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A New Generalized Differential Transform Method for Analytical Solutions of the Bagley-Torvik Equation

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Abstract. This study utilizes the New Generalized Differential Transform method to approximate solutions for the nonlinear Bagley-Torvik equation with fractional order, while considering boundary conditions. Three problems were solved to evaluate the accuracy and efficacy of the proposed numerical methods. The computational results are presented through tables and figures. By comparing these results with alternative methods documented in existing literature, the superiority of the proposed method is demonstrated. The findings suggest that the method is not only effective but also straightforward to implement, providing a high level of accuracy in solving fractional boundary value problems.

1. Introduction

Over the years, fractional-order calculus has been regarded as a vital branch in the mathematical field, with most problems in various fields, such as mechanics [1], physics [2], medicine [3], and other fields modeled through fractional differential equations (FDEs). A particular subset of these equations, known as fractional boundary value problems (FBVPs), combines fractional derivatives with boundary conditions (BCs), posing unique analytical and computational challenges. FBVPs are prominent in numerous applications such as viscoelasticity [4], control theory [5], and fluid dynamics [6]. Their complexity arises from the inherent nonlocality and memory effects introduced by the fractional derivatives, making their analysis and solution more intricate than classical boundary value problems. Addressing these challenges effectively requires robust and efficient numerical methods. A number of numerical techniques have been developed to address FBVPs, such as the Simplified Reproducing Kernel Method (SRKM) [7], the Laplace Transform Homotopy

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Perturbation Method (LT-HPM) [8], and the Bessel Collection Method (BCM) [9]. Each of these techniques offers distinct advantages and limitations, necessitating a thorough evaluation of their performance across different types of FBVPs.

The Differential Transform Method (DTM) is a semi-analytical numerical technique originally developed for solving linear and nonlinear differential equations [10]. Introduced in the early 1980s, DTM transforms differential equations into algebraic equations by employing a Taylor series expansion, allowing for the rapid and accurate determination of solutions in series form. Over time, DTM has been extended and refined to handle a broader class of problems, including partial differential equations, systems of differential equations, and fractional differential equations [11,12].

The New Generalized Differential Transform Method (NGDTM) [13] represents a significant advancement of the traditional DTM, specifically tailored to enhance accuracy and efficiency in solving nonlinear differential equations, including those of fractional order. This method demonstrates particular efficacy in addressing complex boundary value problems, where conventional approaches may encounter difficulties.

In the early 1980s, Torvic and Bagley formulated an equation [14], which is a second order fractional differential equation (FDE), that first appeared when they intended to model the viscoelastic behavior of either the geological strata, metals, or glasses through a fractional differential equation. Since then, the equation has been known to simulate the motion of a rigid plate immersed in a Newtonian fluid. However, as per Diethelm and Ford [15], solutions to this equation require complex mathematical techniques.

This paper aims for the NGDTM to enhance its capability to tackle BVPs. Given the way boundary value problems with fractional order have gained popularity in various fields, this development will come in handy as it will improve the results. The modifications proposed in this essay will involve refining and introducing new techniques to manage the challenges caused by fractional calculus. In this regard, the Bagley-Torvic equation will serve as the primary case study in the application of the modified NGDTM. The results section of this paper will present analytical solutions from the Bagley-Torvic equation that will be obtained once the NGDTM is used. Thereafter, a comparison will be made to compare and improve the efficiency and accuracy of handling non-linearity. The findings will go a long way in contributing to the advancements of fractional calculus and provide valuable insights regarding the behavior of systems governed by the Bagley-Torvic equation.

2. Basic Fractional Calculus Definitions

In this section, some basic definition of fractional calculus.

Definition 1. *For $\alpha \in R^+$, $x > 0$, and $n \in N$, where N are the positive integers, the Riemann-Liouville integral is defined as [16]:*

$$I_x^y f(y) = \frac{a}{\Gamma(\alpha)} \int_a^x f(y)(x-y)^{\alpha-1}\, dy. \tag{2.1}$$

Definition 2. *Consider a real number v and an integer n such that $n-1 < v \leq n$. Then, the Riemann-Liouville fractional derivative of order v is determined as follows [17]:*

$$D^{\nu}_{a+}f(x) = \frac{1}{\Gamma(n-\nu)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\nu-1}f(t)\ dt. \tag{2.2}$$

Theorem 1. *If $D_a^{j\alpha}f$ is continuous on $(a,b]$ for $j = 0,1,\dots,n+1$, where $0 < \alpha \leq 1$, and $x \in [a,b]$, then [18]:*

$$f(x) \cong \sum_{j=0}^{N} \frac{C_j}{\Gamma(\alpha j+\alpha)}(x-a)^{j\alpha+\alpha-1}, \tag{2.3}$$

where

$$c_j = \Gamma(\alpha)[(x-a)^{1-\alpha}D_a^{j\alpha}f(x)](a^+).$$

3. New Generalized Differential Transform Method

In this section, we introduce a NGDTM [13]. Utilizing the generalized Taylor's formula 1, we construct the generalized differential transform for the k^{th} derivative of the function $f(x)$ in a single variable.

$$F(k) = \frac{C_k}{\Gamma(\alpha k+\alpha)} = \frac{\Gamma(\alpha)[(x-a)^{1-\alpha}D_a^{k\alpha}f(x)](a^+)}{\Gamma(\alpha k+\alpha)}, \tag{3.1}$$

The inverse differential transform of $F(k)$ is represented as:

$$f(x) = \sum_{j=0}^{N} \frac{C_j}{\Gamma(\alpha j+\alpha)}(x-a)^{j\alpha+\alpha-1} = \sum_{k=0}^{\infty} F_\alpha(k)(x-a)^{\alpha k+\alpha-1}, \tag{3.2}$$

Hence, Equation (3.2) symbolizes the reversal of the generalized differential transform outlined in Equation (3.1). In practical contexts, we will estimate the function $f(x)$ through a finite series approach, as stipulated in Theorem (1).

$$f(x) = \sum_{k=0}^{n} F_\alpha(k)(x-a)^{\alpha k+\alpha-1}. \tag{3.3}$$

Suppose $H_\alpha(k)$, $F_\alpha(k)$, and $G_\alpha(k)$ be the fractional NGDT of the functions h(x), f(x), and g(x), respectively, for any number $m \in \mathbb{R}$, Table (1) delineates the fundamental properties of the NGDTM, which are derived from definitions (3.1) and (3.3) [13].

4. The Analytical Solutions of the Bagley-Torvik Equation

In this section, we present an analytical solution to the Bagley-Torvik equation using the modified New Generalized Differential Transform Method (NGDTM). Consider the Bagley-Torvik equation:

$$D^2y(x) = \frac{1}{A}\left[f(x) - B\ D^{\frac{3}{2}}y(x) - C\ y(x)\right], \tag{4.1}$$

with BCs:

$$\begin{cases} y(x_0) = \beta_0, \\ y(x_\rho) = \beta_\rho, \end{cases} \tag{4.2}$$

TABLE 1. The fundamental properties of the NGDTM [13].

Original function	Transformed function
$g(x) = m\,h(x)$	$G(x) = m\ H(x)$
$g(x) = f(x) \mp g(x)$	$G(x) = F(x) \mp H(x)$
$g(x) = f(x)h(x)$	$G(k) = x^{\alpha-1}\sum_{i=0}^{k} F(k)H(k-i)$
$g(x) = D_a^{\alpha}h(x)$	$G(k) = \frac{\Gamma(\alpha k+2\alpha)}{\Gamma(\alpha k+\alpha)}H(k+1)$
$g(x) = D_a^{\beta}h(x)$	$G(k) = \frac{\Gamma(\alpha k+\beta+\alpha)}{\Gamma(\alpha k+\alpha)}H(k+\frac{\beta}{\alpha})$
$g(x) = (x-a)^{\alpha m}$	$G(k) = \delta(k-m), \quad \delta(k) = \begin{cases} \frac{\Gamma(\alpha)\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha)}x^{1-\alpha} & \text{if } k=0 \\ 0 & \text{if } k \neq 0 \end{cases}$

where (A$\neq$ 0), B, C, β_0, and β_ρ are constants, $\rho \in \mathbb{Z}^+$, and $f(x)$ is a continuous function.

With the help of the new generalized differential transformation method from equation (4.1), by using the theorems in Table (1), we derive

$$\frac{\Gamma(\alpha k+2+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{2}{\alpha}) = \frac{1}{A}\left[F_\alpha(k) - C\,Y_\alpha(k) - \frac{B\,\Gamma(k\,\alpha+\frac{3}{2}+\alpha)}{\Gamma(\alpha+k\,\alpha)}\,Y_\alpha(k+\frac{3}{2\alpha})\right], \tag{4.3}$$

Simplifying the above equation, we obtain:

$$Y_\alpha(k+\frac{2}{\alpha}) = \frac{\Gamma(\alpha+\alpha k)}{A\,\Gamma(\alpha k+2+\alpha)}\left[F_\alpha(k) - C\,Y_\alpha(k) - \frac{B\,\Gamma(\alpha+\frac{3}{2}+\alpha k)}{\Gamma(\alpha\,k+\alpha)}\,Y_\alpha(k+\frac{3}{2\alpha})\right], \tag{4.4}$$

where $Y_\alpha(k)$ and $F_\alpha(k)$ are the new generalized differential transform functions of $y(x)$ and $f(x)$, respectively.

The NGDTM of BCs Eq.(4.2) is given by

$$\begin{cases} Y_\alpha(x_0) = \beta_0, \\ \sum_{k=0}^{\infty} Y_\alpha(k)(x)^{\alpha(k+1)-1} = \beta_\rho, \end{cases} \tag{4.5}$$

Given that the initial conditions are specified for integer-order derivatives, the transformation of these initial conditions according to the New Generalized Differential Transform Method (NGDTM) is defined as follows:

$$Y_\alpha(k) = \begin{cases} \left(\frac{\Gamma(\alpha)\Gamma(\alpha k+1)\ x^{1-\alpha}}{\Gamma(\alpha k+\alpha)}\right)\beta_{\alpha\,k}, & \text{if} \quad \alpha\,k \in \mathbb{Z}^+ \\ 0, & \text{otherwise} \end{cases} \tag{4.6}$$

where $k = 0, 1, ..., \frac{n}{\alpha} - 1$, whereas the largest integer derivative is n.

It's crucial to note that the number of distinct transformed initial conditions $Y(k)$ is dependent on the order of the higher integer derivative n as well as the parameter α's selected value. Given this consideration, it is essential to exercise care and thoughtful consideration of α when dealing with these initial conditions.

To determine the NGDTM of boundary conditions Eq. (4.2), we will choose $\alpha = \frac{1}{2}$, which is in line with the order of the higher integer derivative $n = 2$. After that, using Eq. (4.6), we can obtain the transformed BCs $Y(k)$ as follows:

$$\begin{cases} Y_{\frac{1}{2}}(0) = x^{0.5}\beta_0, & \text{if} \quad k = 0 \\ Y_{\frac{1}{2}}(1) = 0, & \text{if} \quad k = 1 \\ Y_{\frac{1}{2}}(2) = \gamma, & \text{if} \quad k = 2 \\ Y_{\frac{1}{2}}(3) = 0, & \text{if} \quad k = 3 \end{cases} \tag{4.7}$$

where γ is an unknown parameter.

The inverse NGDTM of Eq. (4.4) and factoring in Eq. (4.7), after making a small simplification, the approximate solution is as follows:

$$y(x) \cong \sum_{k=0}^{N+\frac{n}{\alpha}} Y_{\frac{1}{2}}(k)(x)^{\frac{1}{2}(k+1)-1}, \tag{4.8}$$

where N order of iterations.

In order to calculate the value of γ, it is required that Eq.(4.8) satisfies the boundary condition $y(x_\rho) = \beta_\rho$. This yields a singular algebraic equation for γ, which can be solved using the Maple program to find the value of γ.

5. Numerical Solution of the Bagley Torvik Equation

In this section, we provide a numerical solution of the Bagley-Torvik equation using NGDTM. Three different problems have been examined using this technique. The solutions obtained are very similar to those obtained using exact solution methods and several deformed techniques.

Exampel 1. *Consider the following Bagley Torvik equation [9]:*

$$D_{\alpha}^{\frac{3}{2}}\, y(x) + D_{\alpha}^{2}\, y(x) + y(x) = x + 1, \tag{5.1}$$

where

$$x \in [0,1], \quad 0 \le \alpha \le 1,$$

subject to the BCs,

$$\begin{cases} y(0) = 1, \\ y(1) = 2. \end{cases} \tag{5.2}$$

The exact solution is

$$y(x) = x + 1. \tag{5.3}$$

By applying the NGDTM to both sides of Eq. (5.1), utilizing the theorems from Table (1), the Eq. (5.1) transforms into:

$$\frac{\Gamma(\alpha k+2+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{2}{\alpha})+\frac{\Gamma(\alpha k+\frac{3}{2}+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{\frac{3}{2}}{\alpha})+Y_\alpha(k)=\delta_\alpha(k)+\delta_\alpha(k-\frac{1}{\alpha}), \tag{5.4}$$

Making simple the above equation, we obtain:

$$\frac{\Gamma(\alpha k+2+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{2}{\alpha})=-\frac{\Gamma(\alpha k+\frac{3}{2}+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{\frac{3}{2}}{\alpha})-Y_\alpha(k)+\delta_\alpha(k)+\delta_\alpha(k-\frac{1}{\alpha}). \tag{5.5}$$

In particular, selecting $\alpha=\frac{1}{2}$, the Eq.(5.5) converts to:

$$Y_{\frac{1}{2}}(k+4)=\frac{\Gamma(\frac{k}{2}+\frac{1}{2})}{\Gamma(\frac{k}{2}+\frac{5}{2})}\Bigg[-\frac{\Gamma(\frac{k}{2}+2)}{\Gamma(\frac{k}{2}+\frac{1}{2})}\ Y_{\frac{1}{2}}(k+3)-Y_{\frac{1}{2}}(k)+\delta_{\frac{1}{2}}(k)+\delta_{\frac{1}{2}}(k-2)\Bigg]. \tag{5.6}$$

By using Eq.(4.5), the NGDTM of BCs Eq.(5.2) is given by:

$$\begin{cases} Y_\alpha(0)=1, \\ \sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k)(1)^{\frac{1}{2}(k+1)-1}=2. \end{cases} \tag{5.7}$$

Since the initial conditions are implemented for the integer-order derivatives, the NGDTM defines the transformation of the initial conditions are as follows:

$$Y_{\frac{1}{2}}(k)=\begin{cases} \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)\ x^{0.5}}{\Gamma(\alpha k+\frac{1}{2})}\right)\beta_{\frac{k}{2}}, & \text{if} \quad \frac{k}{2}\in\mathbb{Z}^+ \\ 0, & \text{otherwise.} \end{cases} \tag{5.8}$$

where k = 0, 1,2,3.

By using Eq.(5.8), we obtain the transformed BCs for Eq.(5.2) by:

$$\begin{cases} Y_{\frac{1}{2}}(0)=x^{0.5}, & \text{if} \quad k=0 \\ Y_{\frac{1}{2}}(1)=0, & \text{if} \quad k=1 \\ Y_{\frac{1}{2}}(2)=\gamma\, x^{0.5}, & \text{if} \quad k=2 \\ Y_{\frac{1}{2}}(3)=0, & \text{if} \quad k=3 \end{cases} \tag{5.9}$$

where γ is an unknown parameter.

In this example, there is no need for additional calculations as the transformation of the boundary term is adequate to achieve a precise solution, and no further computations are required. Now substituting Eq. (5.9) into Eq. (4.8), we attain:

$$y(x)=1+\gamma\, x. \tag{5.10}$$

To calculate the value of γ, it is required that Eq.(5.10) satisfies the boundary condition (5.8). The value of γ is:

$$\gamma=1. \tag{5.11}$$

Substituting Eq. (5.11) into Eq. (5.10), we get the subsequent exact solution

$$y(x) = 1 + x. \tag{5.12}$$

Table 2 compares the absolute error between two mathematical methods, NGDT and BCM (Boundary Characteristic Method), over the interval $[0, 1]$ in Example 1. Despite NGDT only employing a boundary condition transformation, it demonstrates superior accuracy compared to BCM, which relies on nine terms. This implies that the exact solution is achieved without additional conversions or complex mathematical operations required by other methods. The advancements in boundary condition selection have led to increased accuracy and faster convergence.

Table 2. Comparison of absolute errors between BCM [9] and NGDTM on [0,1] for example 1.

x	BCM	NGDTM
0	0	0
0.1	9.374×10^{-16}	0
0.2	3.963×10^{-15}	0
0.3	4.283×10^{-15}	0
0.4	3.297×10^{-15}	0
0.5	2.045×10^{-15}	0
0.6	1.027×10^{-15}	0
0.7	3.477×10^{-15}	0
0.8	6.928×10^{-15}	0
0.9	2.394×10^{-15}	0
1.0	1.200×10^{-15}	0

Exampel 2. *Consider the Bagley Torvik Equation BVP [8]:*

$$D_{\alpha}^{\frac{3}{2}}\, y(x) + D_{\alpha}^{2}\, y(x) + y(x) = x^2 + x + \frac{4}{\sqrt{\pi}} x^{0.5} + 3, \tag{5.13}$$

where,

$$1 < \alpha \leq 2, \quad x \in [0, 1],$$

subject to the BCs,

$$\begin{cases} y(0) = 0, \\ y(1) = 3. \end{cases} \tag{5.14}$$

The exact solution is:

$$y(x) = x^2 + x + 1. \tag{5.15}$$

Applying the NGDTM to both sides of Eq.(5.13), by using theorems in Table (1), particularly by choosing $\alpha = \frac{1}{2}$, the equation (5.13) transforms to:

$$\frac{\Gamma(\alpha k + 2 + \alpha)}{\Gamma(\alpha k + \alpha)}\ Y_\alpha(k + \frac{2}{\alpha}) + \frac{\Gamma(\alpha k + \frac{3}{2} + \alpha)}{\Gamma(\alpha k + \alpha)}\ Y_\alpha(k + \frac{\frac{3}{2}}{\alpha}) + Y_\alpha(k) = \delta_\alpha(k - \frac{2}{\alpha}) + \delta_\alpha(k - \frac{1}{\alpha}) + 3\delta_\alpha(k) + \frac{4}{\sqrt{\pi}}\ \delta_\alpha(k - \frac{\frac{1}{2}}{\alpha}). \tag{5.16}$$

Making simple the above equation, we attain:

$$Y_\alpha(k + \frac{2}{\alpha}) = \frac{\Gamma(\alpha k + \alpha)}{\Gamma(\alpha k + 2 + \alpha)}\left[- \frac{\Gamma(\alpha k + \frac{3}{2} + \alpha)}{\Gamma(\alpha k + \alpha)}\ Y_\alpha(k + \frac{\frac{3}{2}}{\alpha}) - Y_\alpha(k) + \delta_\alpha(k - \frac{2}{\alpha}) + \delta_\alpha(k - \frac{1}{\alpha}) + 3\delta_\alpha(k) + \frac{4}{\sqrt{\pi}}\ \delta_\alpha(k - \frac{\frac{1}{2}}{\alpha})\right]. \tag{5.17}$$

In particular, selecting $\alpha = \frac{1}{2}$, the Eq. (5.17) converts to:

$$Y_{\frac{1}{2}}(k+4) = \frac{\Gamma(\frac{k}{2} + \frac{1}{2})}{\Gamma(\frac{k}{2} + \frac{5}{2})}\left[- \frac{\Gamma(\frac{k}{2} + 2)}{\Gamma(\frac{k}{2} + \frac{1}{2})}\ Y_{\frac{1}{2}}(k+3) - Y_{\frac{1}{2}}(k) + \delta_{\frac{1}{2}}(k-4) + \delta_{\frac{1}{2}}(k-2) + \frac{4}{\sqrt{\pi}}\ \delta_{\frac{1}{2}}(k-1) + 3\delta_{\frac{1}{2}}(k)\right]. \tag{5.18}$$

With the assistance of Eq. (4.5), the NGDTM of the BCs Eq. (5.14) is converted to:

$$\begin{cases} Y_\alpha(0) = 1. \\ \sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k)(1)^{\frac{1}{2}(k+1)-1} = 3. \end{cases} \tag{5.19}$$

Since the initial conditions are implemented for the integer-order derivatives, the NGDTM defines the transformation of the initial conditions as per the following technique:

$$Y_{\frac{1}{2}}(k) = \begin{cases} \left(\frac{\Gamma(\frac{1}{2})\Gamma(\frac{k}{2}+1)\ x^{0.5}}{\Gamma(\alpha k + \frac{1}{2})} \right) \beta_{\frac{k}{2}}, & \text{if} \quad \frac{k}{2} \in \mathbb{Z}^+ \\ 0, & \text{otherwise} \end{cases} \tag{5.20}$$

where k = 0, 1,2,3.

Introducing Eq. (5.20), the BCs (5.14) is transformed to:

$$\begin{cases} Y_{\frac{1}{2}}(0) = x^{0.5}, & \text{if} \quad k = 0 \\ Y_{\frac{1}{2}}(1) = 0, & \text{if} \quad k = 1 \\ Y_{\frac{1}{2}}(2) = \gamma\, x^{0.5}, & \text{if} \quad k = 2 \\ Y_{\frac{1}{2}}(3) = 0, & \text{if} \quad k = 3 \end{cases} \tag{5.21}$$

where γ is an unknown parameter.

Exploiting Eq. (5.18) together with Eqs. (5.14), and (5.21), we will find only the first term which is dependent on the value of γ and the first term component of the transformed solution described below.

For $k = 0$

$$Y_{\frac{1}{2}}(4) = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{5}{2})}\left(-Y_{\frac{1}{2}}(3) - Y_{\frac{1}{2}}(0) + \delta_{\frac{1}{2}}(-4) + \delta_{\frac{1}{2}}(-2) + \frac{4}{\sqrt{\pi}}\delta_{\frac{1}{2}}(-1) + 3\delta_{\frac{1}{2}}(0) \right) = 1.0067\, x^{0.5}. \quad (5.22)$$

Substituting Eq. (5.21) and Eq. (5.22) into Eq. (5.18), we obtain

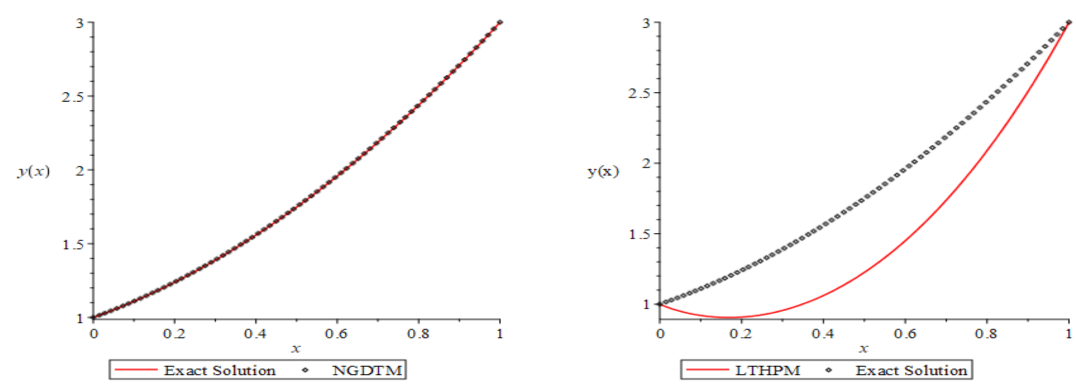
$$y(x) = 1 + \gamma\, x + 1.0067\, x^2. \quad (5.23)$$

To calculate the value of γ, it is required that Eq.(5.23) satisfies the boundary condition (5.19), and thus the value of γ is:

$$\gamma = 0.9933. \quad (5.24)$$

Therefore, Eq. (5.23) becomes

$$y(x) = 1 + 0.9933\, x + 1.0067\, x^2. \quad (5.25)$$



(A) (B)

FIGURE 1. Comparison between the exact solution with Comparing the exact solution against (a) the LT-HPM [8] and (b) the NGDTM for example 2.

Table 3 shows the comparison of the absolute error of the NGDTM and LTHPM using one-step iteration in the interval [0, 1] of example 2. It is evident from the table that the absolute error of the NGDTM is smaller, and therefore it is closer to the exact solution than the LTHPM. Figure 1a illustrates the comparison between the exact solution and standard LTHPM using one-step iteration of example 2, and it is observed that the convergence is slow and one-step iteration is not enough to obtain good results, while the NGDTM provides high accuracy by using one-step iteration. Figure 1b shows the comparison between the exact solutions and NGDTM using one iteration of example 2. This is also evident through the table 3, where it compares the absolute error between NGDTM and LTHPM, and despite the use of the NGDT method with fewer limits than the LTHP method, the results of the NGDT method are better than the LTHP method.

Table 3. Comparison of absolute errors between LT-HPM [8] and NGDTM on [0,1] for example 2.

x	LT-HPM	NGDTM
0	0	0
0.1	1.898×10^{-1}	6.030×10^{-4}
0.2	3.315×10^{-1}	1.072×10^{-3}
0.3	4.340×10^{-1}	1.407×10^{-3}
0.4	4.986×10^{-1}	1.608×10^{-3}
0.5	5.243×10^{-1}	1.675×10^{-3}
0.6	5.095×10^{-1}	1.608×10^{-3}
0.7	4.523×10^{-1}	1.407×10^{-3}
0.8	3.500×10^{-1}	1.072×10^{-3}
0.9	2.002×10^{-1}	6.030×10^{-4}
1.0	0	0

Exampel 3. *Consider the following Bagley Torvik equation with BCs [7]:*

$$D_*^{\frac{3}{2}}\ y(x)+y(x)=x^4+\frac{64}{5\sqrt{\pi}}\ x^{\frac{5}{2}}-8x, \tag{5.26}$$

where,

$$1<\alpha\leq 2,\quad x\in[0,2],$$

subject to the BCs,

$$\begin{cases} y(0)=0, \\ y(2)=0. \end{cases} \tag{5.27}$$

The exact solution is:

$$y(x)=x^4-8x. \tag{5.28}$$

By using the NGDTM on both sides of Eq. (5.26), applying Table (1) theorems, and choosing $\alpha=\frac{1}{2}$, Eq. (5.26) becomes:

$$\frac{\Gamma(\alpha k+\frac{3}{2}+\alpha)}{\Gamma(\alpha k+\alpha)}\ Y_\alpha(k+\frac{\frac{3}{2}}{\alpha})+Y_\alpha(k)=\delta_\alpha(k-\frac{4}{\alpha})+\frac{64}{5\sqrt{\pi}}\ \delta_\alpha(k-\frac{\frac{5}{2}}{\alpha})-8\delta_\alpha(k-\frac{1}{\alpha}), \tag{5.29}$$

Making simple the above equation, we obtain:

$$Y_\alpha(k+\frac{\frac{3}{2}}{\alpha})=\frac{\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+\frac{3}{2}+\alpha)}\left[\delta_\alpha(k-\frac{4}{\alpha})-Y_\alpha(k)+\frac{64}{5\sqrt{\pi}}\ \delta_\alpha(k-\frac{\frac{5}{2}}{\alpha})-8\delta_\alpha(k-\frac{1}{\alpha})\right], \tag{5.30}$$

In particular, selecting $\alpha = \frac{1}{2}$, the Eq. (5.30) converts to:

$$Y_{\frac{1}{2}}(k+3) = \frac{\Gamma(\frac{k+1}{2})}{\Gamma(\frac{k+4}{2})} \left[\delta_{\frac{1}{2}}(k-8) - Y_{\frac{1}{2}}(k) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(k-5) - 8\delta_{\frac{1}{2}}(k-2)\right]. \tag{5.31}$$

The NGDTM of the BCs (5.27) is changed to:

$$\begin{cases} Y_{\frac{1}{2}}(0) = 0, \\ Y_{\frac{1}{2}}(1) = 0, \\ \sum_{k=0}^{\infty} Y_{\frac{1}{2}}(k)(2)^{\frac{1}{2}(k-1)} = 0. \end{cases} \tag{5.32}$$

Considering Eq.(4.7), the transformed final boundary condition (5.32) can be transformed to an unknown prescribed initial condition, yielding:

$$\begin{cases} Y_{\frac{1}{2}}(0) = 0, \\ Y_{\frac{1}{2}}(1) = 0, \\ Y_{\frac{1}{2}}(2) = \gamma\, x^{0.5}, \end{cases} \tag{5.33}$$

where γ is an unknown parameter.

By applying Eq.(5.31) in conjunction with Eq.(5.33), the initial components of the transformed solution are determined as follows:

For $k = 0$

$$Y_{\frac{1}{2}}(3) = \frac{\Gamma(0.5)}{\Gamma(2)} \left[-Y_{\frac{1}{2}}(0) + \delta_{\frac{1}{2}}(-8) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(-5) - 8\delta_{\frac{1}{2}}(-2)\right] = 0. \tag{5.34}$$

For $k = 1$

$$Y_{\frac{1}{2}}(4) = \frac{1}{\Gamma(2.5)} \left[-Y_{\frac{1}{2}}(1) + \delta_{\frac{1}{2}}(-7) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(-4) - 8\delta_{\frac{1}{2}}(-1)\right] = 0. \tag{5.35}$$

For $k = 2$

$$Y_{\frac{1}{2}}(5) = \frac{\Gamma(1.5)}{\Gamma(3)} \left[-Y_{\frac{1}{2}}(2) + \delta_{\frac{1}{2}}(-6) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(-3) - 8\delta_{\frac{1}{2}}(0)\right] = -0.44311346\, Bx^{0.5} - 3.5449077\, x^{0.5}. \tag{5.36}$$

For $k = 3$

$$Y_{\frac{1}{2}}(6) = \frac{\Gamma(2)}{\Gamma(3.5)} \left[-Y_{\frac{1}{2}}(3) + \delta_{\frac{1}{2}}(-5) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(-2) - 8\delta_{\frac{1}{2}}(1)\right] = 0. \tag{5.37}$$

For $k = 4$

$$Y_{\frac{1}{2}}(7) = \frac{\Gamma(2.5)}{\Gamma(4)} \left[-Y_{\frac{1}{2}}(4) + \delta_{\frac{1}{2}}(-4) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(-1) - 8\delta_{\frac{1}{2}}(2)\right] = 0. \tag{5.38}$$

For $k = 5$

$$Y_{\frac{1}{2}}(8) = \frac{\Gamma(3)}{\Gamma(4.5)} \left[-Y_{\frac{1}{2}}(5) + \delta_{\frac{1}{2}}(-3) + \frac{64}{5\sqrt{\pi}}\ \delta_{\frac{1}{2}}(0) - 8\delta_{\frac{1}{2}}(3)\right] = 0.061359\,\gamma\, x^{0.5} + 0.490873x^{0.5} + \frac{1.77245\, x^{0.5}}{\sqrt{\pi}}. \tag{5.39}$$

Substituting the above terms in the NGDTM, Eq.(5.31), the approximate solution of Eq. (5.26) is given by:

$$y(x) = \gamma x^{1.0} + \left(-0.44311346\,\gamma\, x^{0.5} - 3.5449077\, x^{0.5}\right) x^2 + \left(0.061359232\,\gamma\, x^{0.5} + 1.4908739\, x^{0.5}\right) x^{7/2}. \tag{5.40}$$

It is identified that (5.40) provides the approximate solution of Eq. (5.26), but this relies on the unidentified value γ. To determine the value of γ, it is required that Eq. (5.40) satisfy the boundary condition $y(2) = 0$. Subsequently, we obtain a solitary algebraic equation expressed in terms of the variable γ, structured as follows:

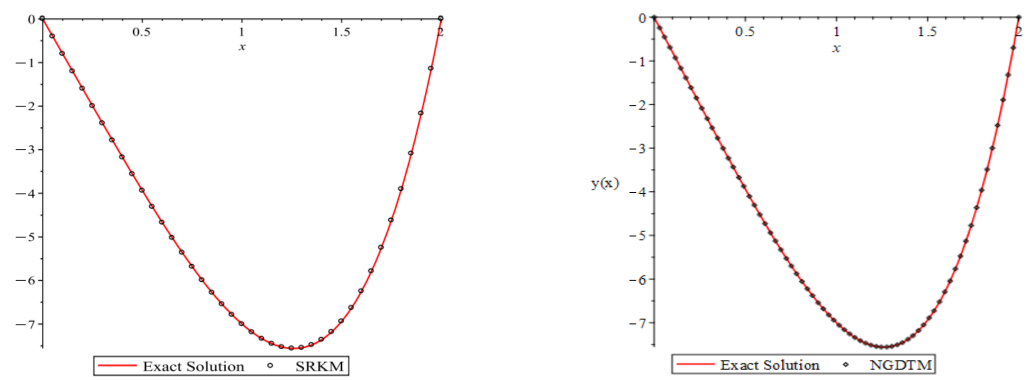
$$\sum_{k=0}^{N+3} Y_{\frac{1}{2}}(k)(2)^{\frac{1}{2}(k+1)-1} = 0. \tag{5.41}$$

Assuming $N = 5$, it can be solved by Eq. (5.41) to obtain the value of the unknown parameter γ:

$$\gamma = -8.0000007. \tag{5.42}$$

Substituting Eq.(5.42) into Eq.(5.40), we obtain the ensuing solution

$$y(x) = -8.0000007\, x + 0.0000003\, x^{2.5} + x^{4.0}. \tag{5.43}$$



(A) (B)

Figure 2. Comparison of the exact solution against (a) SRKM [7] when ($N = 40$) and (b) the NGDTM when ($N = 8$) for Example 3.

Table 4 presents a comparison of the absolute error between the methods NGDTM and SRKM. The absolute error of the proposed method NGDTM is less at each point than the method SRKM. Figure (2a) illustrates the comparison between the exact solution and SRKM using $N = 40$ iterations of this example. We note that the convergence is slow and it doesn't give good results, while the NGDTM provides us with high accuracy using $N = 8$ iterations, as we see in Figure (2b), which shows the comparison between the NGDTM and the exact solution using $N = 8$ iterations of this example. We can see that the absolute error given by NGDTM is less at each point than that given by SRKM.

Table 4. Comparison of absolute errors between SRKM [7] and NGDTM on [0,2], for example 3.

x	SRKM	NGDTM
0	0	0
0.2	5.559×10^{-5}	3.841×10^{-9}
0.4	1.058×10^{-4}	7.120×10^{-9}
0.6	1.476×10^{-4}	1.060×10^{-8}
0.8	1.799×10^{-4}	1.220×10^{-8}
1.0	2.016×10^{-4}	1.300×10^{-8}
1.2	2.126×10^{-4}	1.300×10^{-8}
1.4	2.132×10^{-4}	1.300×10^{-8}
1.6	2.019×10^{-4}	8.000×10^{-9}
1.8	1.777×10^{-4}	1.000×10^{-8}
2.0	0	1.000×10^{-8}

6. Conclusion

This paper aims to enhance the New Generalized Differential Transform Method (NGDTM) to improve its efficiency in solving the Bagley-Torvik equation with boundary conditions. The method provides solutions as a convergent series with easily commutable components, without relying on restrictive assumptions, linearization, or perturbation techniques. The proposed technique's efficiency and reliability have been validated through extensive numerical analyses using several examples from the literature with known exact solutions. The results exhibit excellent agreement with exact solutions and those obtained through other numerical methods, including the Bessel Collection Method, the Laplace Transform Homotopy Perturbation Method, and the Simplified Reproducing Kernel Method.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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