessel sequence* $\{f_n\}_{n\geq 1}$ *in* $\mathcal{H}$ *such that for all* $x \in \mathcal{H}$,

$$Kx = \sum_{n=1}^{+\infty} x_n \langle f_n, x\rangle_A.$$

Such a Bessel sequence is called a K-dual frame to $\{x_n\}_{n\geq 1}$.

Remark 1.2. *Let* $\{x_n\}_{n\geq 1}$ *be a K-frame and* $\{f_n\}_{n\geq 1}$ *be a K-dual to* $\{x_n\}_{n\geq 1}$.

i. *For all* $x \in \mathcal{H}$, *we have*

$$K^* x = \sum_{n=1}^{+\infty} f_n \langle x_n, x\rangle_A.$$

This means that $\{f_n\}_{n\geq 1}$ *is a* K^**-frame.*

ii. $\{x_n\}_{n\geq 1}$ *and* $\{f_n\}_{n\geq 1}$ *are interchangeable if and only if K is self-adjoint.*

Proposition 1.4 ([19]). *Let* $\{x_n\}_{n\geq 1}$ *be a frame for* $\mathcal{H}$ *and* $K \in \mathrm{End}_A^*(\mathcal{H})$. *Then* $\{Kx_n\}_{n\geq 1}$ *is a K-frame for* $\mathcal{H}$.

Definition 1.8 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$. *A K-frame for* $\mathcal{H}$ *is said to be K-minimal if its synthesis operator* T^* *is injective.*

Remark 1.3. *A K-minimal frame does not contain zero elements. In other words, if* $\{x_n\}_{n\geq 1}$ *is a K-minimal frame, then* $x_n \neq 0$ *for all* $n \geq 1$.

Proposition 1.5 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$ *and* $\{x_n\}_{n\geq 1}$ *be a K-frame for* $\mathcal{H}$. *Then the following statements are equivalent:*

i. $\{x_n\}_{n\geq 1}$ *has a unique K-dual frame.*

ii. $\{x_n\}_{n\geq 1}$ *is a K-minimal frame.*

Definition 1.9 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$ *and* $\{x_n\}_{n\geq 1}$ *be a sequence in* $\mathcal{H}$. *The sequence* $\{x_n\}_{n\geq 1}$ *is said to be a K-orthonormal basis if:*

i. $\{x_n\}_{n\geq 1}$ *is an orthonormal system in* $\mathcal{H}$.

ii. $\{x_n\}_{n\geq 1}$ *is a Parseval K-frame.*

Theorem 1.1 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$ *be an isometry and* $\{u_n\}_{n\geq 1}$ *an orthonormal basis for* $\mathcal{H}$. *Then the following statements are equivalent:*

i. $\{x_n\}_{n\geq 1}$ *is a K-orthonormal basis for* $\mathcal{H}$.

ii. *There exists an isometry* $L \in \mathrm{End}_A^*(\mathcal{H})$ *such that* $\mathrm{Range}(L) = \mathrm{Range}(K)$ *and for all* $n \geq 1$, $x_n = Lu_n$.

Proposition 1.6 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$ *and* $\{x_n\}_{n\geq 1}$ *be a K-orthonormal basis for* $\mathcal{H}$. *Then* $\{x_n\}_{n\geq 1}$ *has a unique K-dual frame, which is exactly* $\{K^* x_n\}_{n\geq 1}$.

Proposition 1.7 ([1]). *Let* $K \in \mathrm{End}_A^*(\mathcal{H})$ *and* $\{x_n\}_{n\geq 1}$ *be a K-orthonormal basis. Then the following statements are equivalent:*

i. $\{K^* x_n\}_{n\geq 1}$ *is a* K^**-orthonormal basis for* $\mathcal{H}$.

ii. K *is a co-isometry.*

2. Main Results

2.1. Orthogonal Projections in Super Hilbert C^*-Modules.

Proposition 2.1. *The maps*

$$P_1 : \begin{array}{ccc} \mathcal{H}_1 \oplus \mathcal{H}_2 & \longrightarrow & \mathcal{H}_1 \oplus \mathcal{H}_2 \\ x \oplus y & \mapsto & x \oplus 0 \end{array}$$

and

$$P_2 : \begin{array}{ccc} \mathcal{H}_1 \oplus \mathcal{H}_2 & \longrightarrow & \mathcal{H}_1 \oplus \mathcal{H}_2 \\ x \oplus y & \mapsto & 0 \oplus y \end{array}$$

are orthogonal projections on $\mathcal{H}_1 \oplus \mathcal{H}_2$. *Moreover,* $R(P_1) = \mathcal{H}_1 \oplus 0$ *and* $R(P_2) = 0 \oplus \mathcal{H}_2$.

Proof. We first verify that P_1 is adjointable and satisfies $P_1^2 = P_1$. For any $x \oplus y,\ a \oplus b \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$\begin{aligned} \langle P_1(x \oplus y),\, a \oplus b\rangle_{\mathcal{A}} &= \langle x \oplus 0,\, a \oplus b\rangle_{\mathcal{A}} \\ &= \langle x,\, a\rangle_{\mathcal{A}} + \langle 0,\, b\rangle_{\mathcal{A}} \\ &= \langle x \oplus y,\, a \oplus 0\rangle_{\mathcal{A}}. \end{aligned}$$

Thus, P_1 is self-adjoint ($P_1^* = P_1$) and idempotent ($P_1^2 = P_1$). Therefore, P_1 is an orthogonal projection. Similarly, P_2 is an orthogonal projection with $P_2^* = P_2$ and $P_2^2 = P_2$. Furthermore,

$$R(P_1) = \{x \oplus 0 \mid x \in \mathcal{H}_1\} := \mathcal{H}_1 \oplus 0, \quad R(P_2) = \{0 \oplus y \mid y \in \mathcal{H}_2\} := 0 \oplus \mathcal{H}_2.$$

□

2.2. Bessel Sequences in Super Hilbert Modules.

Proposition 2.2. *Let* $\{x_n\}_{n\geq 1} \subset \mathcal{H}_1$ *and* $\{y_n\}_{n\geq 1} \subset \mathcal{H}_2$ *be sequences in the Hilbert* $\mathcal{A}$*-modules* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively. The following statements are equivalent:*

(i) $\{x_n \oplus y_n\}_{n\geq 1}$ *is a Bessel sequence in* $\mathcal{H}_1 \oplus \mathcal{H}_2$.

(ii) $\{x_n\}_{n\geq 1}$ *and* $\{y_n\}_{n\geq 1}$ *are Bessel sequences in* $\mathcal{H}_1$ *and* $\mathcal{H}_2$*, respectively.*

Proof. (i) ⇒ (ii): Assume that $\{x_n \oplus y_n\}_{n\geq 1}$ is a Bessel sequence in $\mathcal{H}_1 \oplus \mathcal{H}_2$ with Bessel bound B. Then, $\{P_1(x_n \oplus y_n)\}_{n\geq 1} = \{x_n \oplus 0\}_{n\geq 1}$ is a Bessel sequence for $\mathcal{H}_1 \oplus 0$ with the same Bessel bound B. For any $x \oplus 0 \in \mathcal{H}_1 \oplus 0$, we have:

$$\sum_{n=1}^{\infty} \langle x \oplus 0,\, x_n \oplus 0\rangle_{\mathcal{A}} \langle x_n \oplus 0,\, x \oplus 0\rangle_{\mathcal{A}} \leq B\langle x \oplus 0,\, x \oplus 0\rangle_{\mathcal{A}}.$$

This implies:

$$\sum_{n=1}^{\infty} \langle x,\, x_n\rangle_{\mathcal{A}} \langle x,\, x_n\rangle_{\mathcal{A}}^* \leq B\langle x,\, x\rangle_{\mathcal{A}}.$$

Thus, $\{x_n\}_{n\geq 1}$ is a Bessel sequence in $\mathcal{H}_1$ with Bessel bound B. Similarly, using $P_2(x_n \oplus y_n) = 0 \oplus y_n$, we can show that $\{y_n\}_{n\geq 1}$ is a Bessel sequence in $\mathcal{H}_2$ with the same Bessel bound B.

(ii) $\Rightarrow$ (i): Assume that $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ are Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, with Bessel bounds B_1 and B_2, respectively. Let $B = \max\{B_1, B_2\}$. For any $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$\sum_{n=1}^{\infty} \|\langle x \oplus y,\, x_n \oplus y_n\rangle_{\mathcal{A}}\|^2 = \sum_{n=1}^{\infty} \|\langle x,\, x_n\rangle_{\mathcal{A}} + \langle y,\, y_n\rangle_{\mathcal{A}}\|^2.$$

Using the norm inequality $\|a+b\|^2 \leq 2\|a\|^2 + 2\|b\|^2$, this gives:

$$\sum_{n=1}^{\infty} \|\langle x \oplus y,\, x_n \oplus y_n\rangle_{\mathcal{A}}\|^2 \leq 2\sum_{n=1}^{\infty} \left(\|\langle x,\, x_n\rangle_{\mathcal{A}}\|^2 + \|\langle y,\, y_n\rangle_{\mathcal{A}}\|^2\right).$$

Since $\{x_n\}$ and $\{y_n\}$ are Bessel sequences, we have:

$$\sum_{n=1}^{\infty} \|\langle x,\, x_n\rangle_{\mathcal{A}}\|^2 \leq B_1\langle x,\, x\rangle_{\mathcal{A}}, \quad \sum_{n=1}^{\infty} \|\langle y,\, y_n\rangle_{\mathcal{A}}\|^2 \leq B_2\langle y,\, y\rangle_{\mathcal{A}}.$$

Combining these, we obtain:

$$\sum_{n=1}^{\infty} \|\langle x \oplus y,\, x_n \oplus y_n\rangle_{\mathcal{A}}\|^2 \leq 2B\langle x \oplus y,\, x \oplus y\rangle_{\mathcal{A}}.$$

Thus, $\{x_n \oplus y_n\}_{n\geq 1}$ is a Bessel sequence in $\mathcal{H}_1 \oplus \mathcal{H}_2$ with Bessel bound $2B$.

□

2.3. **Operators and Frame Transforms in Super Hilbert Modules.**

Proposition 2.3. *Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be Bessel sequences in $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Let T_1, T_2, and T be the synthesis operators of $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, and $\{x_n \oplus y_n\}_{n\geq 1}$, respectively. Let θ_1, θ_2, and θ be the frame transforms of $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, and $\{x_n \oplus y_n\}_{n\geq 1}$, respectively. Let S_1, S_2, and S be the frame operators of $\{x_n\}_{n\geq 1}$, $\{y_n\}_{n\geq 1}$, and $\{x_n \oplus y_n\}_{n\geq 1}$, respectively. Then:*

(i) *For all $a = \{a_n\} \in \ell^2(\mathcal{A})$, $T(a) = T_1(a) \oplus T_2(a)$.*
(ii) *For all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $\theta(x \oplus y) = \theta_1(x) + \theta_2(y)$.*
(iii) *For all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $S(x \oplus y) = S_1(x) + T_1\theta_2(y) \oplus S_2(y) + T_2\theta_1(x)$.*

Proof. (i) For any $a = \{a_n\} \in \ell^2(\mathcal{A})$, we compute:

$$T(a) = \sum_{n=1}^{\infty} (x_n \oplus y_n)a_n = \left(\sum_{n=1}^{\infty} x_n a_n\right) \oplus \left(\sum_{n=1}^{\infty} y_n a_n\right) = T_1(a) \oplus T_2(a).$$

(ii) For any $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we compute:

$$\theta(x \oplus y) = \{\langle x \oplus y,\, x_n \oplus y_n\rangle_{\mathcal{A}}\}_{n\geq 1} = \{\langle x,\, x_n\rangle_{\mathcal{A}} + \langle y,\, y_n\rangle_{\mathcal{A}}\}_{n\geq 1}.$$

Thus:

$$\theta(x \oplus y) = \theta_1(x) + \theta_2(y).$$

(iii) The frame operator S is given by $S = T\theta$. From (i) and (ii), let $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$\begin{aligned} S(x \oplus y) &= T\theta(x \oplus y) \\ &= T(\theta_1(x) + \theta_2(y)) \\ &= T(\theta_1(x)) + T(\theta_2(y)) \\ &= T_1(\theta_1(x)) \oplus T_2(\theta_1(x)) + T_1(\theta_2(y)) \oplus T_2(\theta_2(y)) \\ &= S_1(x) \oplus T_2\theta_1(x) + T_1\theta_2(y) \oplus S_2(y) \\ &= S_1(x) + T_1\theta_2(y) \oplus S_2(y) + T_2\theta_1(x). \end{aligned}$$

Thus, the result is proven. □

Proposition 2.4. *Let $\mathcal{A}$ be a C^*-algebra, $M \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, and $\{x_n \oplus y_n\}_{n\geq 1}$ a sequence in $\mathcal{H}_1 \oplus \mathcal{H}_2$. If $\{x_n \oplus y_n\}_{n\geq 1}$ is an M-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with bounds A and B, then:*

(i) *For all $x \in \mathcal{H}_1$,*

$$A\langle M_1^* x, M_1^* x\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty} \langle x, x_n\rangle_{\mathcal{A}} \langle x, x_n\rangle_{\mathcal{A}}^* \leq B\langle x, x\rangle_{\mathcal{A}}.$$

(ii) *For all $y \in \mathcal{H}_2$,*

$$A\langle M_2^* y, M_2^* y\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty} \langle y, y_n\rangle_{\mathcal{A}} \langle y, y_n\rangle_{\mathcal{A}}^* \leq B\langle y, y\rangle_{\mathcal{A}}.$$

Here, $M_1 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1$ and $M_2 : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_2$ are linear operators such that $M = M_1 \oplus M_2$ on $\mathcal{H}_1 \oplus \mathcal{H}_2$, and $\langle \cdot, \cdot \rangle_{\mathcal{A}}$ is the inner product taking values in $\mathcal{A}$.

Proof. Since $\{x_n \oplus y_n\}_{n\geq 1}$ is an M-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$, by definition, for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$A\langle M^*(x \oplus y), M^*(x \oplus y)\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty} \langle x \oplus y, x_n \oplus y_n\rangle_{\mathcal{A}} \langle x \oplus y, x_n \oplus y_n\rangle_{\mathcal{A}}^* \leq B\langle x \oplus y, x \oplus y\rangle_{\mathcal{A}}.$$

i. Let $y = 0$. Then for all $x \in \mathcal{H}_1$:

$$A\langle M^*(x \oplus 0), M^*(x \oplus 0)\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty} \langle x \oplus 0, x_n \oplus y_n\rangle_{\mathcal{A}} \langle x \oplus 0, x_n \oplus y_n\rangle_{\mathcal{A}}^* \leq B\langle x, x\rangle_{\mathcal{A}}.$$

Note that:

$$\langle x \oplus 0, x_n \oplus y_n\rangle_{\mathcal{A}} = \langle x, x_n\rangle_{\mathcal{A}} + \langle 0, y_n\rangle_{\mathcal{A}} = \langle x, x_n\rangle_{\mathcal{A}}.$$

Therefore:

$$\sum_{n=1}^{+\infty} \langle x \oplus 0, x_n \oplus y_n\rangle_{\mathcal{A}} \langle x \oplus 0, x_n \oplus y_n\rangle_{\mathcal{A}}^* = \sum_{n=1}^{+\infty} \langle x, x_n\rangle_{\mathcal{A}} \langle x, x_n\rangle_{\mathcal{A}}^*.$$

Moreover, since $M^*(x \oplus 0) = M_1^* x \oplus M_2^* 0 = M_1^* x \oplus 0$, we have:

$$\langle M^*(x\oplus 0), M^*(x\oplus 0)\rangle_{\mathcal{A}} = \langle M_1^*x, M_1^*x\rangle_{\mathcal{A}} + \langle 0,0\rangle_{\mathcal{A}} = \langle M_1^*x, M_1^*x\rangle_{\mathcal{A}}.$$

Also, $\langle x\oplus 0, x\oplus 0\rangle_{\mathcal{A}} = \langle x,x\rangle_{\mathcal{A}} + \langle 0,0\rangle = \langle x,x\rangle_{\mathcal{A}}$.

Therefore, we obtain:

$$A\langle M_1^*x, M_1^*x\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty}\langle x,x_n\rangle_{\mathcal{A}}\langle x,x_n\rangle_{\mathcal{A}}^* \leq B\langle x,x\rangle_{\mathcal{A}}.$$

ii. Similarly, let $x=0$. Then for all $y\in\mathcal{H}_2$:

$$A\langle M^*(0\oplus y), M^*(0\oplus y)\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty}\langle y,y_n\rangle_{\mathcal{A}}\langle y,y_n\rangle_{\mathcal{A}}^* \leq B\langle y,y\rangle_{\mathcal{A}}.$$

Note that:

$$\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}} = \langle 0,x_n\rangle_{\mathcal{A}} + \langle y,y_n\rangle_{\mathcal{A}} = \langle y,y_n\rangle_{\mathcal{A}}.$$

Therefore:

$$\sum_{n=1}^{+\infty}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}^* = \sum_{n=1}^{+\infty}\langle y,y_n\rangle_{\mathcal{A}}\langle y,y_n\rangle_{\mathcal{A}}^*.$$

Moreover, since $M^*(0\oplus y) = M_1^*0\oplus M_2^*y = 0\oplus M_2^*y$, we have:

$$\langle M^*(0\oplus y), M^*(0\oplus y)\rangle_{\mathcal{A}} = \langle 0,0\rangle_{\mathcal{A}} + \langle M_2^*y, M_2^*y\rangle_{\mathcal{A}} = \langle M_2^*y, M_2^*y\rangle_{\mathcal{A}}.$$

Also, $\langle 0\oplus y, 0\oplus y\rangle_{\mathcal{A}} = \langle 0,0\rangle_{\mathcal{A}} + \langle y,y\rangle = \langle y,y\rangle_{\mathcal{A}}$.

Therefore, we obtain:

$$A\langle M_2^*y, M_2^*y\rangle_{\mathcal{A}} \leq \sum_{n=1}^{+\infty}\langle y,y_n\rangle_{\mathcal{A}}\langle y,y_n\rangle_{\mathcal{A}}^* \leq B\langle y,y\rangle_{\mathcal{A}}.$$

□

Corollary 2.1. *Let $\mathcal{A}$ be a C^*-algebra, and let $K\in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_1)$ and $L\in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. If the sequence $\{x_n\oplus y_n\}_{n\geq 1}$ in $\mathcal{H}_1\oplus\mathcal{H}_2$ is a $K\oplus L$-frame for $\mathcal{H}_1\oplus\mathcal{H}_2$, then $\{x_n\}_{n\geq 1}$ is a K-frame for $\mathcal{H}_1$, and $\{y_n\}_{n\geq 1}$ is an L-frame for $\mathcal{H}_2$.*

Proof. Since $\{x_n\oplus y_n\}$ is a $K\oplus L$-frame for $\mathcal{H}_1\oplus\mathcal{H}_2$, there exist constants $A,B>0$ such that, for all $x\oplus y\in\mathcal{H}_1\oplus\mathcal{H}_2$:

$$A\langle (K\oplus L)^*(x\oplus y), (K\oplus L)^*(x\oplus y)\rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty}\langle x\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}\langle x\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}^* \leq B\langle x\oplus y, x\oplus y\rangle_{\mathcal{A}}.$$

Note that $(K\oplus L)^*(x\oplus y) = K^*x\oplus L^*y$.

Let $y=0$. Then, for all $x\in\mathcal{H}_1$:

$$A\langle K^*x\oplus 0, K^*x\oplus 0\rangle_{\mathcal{A}} = A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty}\langle x\oplus 0, x_n\oplus y_n\rangle_{\mathcal{A}}\langle x\oplus 0, x_n\oplus y_n\rangle_{\mathcal{A}}^* \leq B\langle x,x\rangle_{\mathcal{A}}.$$

Since:

$$\langle x\oplus 0, x_n\oplus y_n\rangle_{\mathcal{A}} = \langle x, x_n\rangle_{\mathcal{A}} + \langle 0, y_n\rangle_{\mathcal{A}} = \langle x, x_n\rangle_{\mathcal{A}},$$

we have:

$$\sum_{n=1}^{\infty}\langle x\oplus 0, x_n\oplus y_n\rangle_{\mathcal{A}}\langle x\oplus 0, x_n\oplus y_n\rangle_{\mathcal{A}}^* = \sum_{n=1}^{\infty}\langle x, x_n\rangle_{\mathcal{A}}\langle x, x_n\rangle_{\mathcal{A}}^*.$$

Also, since $\langle x\oplus 0, x\oplus 0\rangle_{\mathcal{A}} = \langle x, x\rangle_{\mathcal{A}} + \langle 0, 0\rangle_{\mathcal{A}} = \langle x, x\rangle_{\mathcal{A}}$, we obtain:

$$A\langle K^*x, K^*x\rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty}\langle x, x_n\rangle_{\mathcal{A}}\langle x, x_n\rangle_{\mathcal{A}}^* \leq B\langle x, x\rangle_{\mathcal{A}}.$$

This shows that $\{x_n\}$ is a K-frame for $\mathcal{H}_1$.

Similarly, let $x = 0$. Then, for all $y\in\mathcal{H}_2$:

$$A\langle 0\oplus L^*y, 0\oplus L^*y\rangle_{\mathcal{A}} = A\langle L^*y, L^*y\rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}^* \leq B\langle y, y\rangle_{\mathcal{A}}.$$

Since:

$$\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}} = \langle 0, x_n\rangle_{\mathcal{A}} + \langle y, y_n\rangle_{\mathcal{A}} = \langle y, y_n\rangle_{\mathcal{A}},$$

we have:

$$\sum_{n=1}^{\infty}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}\langle 0\oplus y, x_n\oplus y_n\rangle_{\mathcal{A}}^* = \sum_{n=1}^{\infty}\langle y, y_n\rangle_{\mathcal{A}}\langle y, y_n\rangle_{\mathcal{A}}^*.$$

Also, since $\langle 0\oplus y, 0\oplus y\rangle_{\mathcal{A}} = \langle 0, 0\rangle_{\mathcal{A}} + \langle y, y\rangle_{\mathcal{A}} = \langle y, y\rangle_{\mathcal{A}}$, we obtain:

$$A\langle L^*y, L^*y\rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty}\langle y, y_n\rangle_{\mathcal{A}}\langle y, y_n\rangle_{\mathcal{A}}^* \leq B\langle y, y\rangle_{\mathcal{A}}.$$

This shows that $\{y_n\}$ is an L-frame for $\mathcal{H}_2$. □

Corollary 2.2. *Let $\mathcal{A}$ be a C^*-algebra, and let $K\in\mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_1)$ and $L\in\mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Then there exist a K-frame $\{x_n\}_{n\geq 1}$ for $\mathcal{H}_1$ and an L-frame $\{y_n\}_{n\geq 1}$ for $\mathcal{H}_2$ such that $\{x_n\oplus y_n\}_{n\geq 1}$ is a $K\oplus L$-frame for $\mathcal{H}_1\oplus\mathcal{H}_2$.*

Proof. Take any frames $\{x_n\}_{n\geq 1}$ in $\mathcal{H}_1$ and $\{y_n\}_{n\geq 1}$ in $\mathcal{H}_2$. Then $\{x_n\oplus y_n\}_{n\geq 1}$ is a frame for $\mathcal{H}_1\oplus\mathcal{H}_2$. By the properties of frames in Hilbert $\mathcal{A}$-modules, $\{Kx_n\oplus Ly_n\}_{n\geq 1}$ is a $K\oplus L$-frame for $\mathcal{H}_1\oplus\mathcal{H}_2$.

By Corollary 2.2 (which states that the components of a $K\oplus L$-frame are K-frames and L-frames, respectively), $\{Kx_n\}_{n\geq 1}$ is a K-frame for $\mathcal{H}_1$ and $\{Ly_n\}_{n\geq 1}$ is an L-frame for $\mathcal{H}_2$. □

This lemma is very useful for what follows.

Lemma 2.1. *Let $K\in\mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_1)$ and $L\in\mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_2)$. Then $(K\oplus L)^* = K^*\oplus L^*$.*

Proof. Let $x \oplus y$, $a \oplus b \in \mathcal{H}_1 \oplus \mathcal{H}_2$. We have:

$$\langle (K \oplus L)(x \oplus y), a \oplus b \rangle_{\mathcal{A}} = \langle Kx \oplus Ly, a \oplus b \rangle_{\mathcal{A}} = \langle Kx, a \rangle_{\mathcal{A}} + \langle Ly, b \rangle_{\mathcal{A}}.$$

Using the adjoint properties in Hilbert $\mathcal{A}$-modules, we know:

$$\langle Kx, a \rangle_{\mathcal{A}} = \langle x, K^* a \rangle_{\mathcal{A}}, \quad \langle Ly, b \rangle_{\mathcal{A}} = \langle y, L^* b \rangle_{\mathcal{A}}.$$

Therefore:

$$\langle (K \oplus L)(x \oplus y), a \oplus b \rangle_{\mathcal{A}} = \langle x, K^* a \rangle_{\mathcal{A}} + \langle y, L^* b \rangle_{\mathcal{A}} = \langle x \oplus y, K^* a \oplus L^* b \rangle_{\mathcal{A}}.$$

Hence:

$$\langle (K \oplus L)(x \oplus y), a \oplus b \rangle_{\mathcal{A}} = \langle x \oplus y, (K^* \oplus L^*)(a \oplus b) \rangle_{\mathcal{A}}.$$

Thus, we conclude that $(K \oplus L)^* = K^* \oplus L^*$. □

The following proposition shows that there is no $K \oplus L$-frame for $\mathcal{H} \oplus \mathcal{H}$ of the form $\{x_n \oplus x_n\}_{n \geq 1}$ whenever $K, L \neq 0$.

Proposition 2.5. *Let $\mathcal{A}$ be a C^*-algebra, and let $K, L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H})$, where $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module. Let $\{x_n \oplus x_n\}_{n \geq 1}$ be a Bessel sequence for $\mathcal{H} \oplus \mathcal{H}$. Then:*

$$\{x_n \oplus x_n\}_{n \geq 1} \text{ is a } K \oplus L\text{-frame for } \mathcal{H} \oplus \mathcal{H} \iff K = L = 0.$$

Proof. (i) Assume that $K \neq 0$ or $L \neq 0$. Without loss of generality, suppose $K \neq 0$. Let $x \in \mathcal{H}$ such that $K^* x \neq 0$. Then:

$$\|(K^* \oplus L^*)(x \oplus (-x))\|^2 = \langle K^* x \oplus L^*(-x), K^* x \oplus L^*(-x) \rangle_{\mathcal{A}}$$

$$= \langle K^* x, K^* x \rangle_{\mathcal{A}} + \langle L^*(-x), L^*(-x) \rangle_{\mathcal{A}} \geq \langle K^* x, K^* x \rangle_{\mathcal{A}} \neq 0.$$

However, for each $n \geq 1$:

$$\langle x \oplus (-x), x_n \oplus x_n \rangle_{\mathcal{A}} = \langle x, x_n \rangle_{\mathcal{A}} + \langle -x, x_n \rangle_{\mathcal{A}} = \langle x, x_n \rangle_{\mathcal{A}} - \langle x, x_n \rangle_{\mathcal{A}} = 0.$$

Therefore:

$$\sum_{n=1}^{\infty} \langle x \oplus (-x), x_n \oplus x_n \rangle_{\mathcal{A}} \langle x \oplus (-x), x_n \oplus x_n \rangle_{\mathcal{A}}^* = 0.$$

Thus, the frame inequality:

$$A \langle (K^* \oplus L^*)(x \oplus (-x)), (K^* \oplus L^*)(x \oplus (-x)) \rangle_{\mathcal{A}} \leq \sum_{n=1}^{\infty} \langle x \oplus (-x), x_n \oplus x_n \rangle_{\mathcal{A}} \langle x \oplus (-x), x_n \oplus x_n \rangle_{\mathcal{A}}^*$$

cannot hold because the left-hand side is non-zero while the right-hand side is zero. Therefore, $\{x_n \oplus x_n\}$ is not a $K \oplus L$-frame for $\mathcal{H} \oplus \mathcal{H}$.

(ii) Conversely, suppose that $K = L = 0$. Then $K^* = L^* = 0$. For any $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$, we have:

$$\langle (K^* \oplus L^*)(x \oplus y),\, (K^* \oplus L^*)(x \oplus y)\rangle_{\mathcal{A}} = \langle 0 \oplus 0,\, 0 \oplus 0\rangle_{\mathcal{A}} = 0.$$

Since $\{x_n \oplus x_n\}$ is a Bessel sequence, the right-hand side of the frame inequality is finite. Therefore, the frame inequality:

$$0 \leq \sum_{n=1}^{\infty} \langle x \oplus y,\, x_n \oplus x_n\rangle_{\mathcal{A}} \langle x \oplus y,\, x_n \oplus x_n\rangle_{\mathcal{A}}^* \leq B\langle x \oplus y,\, x \oplus y\rangle_{\mathcal{A}}$$

holds trivially for all $x \oplus y \in \mathcal{H} \oplus \mathcal{H}$. Thus, $\{x_n \oplus x_n\}$ is a $0 \oplus 0$-frame for $\mathcal{H} \oplus \mathcal{H}$.

□

This proposition demonstrates that for two frames, the direct sum of which does not constitute a frame for the super Hilbert module $\mathcal{H} \oplus \mathcal{H}$, if $\{x_n\}$ is a K-frame and $\{y_n\}$ is an L-frame, their combined frames $\{x_n \oplus y_n\}$ may not necessarily form a $K \oplus L$-frame.

Here is a simple example illustrating this principle, which directly follows from Proposition 2.5.

Example 2.1. *Let $K \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H})$ such that $K \neq 0$. Take any K-frame $\{x_n\}_{n\geq 1}$ for $\mathcal{H}$. Then $\{x_n \oplus x_n\}_{n\geq 1}$ is not a $K \oplus K$-frame for the super Hilbert module $\mathcal{H} \oplus \mathcal{H}$.*

Proposition 2.6. *Let $\mathcal{A}$ be a C^*-algebra, and let $K \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_1)$ and $L \in \mathrm{End}_{\mathcal{A}}^*(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $\{x_n\}_{n\geq 1}$ be a K-frame for $\mathcal{H}_1$ and $\{y_n\}_{n\geq 1}$ be an L-frame for $\mathcal{H}_2$. Let θ_1 and θ_2 be the analysis operators (frame transforms) of $\{x_n\}$ and $\{y_n\}$, respectively.*

If $R(\theta_1) \perp R(\theta_2)$ in $\ell^2(\mathcal{A})$, then $\{x_n \oplus y_n\}_{n\geq 1}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Proof. Denote by A_1 and A_2 the lower frame bounds for $\{x_n\}$ and $\{y_n\}$, respectively, and let $A = \min\{A_1, A_2\}$.

Since $R(\theta_1) \perp R(\theta_2)$ in $\ell^2(\mathcal{A})$, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, we have

$$\langle \theta_1(x), \theta_2(y)\rangle_{\ell^2(\mathcal{A})} = 0.$$

Recall that the inner product in $\ell^2(\mathcal{A})$ is defined by

$$\langle \{a_n\}, \{b_n\}\rangle_{\ell^2(\mathcal{A})} = \sum_{n=1}^{\infty} a_n^* b_n.$$

Therefore, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

$$\sum_{n=1}^{\infty} \langle x_n, x\rangle_{\mathcal{A}}^* \langle y_n, y\rangle_{\mathcal{A}} = 0.$$

This implies that

$$\langle T_2\theta_1(x), y\rangle_{\mathcal{A}} = \sum_{n=1}^{\infty} \langle y_n, y\rangle_{\mathcal{A}}^* \langle x_n, x\rangle_{\mathcal{A}} = 0,$$

where T_2 is the synthesis operator of $\{y_n\}$.

Similarly,

$$\langle T_1\theta_2(y), x\rangle_{\mathcal{A}} = \sum_{n=1}^{\infty}\langle x_n, x\rangle_{\mathcal{A}}^*\langle y_n, y\rangle_{\mathcal{A}} = 0,$$

where T_1 is the synthesis operator of $\{x_n\}$.

Therefore, $T_2\theta_1 = 0$ and $T_1\theta_2 = 0$.

The frame operator S of $\{x_n \oplus y_n\}$ is given by

$$S(x \oplus y) = T\theta(x \oplus y) = T(\theta_1 x \oplus \theta_2 y) = T_1\theta_1 x \oplus T_2\theta_2 y = S_1 x \oplus S_2 y,$$

where $T = T_1 \oplus T_2$ is the synthesis operator for $\{x_n \oplus y_n\}$, and $S_1 = T_1\theta_1$ and $S_2 = T_2\theta_2$ are the frame operators for $\{x_n\}$ and $\{y_n\}$, respectively.

Then, for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$,

$$\langle S(x \oplus y), x \oplus y\rangle_{\mathcal{A}} = \langle S_1 x, x\rangle_{\mathcal{A}} + \langle S_2 y, y\rangle_{\mathcal{A}}.$$

Using the frame inequalities for $\{x_n\}$ and $\{y_n\}$, we have

$$\langle S_1 x, x\rangle_{\mathcal{A}} \geq A_1\langle K^* x, K^* x\rangle_{\mathcal{A}},$$

and

$$\langle S_2 y, y\rangle_{\mathcal{A}} \geq A_2\langle L^* y, L^* y\rangle_{\mathcal{A}}.$$

Therefore,

$$\langle S(x \oplus y), x \oplus y\rangle_{\mathcal{A}} \geq A_1\langle K^* x, K^* x\rangle_{\mathcal{A}} + A_2\langle L^* y, L^* y\rangle_{\mathcal{A}} \geq A\langle K^* x \oplus L^* y, K^* x \oplus L^* y\rangle_{\mathcal{A}}.$$

Since $K \oplus L$ acts on $\mathcal{H}_1 \oplus \mathcal{H}_2$, and $(K \oplus L)^* = K^* \oplus L^*$, we have

$$\langle S(x \oplus y), x \oplus y\rangle_{\mathcal{A}} \geq A\langle (K \oplus L)^*(x \oplus y), (K \oplus L)^*(x \oplus y)\rangle_{\mathcal{A}}.$$

Thus, the lower frame condition holds for $\{x_n \oplus y_n\}$ as a $K \oplus L$-frame. Since $\{x_n \oplus y_n\}$ is the direct sum of Bessel sequences (due to $\{x_n\}$ and $\{y_n\}$ being frames), the upper frame condition is also satisfied. Therefore, $\{x_n \oplus y_n\}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. □

Example 2.2. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module that admits a countable orthonormal basis $\{e_n\}_{n\geq 1}$. Define submodules M and N of $\mathcal{H}$ by:*

$$M = \overline{span}\{e_{2n} : n \geq 1\}, \quad N = \overline{span}\{e_{2n-1} : n \geq 1\}.$$

Let P and Q be the orthogonal projections from $\mathcal{H}$ onto M and N, respectively. Then $\{Pe_n \oplus Qe_n\}_{n\geq 1}$ is a $P \oplus Q$-frame for $\mathcal{H} \oplus \mathcal{H}$.

In fact, $\{Pe_n\}_{n\geq 1}$ and $\{Qe_n\}_{n\geq 1}$ are P-frames and Q-frames for $\mathcal{H}$, respectively. Let θ_1 and θ_2 be their analysis operators (frame transforms). Then for all $x, y \in \mathcal{H}$, we have:

$$\langle \theta_1(x),\, \theta_2(y)\rangle_{\ell^2(\mathcal{A})} = \sum_{n=1}^{\infty}\langle x,\, Pe_n\rangle_{\mathcal{A}}^*\langle y,\, Qe_n\rangle_{\mathcal{A}} = 0,$$

since $\langle x,\, Pe_n\rangle_{\mathcal{A}}^*\langle y,\, Qe_n\rangle_{\mathcal{A}} = 0$ because M and N are orthogonal submodules.

Therefore, $R(\theta_1) \perp R(\theta_2)$ in $\ell^2(\mathcal{A})$. By Proposition 2.6, $\{Pe_n \oplus Qe_n\}_{n\geq 1}$ is a $P \oplus Q$-frame for $\mathcal{H} \oplus \mathcal{H}$.

Proposition 2.7. *Let $\mathcal{A}$ be a C^*-algebra, and let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $\{x_n\}_{n\geq 1}$ be a K-frame for $\mathcal{H}_1$ and $\{y_n\}_{n\geq 1}$ be an L-frame for $\mathcal{H}_2$. If $\{x_n \oplus y_n\}_{n\geq 1}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$, then:*

$$\begin{cases} R(K) \subseteq T_1(N(T_2)), \\ R(L) \subseteq T_2(N(T_1)), \end{cases}$$

where T_1 and T_2 are the synthesis operators for $\{x_n\}$ and $\{y_n\}$, respectively.

Proof. Assume that $\{x_n \oplus y_n\}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$, and let T be its synthesis operator. By the properties of frames in Hilbert $\mathcal{A}$-modules (similar to Proposition 1.9 in the Hilbert space case), we have $R(K \oplus L) = R(T)$. Then for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, there exists $a \in \ell^2(\mathcal{A})$ such that:

$$Kx \oplus Ly = Ta.$$

This implies that:

$$Kx = T_1 a, \quad Ly = T_2 a,$$

since $T = T_1 \oplus T_2$ and $a = \{a_n\}_{n\geq 1}$ is in $\ell^2(\mathcal{A})$.

Now, by taking $y = 0$, for any $x \in \mathcal{H}_1$, there exists $a \in \ell^2(\mathcal{A})$ such that:

$$Kx = T_1 a, \quad 0 = Ly = T_2 a.$$

Thus, $a \in N(T_2)$ (the kernel of T_2). Therefore:

$$R(K) \subseteq T_1(N(T_2)).$$

Similarly, by taking $x = 0$, for any $y \in \mathcal{H}_2$, there exists $a \in \ell^2(\mathcal{A})$ such that:

$$0 = Kx = T_1 a, \quad Ly = T_2 a.$$

Thus, $a \in N(T_1)$, and therefore:

$$R(L) \subseteq T_2(N(T_1)).$$

□

This proposition shows that the non-minimality of the two sequences $\{x_n\}$ and $\{y_n\}$ is necessary for their direct sum to be a $K \oplus L$-frame whenever $K, L \neq 0$.

Corollary 2.3. *Let $\mathcal{A}$ be a C^*-algebra, and let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $\{x_n \oplus y_n\}_{n\geq 1}$ be a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then:*

(i) *If $\{x_n\}_{n\geq 1}$ is K-minimal, then $L = 0$.*
(ii) *If $\{y_n\}_{n\geq 1}$ is L-minimal, then $K = 0$.*
(iii) *If $\{x_n\}_{n\geq 1}$ is K-minimal and $\{y_n\}_{n\geq 1}$ is L-minimal, then $K = 0$ and $L = 0$.*

In particular: If $K \neq 0$ and $L \neq 0$, and at least one of $\{x_n\}$ or $\{y_n\}$ is minimal, then $\{x_n \oplus y_n\}$ is never a $K \oplus L$-frame for the super Hilbert module $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Proof. (i) Assume that $\{x_n\}$ is K-minimal. This means that the synthesis operator $T_1 : \ell^2(\mathcal{A}) \to \mathcal{H}_1$ is injective, so $N(T_1) = \{0\}$. By Proposition 2.7, we have:

$$R(L) \subseteq T_2(N(T_1)) = T_2(\{0\}) = \{0\}.$$

Therefore, $R(L) = \{0\}$, which implies $L = 0$.

(ii) Similarly, if $\{y_n\}$ is L-minimal, then $N(T_2) = \{0\}$. By Proposition 2.7:

$$R(K) \subseteq T_1(N(T_2)) = T_1(\{0\}) = \{0\}.$$

Hence, $R(K) = \{0\}$, so $K = 0$.

(iii) This follows directly from parts (i) and (ii). If both sequences are minimal, then $K = 0$ and $L = 0$.

□

Example 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra, and let $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$ be orthonormal bases for Hilbert $\mathcal{A}$-modules $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Define operators $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$ by:*

$$K(e_n) = e_{2n}, \quad L(f_n) = f_{2n-1}, \quad \forall n \geq 1.$$

Let $\{x_n\}_{n\geq 1} = \{e_{2n}\}$ and $\{y_n\}_{n\geq 1} = \{f_{2n-1}\}$. Then $\{x_n\}$ is a K-frame for $\mathcal{H}_1$, and $\{y_n\}$ is an L-frame for $\mathcal{H}_2$. However, $\{x_n \oplus y_n\}$ is not a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Explanation: The sequence $\{x_n\}$ is minimal because its synthesis operator T_1 is injective (since $\{e_{2n}\}$ are linearly independent over $\mathcal{A}$). By Corollary 2.3, since $\{x_n\}$ is K-minimal and $K \neq 0$, it must be that $L = 0$. However, $L(f_n) = f_{2n-1} \neq 0$, which contradicts $L = 0$. Therefore, $\{x_n \oplus y_n\}$ cannot be a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

Corollary 2.4. *If $\{x_n\}_{n\geq 1}$ is a Riesz basis for a Hilbert $\mathcal{A}$-module $\mathcal{H}_1$, then there is no sequence $\{y_n\}_{n\geq 1}$ in any Hilbert $\mathcal{A}$-module $\mathcal{H}_2$ such that $\{x_n \oplus y_n\}$ is a frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

Proof. In this case, K is the identity operator $I_{\mathcal{H}_1}$ on $\mathcal{H}_1$, and L is the identity on $\mathcal{H}_2$. Since $\{x_n\}$ is a Riesz basis, it is minimal, and its synthesis operator T_1 is invertible (hence injective). By Corollary 2.3 (part (i)), since $\{x_n\}$ is K-minimal and $K \neq 0$, it must be that $L = 0$. However, L is the identity operator on $\mathcal{H}_2$ and cannot be zero unless $\mathcal{H}_2 = \{0\}$. Therefore, no such sequence $\{y_n\}$ exists. □

2.4. K-Duality, L-Duality, and $K \oplus L$-Duality.

We examine the relationship between K-duality, L-duality, and $K \oplus L$-duality in the context of Hilbert $\mathcal{A}$-modules. This subsection focuses on establishing connections between these concepts and their role in the structure of frames and operators within super Hilbert modules.

Proposition 2.8. *Let $K \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_2)$. Let $\{x_n \oplus y_n\}_{n\geq 1}$ be a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then $\{a_n \oplus b_n\}_{n\geq 1}$ is a $K \oplus L$-dual frame to $\{x_n \oplus y_n\}$ if and only if $\{a_n\}$ is a K-dual frame to $\{x_n\}$ and $\{b_n\}$ is an L-dual frame to $\{y_n\}$.*

Proof. Suppose $\{a_n \oplus b_n\}$ is a $K \oplus L$-dual frame to $\{x_n \oplus y_n\}$. Then for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, a_n \oplus b_n \rangle_{\mathcal{A}} (x_n \oplus y_n).$$

This equality decomposes into:

$$\begin{cases} Kx = \sum_{n=1}^{\infty} \langle x,\, a_n \rangle_{\mathcal{A}} x_n + \langle y,\, b_n \rangle_{\mathcal{A}} x_n, \\ Ly = \sum_{n=1}^{\infty} \langle x,\, a_n \rangle_{\mathcal{A}} y_n + \langle y,\, b_n \rangle_{\mathcal{A}} y_n. \end{cases}$$

However, since $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$ are independent, and $a_n \in \mathcal{H}_1$, $b_n \in \mathcal{H}_2$, the cross terms vanish (because $\langle y,\, a_n \rangle_{\mathcal{A}} = 0$ and $\langle x,\, b_n \rangle_{\mathcal{A}} = 0$). Therefore, we have:

$$\begin{cases} Kx = \sum_{n=1}^{\infty} \langle x,\, a_n \rangle_{\mathcal{A}} x_n, \\ Ly = \sum_{n=1}^{\infty} \langle y,\, b_n \rangle_{\mathcal{A}} y_n. \end{cases}$$

Thus, $\{a_n\}$ is a K-dual frame to $\{x_n\}$, and $\{b_n\}$ is an L-dual frame to $\{y_n\}$.

Conversely, if $\{a_n\}$ is a K-dual frame to $\{x_n\}$ and $\{b_n\}$ is an L-dual frame to $\{y_n\}$, then for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$:

$$\begin{cases} Kx = \sum_{n=1}^{\infty} \langle x,\, a_n \rangle_{\mathcal{A}} x_n, \\ Ly = \sum_{n=1}^{\infty} \langle y,\, b_n \rangle_{\mathcal{A}} y_n. \end{cases}$$

Therefore, for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \left(\langle x,\, a_n \rangle_{\mathcal{A}} x_n \oplus \langle y,\, b_n \rangle_{\mathcal{A}} y_n \right) = \sum_{n=1}^{\infty} \langle x \oplus y,\, a_n \oplus b_n \rangle_{\mathcal{A}} (x_n \oplus y_n).$$

Thus, $\{a_n \oplus b_n\}$ is a $K \oplus L$-dual frame to $\{x_n \oplus y_n\}$. □

Proposition 2.9. *Let $K \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \text{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $\{x_n\}_{n\geq 1}$ be a K-frame for $\mathcal{H}_1$ with K-dual frame $\{f_n\}_{n\geq 1}$, and $\{y_n\}_{n\geq 1}$ be an L-frame for $\mathcal{H}_2$ with L-dual frame $\{g_n\}_{n\geq 1}$. Then the following statements are equivalent:*

i. *$\{x_n \oplus y_n\}_{n\geq 1}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with $K \oplus L$-dual frame $\{f_n \oplus g_n\}_{n\geq 1}$.*
ii. *$T_2\theta_1 = 0_{H_1}$ and $T_1\theta_2 = 0_{H_2}$, where T_1 and T_2 are the synthesis operators of $\{x_n\}$ and $\{y_n\}$, respectively, and θ_1 and θ_2 are the analysis operators of $\{f_n\}$ and $\{g_n\}$, respectively.*

Proof.

(i) ⇒ (ii): Assume that $\{x_n \oplus y_n\}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with $K \oplus L$-dual frame $\{f_n \oplus g_n\}$. Then for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, f_n \oplus g_n \rangle_{\mathcal{A}} \cdot (x_n \oplus y_n).$$

The inner product in the Hilbert $\mathcal{A}$-module $\mathcal{H}_1 \oplus \mathcal{H}_2$ splits as:

$$\langle x \oplus y,\, f_n \oplus g_n \rangle_{\mathcal{A}} = \langle x,\, f_n \rangle_{\mathcal{A}} + \langle y,\, g_n \rangle_{\mathcal{A}}.$$

Therefore,

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \left(\langle x,\, f_n \rangle_{\mathcal{A}} + \langle y,\, g_n \rangle_{\mathcal{A}} \right) \cdot (x_n \oplus y_n).$$

Expanding the right-hand side, we get:

$$Kx = \sum_{n=1}^{\infty} \langle x,\, f_n \rangle_{\mathcal{A}} x_n + \sum_{n=1}^{\infty} \langle y,\, g_n \rangle_{\mathcal{A}} x_n,$$
$$Ly = \sum_{n=1}^{\infty} \langle x,\, f_n \rangle_{\mathcal{A}} y_n + \sum_{n=1}^{\infty} \langle y,\, g_n \rangle_{\mathcal{A}} y_n.$$

However, since $x_n \in \mathcal{H}_1$ and $y_n \in \mathcal{H}_2$, and $\langle y,\, g_n \rangle_{\mathcal{A}} x_n \in \mathcal{H}_1$, the term $\sum_n \langle y,\, g_n \rangle_{\mathcal{A}} x_n$ is in $\mathcal{H}_1$.

Similarly, $\langle x,\, f_n \rangle_{\mathcal{A}} y_n \in \mathcal{H}_2$.

Therefore, we have:

$$\begin{cases} Kx = T_1\theta_1(x) + T_1\theta_2(y), \\ Ly = T_2\theta_1(x) + T_2\theta_2(y). \end{cases}$$

Since $Kx = T_1\theta_1(x) + T_1\theta_2(y)$ and $Kx = \sum_n \langle x,\, f_n \rangle_{\mathcal{A}} x_n + \sum_n \langle y,\, g_n \rangle_{\mathcal{A}} x_n$, we can write:

$$Kx = T_1\theta_1(x) + T_1\theta_2(y).$$

Similarly,

$$Ly = T_2\theta_1(x) + T_2\theta_2(y).$$

But since $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, and $\theta_2(y) \in \ell^2(\mathcal{A})$, $T_1\theta_2(y) \in \mathcal{H}_1$, and $T_2\theta_1(x) \in \mathcal{H}_2$.

For the equality to hold for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, it must be that:

$$\begin{cases} T_1\theta_2(y) = 0_{H_1}, \quad \forall y \in \mathcal{H}_2, \\ T_2\theta_1(x) = 0_{H_2}, \quad \forall x \in \mathcal{H}_1. \end{cases}$$

Therefore, $T_1\theta_2 = 0_{H_1}$ and $T_2\theta_1 = 0_{H_2}$.

(ii) $\Rightarrow$ (i): Assume that $T_1\theta_2 = 0$ and $T_2\theta_1 = 0$. Then for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, we have:

$$Kx = T_1\theta_1(x) + T_1\theta_2(y) = T_1\theta_1(x) + 0 = \sum_{n=1}^{\infty} \langle x,\, f_n \rangle_{\mathcal{A}} x_n,$$
$$Ly = T_2\theta_1(x) + T_2\theta_2(y) = 0 + T_2\theta_2(y) = \sum_{n=1}^{\infty} \langle y,\, g_n \rangle_{\mathcal{A}} y_n.$$

Therefore,

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \left(\langle x,\, f_n \rangle_{\mathcal{A}} x_n \oplus \langle y,\, g_n \rangle_{\mathcal{A}} y_n \right).$$

Since $\langle x \oplus y,\, f_n \oplus g_n \rangle_{\mathcal{A}} = \langle x,\, f_n \rangle_{\mathcal{A}} + \langle y,\, g_n \rangle_{\mathcal{A}}$, we can write:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, f_n \oplus g_n \rangle_{\mathcal{A}} \cdot (x_n \oplus y_n).$$

Therefore, $\{x_n \oplus y_n\}$ is a $K \oplus L$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$ with $K \oplus L$-dual frame $\{f_n \oplus g_n\}$.

□

2.5. $K \oplus L$-**Minimal Frames.**

In this part, we study $K \oplus L$-minimal frames for super Hilbert modules over a C^*-algebra $\mathcal{A}$.

The following result establishes a necessary and sufficient condition for an M-frame in $\mathcal{H}_1 \oplus \mathcal{H}_2$ to be an M-minimal frame.

Proposition 2.10. *Let $M \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Let $\{x_n \oplus y_n\}_{n \geq 1}$ be an M-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then the following statements are equivalent:*

(i) *$\{x_n \oplus y_n\}_{n \geq 1}$ is an M-minimal frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

(ii) *$N(T_1) \cap N(T_2) = \{0\}$,*

where T_1 and T_2 are the synthesis operators of $\{x_n\}$ and $\{y_n\}$, respectively.

Proof. Let $T : \ell^2(\mathcal{A}) \to \mathcal{H}_1 \oplus \mathcal{H}_2$ be the synthesis operator of $\{x_n \oplus y_n\}$, defined by

$$T(a) = \sum_{n=1}^{\infty} (x_n \oplus y_n) a_n = T_1(a) \oplus T_2(a),$$

for all $a = \{a_n\} \in \ell^2(\mathcal{A})$.

Then,

$$N(T) = \{a \in \ell^2(\mathcal{A}) : T(a) = 0\} = \{a \in \ell^2(\mathcal{A}) : T_1(a) = 0 \text{ and } T_2(a) = 0\} = N(T_1) \cap N(T_2).$$

Recall that a frame is minimal if its synthesis operator is injective, i.e., $N(T) = \{0\}$.

Therefore,

$$\{x_n \oplus y_n\} \text{ is an } M\text{-minimal frame} \iff N(T) = \{0\} \iff N(T_1) \cap N(T_2) = \{0\}.$$

□

The following result provides a sufficient condition for $\{x_n \oplus y_n\}$ to be a $K \oplus L$-minimal frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

We will need the following lemma.

Lemma 2.2. *Let $\mathcal{H}$ be a Hilbert $\mathcal{A}$-module, and let $A, B \subseteq \mathcal{H}$ be closed submodules. Then the following statements are equivalent:*

(i) *$A = B^{\perp}$.*

(ii) (a) *$A \perp B$,*

(b) *$A^{\perp} \cap B^{\perp} = \{0\}$.*

Proof. (i) $\Rightarrow$ (ii): Assume that $A = B^{\perp}$. Then $A \subseteq B^{\perp}$, so $A \perp B$. Also, $A^{\perp} = (B^{\perp})^{\perp} = \overline{B}$, so

$$A^{\perp} \cap B^{\perp} = \overline{B} \cap B^{\perp} = B \cap B^{\perp} = \{0\}.$$

(ii) $\Rightarrow$ (i): Assume that $A \perp B$ and $A^{\perp} \cap B^{\perp} = \{0\}$. Since $A \perp B$, we have $A \subseteq B^{\perp}$. Let $x \in B^{\perp}$. Then $x \in (A^{\perp})^{\perp}$, because if $x \in A^{\perp}$, then $x \in A^{\perp} \cap B^{\perp} = \{0\}$, so $x = 0$. Therefore, $x \in A$. Thus, $B^{\perp} \subseteq A$, and since $A \subseteq B^{\perp}$, we have $A = B^{\perp}$.

□

Proposition 2.11. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, and let $\{x_n\}$ be a K-frame for $\mathcal{H}_1$ and $\{y_n\}$ be an L-frame for $\mathcal{H}_2$. Let θ_1 and θ_2 be the analysis operators of $\{x_n\}$ and $\{y_n\}$, respectively. If $R(\theta_1) = R(\theta_2)^{\perp}$ in $\ell^2(\mathcal{A})$, then $\{x_n \oplus y_n\}$ is a $K \oplus L$-minimal frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

Proof. Since $R(\theta_1) = R(\theta_2)^{\perp}$, we have $R(\theta_1) \perp R(\theta_2)$ and, by Lemma 2.2, $R(\theta_1)^{\perp} \cap R(\theta_2)^{\perp} = \{0\}$.

Recall that θ_i^* is the synthesis operator T_i, so $N(T_i) = N(\theta_i^*) = (R(\theta_i))^{\perp}$.

Thus,

$$N(T_1) = R(\theta_1)^{\perp} = R(\theta_2), \quad N(T_2) = R(\theta_2)^{\perp} = R(\theta_1).$$

Since $R(\theta_1)^{\perp} \cap R(\theta_2)^{\perp} = \{0\}$, it follows that $N(T_1) \cap N(T_2) = \{0\}$.

By Proposition 2.10, $\{x_n \oplus y_n\}$ is a $K \oplus L$-minimal frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. □

Example 2.4. *The following example illustrates an instance of a $K \oplus L$-minimal frame for a super Hilbert module. Let $\mathcal{A}$ be a unital C^*-algebra, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be countably generated Hilbert $\mathcal{A}$-modules with orthonormal bases $\{e_n\}_{n\geq 1}$ and $\{f_n\}_{n\geq 1}$, respectively.*

Define sequences $\{x_n\}_{n\geq 1} \subset \mathcal{H}_1$ and $\{y_n\}_{n\geq 1} \subset \mathcal{H}_2$ as follows, for all $n \geq 1$:

$$x_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ e_n & \text{if } n \text{ is even.} \end{cases}$$

$$y_n = \begin{cases} 0 & \text{if } n \text{ is even,} \\ f_n & \text{if } n \text{ is odd.} \end{cases}$$

Define operators $K \in B(\mathcal{H}_1)$ and $L \in B(\mathcal{H}_2)$ by:

$$K(e_n) = e_{2n}, \quad \forall n \geq 1,$$

$$L(f_n) = f_{2n-1}, \quad \forall n \geq 1.$$

We need to verify that K and L are adjointable operators.

Firstly, define K^* on the basis elements e_{2n} by:

$$K^*(e_{2n}) = e_n, \quad \forall n \geq 1,$$

and $K^*(e_k) = 0$ if k is odd.

Similarly for L^*:

$$L^*(f_{2n-1}) = f_n, \quad \forall n \geq 1,$$

and $L^*(f_k) = 0$ *if* k *is even.*

One can verify that K *and* K^* *satisfy the adjoint relationship:*

$$\langle Kx, y\rangle_{\mathcal{A}} = \langle x, K^*y\rangle_{\mathcal{A}}, \quad \forall x, y \in \mathcal{H}_1,$$

and similarly for L *and* L^* *on* $\mathcal{H}_2$*.*

Now, for any $x \in \mathcal{H}_1$*:*

$$\|K^*x\|^2 = \|\langle K^*x, K^*x\rangle_{\mathcal{A}}\| = \left\|\sum_{n=1}^{\infty}\langle K^*x, e_n\rangle_{\mathcal{A}}^*\langle K^*x, e_n\rangle_{\mathcal{A}}\right\|.$$

But $K^*x = \sum_{n=1}^{\infty}\langle x, Ke_n\rangle_{\mathcal{A}}e_n$*, and since* $Ke_n = e_{2n}$*, we have:*

$$K^*x = \sum_{n=1}^{\infty}\langle x, e_{2n}\rangle_{\mathcal{A}}e_n.$$

Therefore,

$$\|K^*x\|^2 = \left\|\sum_{n=1}^{\infty}\langle x, e_{2n}\rangle_{\mathcal{A}}^*\langle x, e_{2n}\rangle_{\mathcal{A}}\right\| = \sum_{n=1}^{\infty}\|\langle x, e_{2n}\rangle_{\mathcal{A}}\|^2.$$

Similarly, for $y \in \mathcal{H}_2$*:*

$$\|L^*y\|^2 = \sum_{n=1}^{\infty}\|\langle y, f_{2n-1}\rangle_{\mathcal{A}}\|^2.$$

Therefore, for all $x \in \mathcal{H}_1$*:*

$$\|K^*x\|^2 = \sum_{n=1}^{\infty}\|\langle x, x_n\rangle_{\mathcal{A}}\|^2,$$

since $x_n = e_n$ *when* n *is even and* $x_n = 0$ *when* n *is odd.*

Similarly for $y \in \mathcal{H}_2$*:*

$$\|L^*y\|^2 = \sum_{n=1}^{\infty}\|\langle y, y_n\rangle_{\mathcal{A}}\|^2.$$

Thus, $\{x_n\}$ *is a K-frame for* $\mathcal{H}_1$*, and* $\{y_n\}$ *is an L-frame for* $\mathcal{H}_2$*.*

Now, let's compute the analysis operators θ_1 *and* θ_2*.*

For $x \in \mathcal{H}_1$*:*

$$\theta_1(x) = \{\langle x_n, x\rangle_{\mathcal{A}}\}_{n\geq 1} = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ \langle e_n, x\rangle_{\mathcal{A}}, & \text{if } n \text{ is even.} \end{cases}$$

Thus, $\theta_1(x)$ *has non-zero entries only at even positions.*

Therefore, the range $R(\theta_1)$ *is the submodule of* $\ell^2(\mathcal{A})$ *spanned by* $\{\delta_{2n}\}_{n\geq 1}$*, where* $\{\delta_n\}$ *is the standard orthonormal basis of* $\ell^2(\mathcal{A})$*.*

Similarly, for $y \in \mathcal{H}_2$*:*

$$\theta_2(y) = \{\langle y_n, y\rangle_{\mathcal{A}}\}_{n\geq 1} = \begin{cases} \langle f_n, y\rangle_{\mathcal{A}}, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases}$$

Thus, $\theta_2(y)$ has non-zero entries only at odd positions.

Therefore, $R(\theta_2)$ is the submodule of $\ell^2(\mathcal{A})$ spanned by $\{\delta_{2n-1}\}_{n\geq 1}$.

Since $R(\theta_1)$ and $R(\theta_2)$ are orthogonal in $\ell^2(\mathcal{A})$, and $R(\theta_1) = R(\theta_2)^{\perp}$, we can apply Proposition 2.11 to conclude that $\{x_n \oplus y_n\}$ is a $K \oplus L$-minimal frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

2.6. $K \oplus L$-Orthonormal Bases.

In this section, we investigate $K \oplus L$-orthonormal bases for super Hilbert modules over a C^*-algebra $\mathcal{A}$.

Proposition 2.12. *Let $\{x_n\}_{n\geq 1}$ and $\{y_n\}_{n\geq 1}$ be two orthonormal systems in Hilbert $\mathcal{A}$-modules $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then $\{x_n \oplus y_n\}_{n\geq 1}$ is never an orthonormal system for the super Hilbert module $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

Proof. For any $n \geq 1$,

$$\langle x_n \oplus y_n, x_n \oplus y_n\rangle_{\mathcal{A}} = \langle x_n, x_n\rangle_{\mathcal{A}} + \langle y_n, y_n\rangle_{\mathcal{A}} = 1_{\mathcal{A}} + 1_{\mathcal{A}} = 2 \cdot 1_{\mathcal{A}} \neq 1_{\mathcal{A}}.$$

Thus, $\{x_n \oplus y_n\}$ cannot be an orthonormal system for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

□

The following result provides necessary conditions on $\{x_n\}$ and $\{y_n\}$ for $\{x_n \oplus y_n\}$ to be a $K \oplus L$-orthonormal basis.

Proposition 2.13. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Let $\{x_n \oplus y_n\} \subset \mathcal{H}_1 \oplus \mathcal{H}_2$.*

(i) *If $K = 0$ and $L \neq 0$, then:*

$$\{x_n \oplus y_n\} \text{ is a } K \oplus L\text{-orthonormal basis} \iff \begin{cases} \forall n \geq 1,\ x_n = 0, \\ \{y_n\} \text{ is an } L\text{-orthonormal basis for } \mathcal{H}_2. \end{cases}$$

(ii) *If $K \neq 0$ and $L = 0$, then:*

$$\{x_n \oplus y_n\} \text{ is a } K \oplus L\text{-orthonormal basis} \iff \begin{cases} \forall n \geq 1,\ y_n = 0, \\ \{x_n\} \text{ is a } K\text{-orthonormal basis for } \mathcal{H}_1. \end{cases}$$

(iii) *If $K \neq 0$ and $L \neq 0$, and $\{x_n \oplus y_n\}$ is a $K \oplus L$-orthonormal basis, then:*
   (a) *$\{K^*(x_n) \oplus L^*(y_n)\}$ is the unique $K \oplus L$-dual frame to $\{x_n \oplus y_n\}$.*
   (b) *$\{x_n\}$ is a non-minimal K-frame for $\mathcal{H}_1$ whose $\{K^*(x_n)\}$ is a K-dual frame.*
   (c) *$\{y_n\}$ is a non-minimal L-frame for $\mathcal{H}_2$ whose $\{L^*(y_n)\}$ is an L-dual frame.*

Proof. (i) Assume $K = 0$ and $L \neq 0$, and that $\{x_n \oplus y_n\}$ is a $K \oplus L$-orthonormal basis. Then, for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$\|L^*(y)\|^2 = \|K^*(x) \oplus L^*(y)\|^2 = \left\| \sum_{n=1}^{\infty} \langle x \oplus y,\, x_n \oplus y_n \rangle_{\mathcal{A}} \cdot (x_n \oplus y_n) \right\|.$$

Taking $y = 0$, we have

$$\|K^*(x)\|^2 = \|K^*(x) \oplus 0\|^2 = \sum_{n=1}^{\infty} \|\langle x,\, x_n \rangle_{\mathcal{A}}\|^2 = 0.$$

Since $K = 0$, $K^* = 0$, so $\|K^*(x)\|^2 = 0$ for all $x \in \mathcal{H}_1$, which is consistent. However, the sum $\sum_{n=1}^{\infty} \|\langle x,\, x_n \rangle_{\mathcal{A}}\|^2 = 0$ implies $\langle x,\, x_n \rangle_{\mathcal{A}} = 0$ for all n and all $x \in \mathcal{H}_1$. Therefore, $x_n = 0$ for all $n \geq 1$.

Next, for all $y \in \mathcal{H}_2$,

$$\|L^*(y)\|^2 = \sum_{n=1}^{\infty} \|\langle y,\, y_n \rangle_{\mathcal{A}}\|^2.$$

Also, for all $n, m \geq 1$,

$$\langle y_n,\, y_m \rangle_{\mathcal{A}} = \langle x_n \oplus y_n,\, x_m \oplus y_m \rangle_{\mathcal{A}} = \delta_{n,m} \cdot 1_{\mathcal{A}}.$$

Hence, $\{y_n\}$ is an L-orthonormal basis for $\mathcal{H}_2$.

Conversely, if $x_n = 0$ for all $n \geq 1$ and $\{y_n\}$ is an L-orthonormal basis for $\mathcal{H}_2$, then $\{x_n \oplus y_n\}$ is a $K \oplus L$-orthonormal basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$.

(ii) The proof is similar to that of part (i) by interchanging the roles of K and L, and $\mathcal{H}_1$ and $\mathcal{H}_2$.

(iii) Assume $K \neq 0$ and $L \neq 0$, and $\{x_n \oplus y_n\}$ is a $K \oplus L$-orthonormal basis. Then, by definition, for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$,

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, x_n \oplus y_n \rangle_{\mathcal{A}} \cdot (x_n \oplus y_n).$$

This implies that

$$Kx = \sum_{n=1}^{\infty} \langle x,\, x_n \rangle_{\mathcal{A}} x_n + \langle y,\, y_n \rangle_{\mathcal{A}} x_n,$$

$$Ly = \sum_{n=1}^{\infty} \langle x,\, x_n \rangle_{\mathcal{A}} y_n + \langle y,\, y_n \rangle_{\mathcal{A}} y_n.$$

However, since $x_n \in \mathcal{H}_1$ and $y_n \in \mathcal{H}_2$ are orthogonal elements (modules over $\mathcal{A}$), the cross terms vanish, and we have:

$$Kx = \sum_{n=1}^{\infty} \langle x,\, x_n \rangle_{\mathcal{A}} x_n,$$

$$Ly = \sum_{n=1}^{\infty} \langle y,\, y_n \rangle_{\mathcal{A}} y_n.$$

Thus, $\{x_n\}$ and $\{y_n\}$ are frames for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

(a) The dual frame $\{K^*(x_n) \oplus L^*(y_n)\}$ satisfies:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, K^*(x_n) \oplus L^*(y_n)\rangle_{\mathcal{A}} \cdot (x_n \oplus y_n).$$

Since $K^*(x_n) = K^*(x_n)$ and $L^*(y_n) = L^*(y_n)$, this shows that $\{K^*(x_n) \oplus L^*(y_n)\}$ is the unique $K \oplus L$-dual frame of $\{x_n \oplus y_n\}$.

(b) Since the synthesis operator T_1 of $\{x_n\}$ is not injective (as $\{x_n \oplus y_n\}$ is an orthonormal basis but x_n alone does not span $\mathcal{H}_1$ minimally due to the presence of y_n), $\{x_n\}$ is a non-minimal K-frame for $\mathcal{H}_1$. The set $\{K^*(x_n)\}$ serves as a K-dual frame to $\{x_n\}$.

(c) Similarly, $\{y_n\}$ is a non-minimal L-frame for $\mathcal{H}_2$, and $\{L^*(y_n)\}$ is an L-dual frame to $\{y_n\}$.

□

By the above proposition, we deduce that if $\{x_n \oplus y_n\}_{n\geq 1}$ is a $K \oplus L$-orthonormal basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$, then $\{K^*(x_n) \oplus L^*(y_n)\}_{n\geq 1}$ is a $K^* \oplus L^*$-frame for $\mathcal{H}_1 \oplus \mathcal{H}_2$. One may wonder under which conditions $\{K^*(x_n) \oplus L^*(y_n)\}$ is a $K^* \oplus L^*$-orthonormal basis.

We will need the following lemmas.

Lemma 2.3. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert $\mathcal{A}$-modules. Then the following statements are equivalent:*

(i) *$K \oplus L$ is an isometry in $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

(ii) *K and L are both isometries.*

Proof. (i) $\Rightarrow$ (ii) Assume that $K \oplus L$ is an isometry in $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$, we have:

$$\langle Kx \oplus Ly,\, Kx \oplus Ly\rangle_{\mathcal{A}} = \langle x \oplus y,\, x \oplus y\rangle_{\mathcal{A}}.$$

This implies:

$$\langle Kx,\, Kx\rangle_{\mathcal{A}} + \langle Ly,\, Ly\rangle_{\mathcal{A}} = \langle x,\, x\rangle_{\mathcal{A}} + \langle y,\, y\rangle_{\mathcal{A}}.$$

Since this holds for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$, we have:

$$\langle Kx,\, Kx\rangle_{\mathcal{A}} = \langle x,\, x\rangle_{\mathcal{A}}, \quad \forall x \in \mathcal{H}_1,$$

and

$$\langle Ly,\, Ly\rangle_{\mathcal{A}} = \langle y,\, y\rangle_{\mathcal{A}}, \quad \forall y \in \mathcal{H}_2.$$

Therefore, K and L are isometries.

(ii) $\Rightarrow$ (i) Assume that K and L are both isometries. Then for all $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_2$:

$$\langle Kx \oplus Ly,\, Kx \oplus Ly\rangle_{\mathcal{A}} = \langle Kx,\, Kx\rangle_{\mathcal{A}} + \langle Ly,\, Ly\rangle_{\mathcal{A}} = \langle x,\, x\rangle_{\mathcal{A}} + \langle y,\, y\rangle_{\mathcal{A}} = \langle x \oplus y,\, x \oplus y\rangle_{\mathcal{A}}.$$

Therefore, $K \oplus L$ is an isometry in $\mathcal{H}_1 \oplus \mathcal{H}_2$.

□

Lemma 2.4. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$. Then the following statements are equivalent:*

(i) *$K \oplus L$ is a co-isometry in $\mathcal{H}_1 \oplus \mathcal{H}_2$.*

(ii) *K and L are both co-isometries.*

Proof. Recall that an adjointable operator K is a co-isometry if $KK^* = I_{\mathcal{H}_1}$.

We have:

$$(K \oplus L)(K^* \oplus L^*) = KK^* \oplus LL^*.$$

Thus, $K \oplus L$ is a co-isometry if and only if $KK^* = I_{\mathcal{H}_1}$ and $LL^* = I_{\mathcal{H}_2}$, i.e., K and L are both co-isometries.

□

Proposition 2.14. *Let $K \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_1)$ and $L \in \mathrm{End}^*_{\mathcal{A}}(\mathcal{H}_2)$. Let $\{x_n \oplus y_n\}_{n\geq 1}$ be a $K \oplus L$-orthonormal basis for the super Hilbert module $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then the following statements are equivalent:*

(i) *$\{K^*(x_n) \oplus L^*(y_n)\}_{n\geq 1}$ is a $K^* \oplus L^*$-orthonormal basis.*

(ii) *K and L are both co-isometries.*

Proof. First, note that since $\{x_n \oplus y_n\}$ is a $K \oplus L$-orthonormal basis, we have for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$Kx \oplus Ly = \sum_{n=1}^{\infty} \langle x \oplus y,\, x_n \oplus y_n \rangle_{\mathcal{A}} \cdot (x_n \oplus y_n).$$

Consider the sequence $\{K^*(x_n) \oplus L^*(y_n)\}$. We aim to show that this sequence is a $K^* \oplus L^*$-orthonormal basis if and only if K and L are co-isometries.

(i) Suppose $\{K^*(x_n) \oplus L^*(y_n)\}$ is a $K^* \oplus L^*$-orthonormal basis for $\mathcal{H}_1 \oplus \mathcal{H}_2$. Then, for all $x \oplus y \in \mathcal{H}_1 \oplus \mathcal{H}_2$:

$$K^*x \oplus L^*y = \sum_{n=1}^{\infty} \langle x \oplus y,\, K^*(x_n) \oplus L^*(y_n) \rangle_{\mathcal{A}} \cdot (K^*(x_n) \oplus L^*(y_n)).$$

Since $K^*(x_n) \oplus L^*(y_n) = (K^* \oplus L^*)(x_n \oplus y_n)$, the above equation becomes:

$$K^*x \oplus L^*y = \sum_{n=1}^{\infty} \langle x \oplus y,\, (K^* \oplus L^*)(x_n \oplus y_n) \rangle_{\mathcal{A}} \cdot (K^*x_n \oplus L^*y_n).$$

This suggests that $K^* \oplus L^*$ acts similarly to a frame operator associated with $\{x_n \oplus y_n\}$.

Since $\{K^*(x_n) \oplus L^*(y_n)\}$ is a $K^* \oplus L^*$-orthonormal basis, it follows that $K^* \oplus L^*$ is an isometry. Therefore, K^* and L^* are isometries, which implies that K and L are co-isometries (since the adjoint of an isometry is a co-isometry).

(ii) Conversely, if K and L are co-isometries, then K^* and L^* are isometries. Therefore, $K^* \oplus L^*$ is an isometry, and the sequence $\{K^*(x_n) \oplus L^*(y_n)\}$ is an orthonormal system.

To show that it is a $K^* \oplus L^*$-orthonormal basis, we need to verify that it satisfies the frame condition. Since $\{x_n \oplus y_n\}$ is a complete system in $\mathcal{H}_1 \oplus \mathcal{H}_2$, and $K^* \oplus L^*$ is an isometry, it follows that $\{K^*(x_n) \oplus L^*(y_n)\}$ is also complete.

Therefore, $\{K^*(x_n) \oplus L^*(y_n)\}$ is a $K^* \oplus L^*$-orthonormal basis.

□

Conclusion

In this paper, we have explored the concept of K-frames in the context of super Hilbert modules over a C^*-algebra $\mathcal{A}$. We have established relationships between K-frames, L-frames, and $K \oplus L$-frames, and provided several propositions and examples to illustrate these concepts. The results extend known theories in Hilbert spaces to Hilbert $\mathcal{A}$-modules, opening avenues for further research in operator theory and functional analysis within the framework of C^*-algebras.

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