

Fixed Point for Some Contractive Conditions in 2-Normed Spaces Involving a Graph**Neeraj Kumar¹, Seema Mehra¹, Ahmad Aloqaily², Nabil Mlaiki^{2,*}**¹*Department of Mathematics, Maharshi Dayanand University, Rohtak, 124001, India*²*Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586 Saudi Arabia***Corresponding author: nmlaiki2012@gmail.com, nmlaiki@psu.edu.sa*

Abstract. This note focuses on contraction type mappings in 2 - normed linear spaces with a graph. We extend Chatterjea, Kannan's extensions and (N - S - G), (N - S - A - G) to build contractive mappings respectively for a single map and for two maps in 2 - normed linear spaces which is complete (2 - Banach space) via graph. The findings broaden, generalize, and augment previously established results in the literature. We outline the future scope of our determined outcomes at the end of this note.

1. INTRODUCTION

Mappings of a whole metric space into itself are associated with the Banach [1] fixed point theorem. It lays out the prerequisites for a fixed point existence and uniqueness. Furthermore, it provides an iterative method where we can acquire estimates for fixed point and error boundaries. This result carries important implications for implicit function theory as well as for determining the unique solution of differential, integral, and linear algebraic equations. Additionally, it has proven useful in a variety of scientific fields, including computer science, biology, chemistry, economics, and engineering [2–4]. Gahler [5] was the one who originally proposed the concept of linear 2 - normed space. White and Cho & Freese [6,7] established the linear 2 - normed spaces geometric structure. Numerous investigators have demonstrated some significant findings in classical normed spaces into a 2 - norm environment in recent times. In [6] the idea of 2 - Banach space is briefly explained. For $n > 2$ a linear 2 - normed space was expanded by Gunawan and Mashadi[5].

In this study, inspired by [9, 10] we propose (N - S - G) and (N - S - A - G) contractions in 2 - Banach space equipped with a graph as well as an extension of Chatterjea and Kannan contractions. Furthermore, we outline the future scope of our determined outcomes at the end of this note.

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2. DEFINITIONS AND PRELIMINARIES

Definition 2.1. ([24]). For any $s_n \in F$, the sequence $\{s_n\} \subset F$ defined as

$$s_n = Ss_{n-1} = S^n s_0 = 1, 2, 3, \dots$$

is called the sequence of successive approximations with initial value s_0 and S is a self map on F . This is known as Picard iteration at s_0 .

Definition 2.2. ([7]). Let F be a real linear space with $\|\cdot, \cdot\| : F \times F \rightarrow [0, \infty)$ be a function. Then $(F, \|\cdot, \cdot\|)$ is called linear 2 - space for all $s, t, u \in F$ and $\alpha \in \mathbb{R}$:

(N1) $\|s, t\| = 0$ iff s and t are linearly dependent;

(N2) $\|s, t\| = \|t, s\|$;

(N3) $\|\alpha s, t\| = |\alpha| \|s, t\|$;

(N4) $\|s+t, u\| \leq \|s, u\| + \|t, u\|$.

Example 2.1. ([8]). A standard example of a 2 - normed space is \mathbb{R} equipped with the following 2 - norm, $\|s, t\| =$ the area of the triangle having vertices $0, s, t$. Clearly in any 2 - normed space $\|s, t\| \geq 0$ and $\|\alpha s, t\| = |\alpha| \|s, t\|, \alpha \in \mathbb{R}$. Also, if s, t, u are linearly dependent then $\|s+t, u\| = \|s, u\| + \|t, u\|$ or $\|s-t, u\| = \|s, u\| - \|t, u\|$.

Definition 2.3. ([20]). A sequence $\{s_n\}$ in 2 - normed space $(F, \|\cdot, \cdot\|)$ is said to be convergent if there exists a point $s \in F$ such that $\lim_{n \rightarrow \infty} \|s_n - s, u\| = 0$ for all $u \in F$. If $\{s_n\}$ converges to s then we write $\{s_n\} \rightarrow s$ as $n \rightarrow \infty$.

Definition 2.4. ([20]). A sequence $\{s_n\}$ in 2 - normed space $(F, \|\cdot, \cdot\|)$ is said to be Cauchy if $\lim_{n, m \rightarrow \infty} \|s_n - s_m, u\| = 0$ for all $u \in F$.

Definition 2.5. ([20]). A linear 2 - normed space is said to be complete if every Cauchy sequence is convergent to an element of F . A complete 2 - normed linear space F is called 2 - Banach space.

Definition 2.6. ([11]). Let $(F, \|\cdot, \cdot\|)$ be linear 2 - normed space, C be a subset of F then the closure of C is $\bar{C} = \{s \in F; \text{there is a sequence } \{s_n\} \text{ of } C \text{ such that } \{s_n\} \text{ converges to } s\}$. We say C is sequentially closed if $\bar{C} = C$.

Definition 2.7. ([11]). Let $(F, \|\cdot, \cdot\|)$ be linear 2 - normed space, B be a non empty subset of F and $e \in B$ then B is said to be e - bounded if there exists some positive M such that $\|s, e\| \leq M$ for all $s \in B$. If for all e of B , B is e - bounded then B is bounded set.

Definition 2.8. ([11]). Let $(F, \|\cdot, \cdot\|)$ be linear 2 - normed space. Then the mapping $S: F \rightarrow F$ is said to be a contraction if there exists $s \in [0, 1)$ such that

$$\|Ss - St, u\| \leq s \|s - t, u\| \quad \forall s, t, u \in F. \quad (2.1)$$

The notion of G - contraction was created by Jachymski [12] who also brought two concepts of graph and fixed point theories together. Since then, multiple researchers have extensively

investigated Hilbert and Banach via Graph fixed point theorems in metric space see in [13–20]. Motivated by Jachymski [12] throughout in this note, let Σ be the diagonal product of $F \times F$ and G stand for a graph and $V(G)$ be the set of vertices coincides with F i.e. $V(G) = F$ and $E(G) = \{(s, t): s, t \in F \times F\}$ be the collection of edges with every loop in it and fulfill transitive axiom. Consider the set $E(G^{-1}) = \{(t, s) \in F \times F: (s, t) \in E(G)\}$ and $E(\dot{G}) = E(G) \cup E(G^{-1})$. See in [21–23] for further fixed point and graph axioms.

Here, we confer about convergence, Cauchy and completeness in 2-normed space endowed with a graph.

Definition 2.9. Let $(F, \|\cdot, \cdot\|)$ be linear 2-normed space endowed with a Graph, $G = (V, E)$ then

- (1) A sequence $\{s_n\}$ of $F = V(G)$ is converges to a point s of F if, $\lim_{n \rightarrow \infty} \|s_n - s, u\| = 0, \forall u \in V(G)$.
- (2) A Cauchy sequence is convergent iff $\lim_{n, m \rightarrow \infty} \|s_n - s_m, u\| = 0, \forall u \in V(G)$.
- (3) If every Cauchy sequence converges to an element of $F = V(G)$, then the space is considered complete (2-Banach space) $F = V(G)$.

3. MAIN RESULTS

Here we introduce Generalized Kannan Contraction (GKC) with the help of graph i.e.

Definition 3.1. Let $(F, \|\cdot, \cdot\|)$ be linear 2-normed space ($F \neq \emptyset$) endowed with a Graph i.e. $G = (V, E)$ where $V(G) = F$ and $E(G) = \{(s, t): s, t \in F \times F\}$. A self map S on F is Generalized Kannan Contraction (GKC) if S preserves edges of G for any $(u, u), (s, t) \in E(G)$ and

$$\|Ss - St, u\| \leq k[\|s - Ss, u\| + \|t - St, u\| + \|t - Ss, u\|], \text{ where } k \in [0, \frac{1}{2}) \tag{3.1}$$

Theorem 3.1. Assume that $(F, \|\cdot, \cdot\|)$ is linear 2-Banach space ($F \neq \emptyset$) equipped with a graph, such as $G = (V, E)$, where $Q \neq \emptyset$ is a closed, bounded subset of F . If S satisfies Generalized Kannan Contraction (GKC) then S has a unique fixed point s within Q .

Proof. After defining a sequence $\{s_n\}$ in Q using Picard iteration and letting $s_0 \in F = V(G)$, we write

$$\begin{aligned} \|s_n - s_{n+1}, u\| &= \|Ss_{n-1} - Ss_n, u\| \\ &\leq k[\|s_{n-1} - s_n, u\| + \|s_n - s_{n+1}, u\| + \|s_{n+1} - s_{n+1}, u\|] \end{aligned} \tag{3.2}$$

Thus we derive

$$\|s_n - s_{n+1}, u\| \leq \frac{k}{1-k} \|s_n - s_{n+1}, u\| \tag{3.3}$$

As $k \in [0, \frac{1}{2})$ so $\frac{k}{1-k} \in [0, \frac{1}{2})$. Hence, S is a contraction mapping and from (3.3) we get

$$\|s_n - s_{n+1}, u\| \leq \left(\frac{k}{1-k}\right)^n \|s_n - s_{n+1}, u\| \tag{3.4}$$

Furthermore, we demonstrate that $\{s_n\}$ is a Cauchy sequence. Choose $n, m > 0$ such that $m > n$ with $m = n+q$

$$\begin{aligned} \|s_n - s_{n+1}, u\| &= \|s_{n-1} - s_{n+q}, u\| \\ &\leq \|s_n - s_{n+1}, u\| + \|s_{n+2} - s_{n+1}, u\| + \dots + \|s_{n+q-1} - s_{n+q}, u\| \end{aligned} \quad (3.5)$$

Using(3.5) we deduce,

$$\|s_n - s_m, u\| \leq k^n \|s_0 - s_1, u\| + k^{n+1} \|s_0 - s_1, u\| + \dots + k^{n+q-1} \|s_0 - s_1, u\| \quad (3.6)$$

For $n, m \rightarrow \infty$, we get

$$\|s_n - s_m, u\| \leq 0 \quad (3.7)$$

Hence, $\{s_n\}$ is a Cauchy sequence in Q so there $\exists t$ of Q such that

$$\lim_{n \rightarrow \infty} s_n = 0 \quad (3.8)$$

Here, we demonstrate that $t \in Q$ is a fixed point of S . By using the triangle inequality of the 2 - norm, we get

$$\|St - t, u\| \leq \|St - s_n, u\| + \|s_n - t, u\|. \quad (3.9)$$

As $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \|St - t, u\| = 0 \quad (3.10)$$

This implies $St = t$ so t is a fixed point of S . Let s be any other fixed point of S ,

$$\begin{aligned} \|t - s, u\| &\leq \|St - Ss, u\| \\ &\leq k[\|t - St, u\| + \|s - Ss, u\| + \|s - t, u\|] \\ &= 0 \end{aligned} \quad (3.11)$$

It is implied by inequality (3.11) i.e., there is just one fixed point ($s = t$). \square

If we are employing $\|t - St, u\| = 0$ in above Theorem 3.1 we obtain:

Corollary 3.1. Assume $(F, \|\cdot, \cdot\|)$ is 2 - Banach space ($F \neq \emptyset$) with a graph i.e. $G = (V, E)$, where $Q \neq \emptyset$ is closed and bounded subset of F . If a self map S on F satisfies:

$$\|Ss - St, u\| \leq k[\|s - Ss, u\| + \|t - St, u\|], \text{ where } k \in [0, \frac{1}{2}) \quad (3.12)$$

Then $Ss = s \in Q$ unique as well.

Definition 3.2. Let $(F, \|\cdot, \cdot\|)$ be linear 2 - normed space ($F \neq \emptyset$) employing a graph i.e., $G = (V, E)$ where $V(G) = F$ and $E(G) = \{(s, t) : s, t \in F \times F\}$. A self map S on F is Generalized Kannan type contraction If edges are preserved by S of G for any $(u, u), (s, t) \in E(G)$ and

$$\|Ss - St, u\| \leq l[\|s - St, u\| + l\|t - Ss, u\| + p\|s - t, u\|] \text{ where } l, m, p \in [0, \frac{1}{2}) \quad (3.13)$$

Then $Ss = s \in F$ and s is unique.

Theorem 3.2. Let $(F, \|\cdot, \cdot\|)$ be a 2 - Banach space ($F \neq \phi$) that is equipped with a graph, such that $G = (V, E)$, and let $Q \neq \phi$ be a bounded as well as closed subset of F . S has a unique fixed point s within Q if S satisfies the Generalized Kannan type contraction.

Proof. Let $s_0 \in F = V(G)$ and a sequence $\{s_n\} \in Q$ defined by Picard iteration then we write

$$\begin{aligned} \|s_n - s_{n+1}, u\| &= \|Ss_{n-1} - Ss_n, u\| \\ &\leq l\|s_{n-1} - Ss_n, u\| + m\|s_n - s_n, u\| + p\|s_{n-1} - s_n, u\| \\ &= l\|s_{n-1} - s_{n+1}, u\| + p\|s_{n-1} - s_n, u\| \\ &\leq l[\|s_{n-1} - s_n, u\| + \|s_n - s_{n+1}, u\|] + p\|s_{n-1} - s_n, u\| \end{aligned} \tag{3.14}$$

The result of (3.14) is that

$$\|s_n - s_{n+1}, u\| \leq \left(\frac{l+p}{1-l}\right)^n \|s_{n-1} - s_n, u\| \tag{3.15}$$

As $l, p \in [0, \frac{1}{2})$ so $r = \frac{l+p}{1-l} \in [0, \frac{1}{2})$. Furthermore, from (3.14), we have

$$\|s_n - s_{n+1}, u\| \leq r^n \|s_0 - s_1, u\| \tag{3.16}$$

By applying the 2 - norm triangle inequality, we obtain

$$\begin{aligned} \|s_n - s_m, u\| &\leq \|s_n - s_{n+1}, u\| + \|s_{n+1} - s_{n+2}, u\| + \dots + \|s_{m-2} - s_{m-1}, u\| \\ &\leq r^n \|s_0 - s_1, u\| [1 + r + r^2 + \dots] \end{aligned} \tag{3.17}$$

Using $m, n \rightarrow \infty$ in (3.17), we get

$$\lim_{n, m \rightarrow \infty} \|s_n - s_m, u\| = 0. \tag{3.18}$$

As a result, we can conclude that the sequence $\{s_n\}$ in Q is Cauchy. □

Hence, $\{s_n\}$ is a Cauchy sequence in Q so there $\exists t$ of Q such that

$$\lim_{n \rightarrow \infty} s_n = t \tag{3.19}$$

We shall now demonstrate that t of Q is a fixed point of S .

$$\begin{aligned} \|t - St, u\| &= \|t - s_n, u\| + \|s_n - St, u\| \\ &\leq \|t - s_n, u\| + \|s_n - s_{n+1}, u\| + \|s_{n+1} - St, u\| \\ &\leq \|s - s_n, u\| + r^n \|s_0 - s_1, u\| + \|Ss_n - St, u\| \\ &\leq \|s - s_n, u\| + r^n \|s_0 - s_1, u\| + l\|s_n - St, u\| + m\|t - Ss_n, u\| + p\|s_n - t, u\| \end{aligned} \tag{3.20}$$

If we allow $n \rightarrow \infty$ in (3.20), we get that

$$\|t - St, u\| \leq (l + m)\|t - St, u\| \tag{3.21}$$

A contradiction exists in inequality (3.21) unless $\|t - St, u\| = 0$. Thus, we can conclude that $t = St$ and $t \in Q$ is a fixed point of S . We now demonstrate that the fixed point is unique to finish the proof. Let s be an additional fixed point of T . Therefore, $Ss = s$ so

$$\begin{aligned} \|t - s, u\| &\leq \|St - Ss, u\| \\ &\leq l\|t - Ss, u\| + m\|s - St, u\| + p\|t - s, u\| \\ &\leq l + m + p[\|t - Ss, u\| + \|s - St, u\| + \|t - s, u\|] \end{aligned} \quad (3.22)$$

Unless $\|t - s, u\| = 0$, the inequality (3.21) is contradictory for $l, m, p \in [0, \frac{1}{2})$. Consequently, $s = t$. The proof is now complete.

Definition 3.3. ([24]). Let $\psi: (0, \infty) \rightarrow (0, \infty)$ be a function satisfies the following properties:

- (1) ψ is monotone increasing ;
- (2) $\psi(s) < s$ for all $s > 0$;
- (3) $\psi(0) = 0$
- (4) ψ is continuous.

Definition 3.4. Let $(F, \|\cdot, \cdot\|)$ be linear 2 - normed space ($F \neq \phi$) employing a Graph i.e. $G = (V, E)$ where $V(G) = F$ and $E(G) = \{(s, t) : s, t \in F \times F\}$. Then the self maps S, T on F is $(S - N - G)$ contraction if the edges of G preserved S, T i.e., for any $(u, u), (s, t) \in E(G)$ and

$$\|Ss - Tt, u\| \leq l[\|s - Ss, u\| + \|t - Tt, u\|] + \psi(p(\|s - t, u\|)) \text{ where, } l > 0, p \geq 0 \text{ and } 2l + m < 1. \quad (3.23)$$

Theorem 3.3. Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \phi$) endowed with a graph i.e. $G = (V, E)$. If S, T satisfy $(S - N - G)$ contraction then S, T have a unique fixed point within F .

Proof. Any given point s in F can serve as s_0 . For every $n = 0, 1, 2, \dots$, we define a sequence $\{s_n\}$ in F such that $s_{2n+1} = Ss_{2n}$, $s_{2n+2} = Ts_{2n+1}$. It is easily demonstrated that $u = s_n$ in F is a common fixed point of S and T provided $n \geq 0$ is such that $s_{2n} = s_{n+1} = s_{n+2}$. Thus, let's assume that the sequence $\{s_n\}$ does not include three consecutive equivalent terms. Consequently,

$$\begin{aligned} \|s_{2n+1} - s_{2n}, u\| &= \|Ss_{2n} - s_{2n-1}, u\| \\ &\leq l[\|s_{2n} - Ss_{2n}, u\| + \|s_{2n-1} - Ts_{2n-1}, u\|] + \psi(p(\|s_{2n} - s_{2n-1}, u\|)) \\ &< l[\|s_{2n+1} - s_{2n}, u\| + \|s_{2n} - s_{2n-1}, u\|] + p\|s_{2n} - s_{2n-1}, u\|, \end{aligned}$$

is valid for every $n \geq 1$ and every u in $F - \{0\}$. Similarly, the following inequality can be shown to be valid

$$\|s_{2n+1} - s_{2n}, u\| < l[\|s_{2n-2} - s_{2n-1}, u\| + \|s_{2n} - s_{2n-1}, u\|] + p\|s_{2n-2} - s_{2n-1}, u\| \quad (3.24)$$

The latter suggests that

$$\|s_{n+1} - s_n, u\| < k[\|s_n - s_{n-1}, u\|] \quad (3.25)$$

is true for every $n \geq 1$ and every u in Q , for $k = \frac{l+p}{1-l}$. Moreover, the inequality (3.24), which is equivalent to the Theorem 3.1 in this case, suggests that the sequence $\{s_n\}$ is a Cauchy sequence, i.e., there $\exists t \in F$ such that

$$\lim_{n \rightarrow \infty} s_n = t$$

We intend to show that $St = t$. Consequently,

$$\begin{aligned} \|t - St, u\| &\leq \|t - s_{2n+2}, u\| + \|s_{2n+2} - St, u\| = \|t - s_{2n+2}, u\| + \|Ts_{2n+1} - St, u\| \\ &\leq \|t - s_{2n+2}, u\| + l[\|s_{2n+1} - Ts_{2n+1}, u\| + \|t - St, u\|] + \psi(p(\|t - s_{2n+1}, u\|)) \\ &< \|t - s_{2n+2}, u\| + l[\|s_{2n+1} - s_{2n+2}, u\| + \|t - St, u\|] + \psi(p(\|t - s_{2n+1}, u\|)), \end{aligned} \tag{3.26}$$

is valid for every $n \geq 1$ and every u in $F - \{0\}$. The latter inequality is converted as $\|t - St, u\| < k\|t - St, u\|$ for any $u \in F$. Nevertheless, $k < 1$ since for every u in Q this also implies that $\|t - St, u\| = 0$. Hence $St = t$, which indicates that t is a fixed point of S . Comparably, it is possible to show that $Tt = t$ in addition to S . We will show that S and T have exactly one fixed point in common. We shall demonstrate that there is just one common fixed point between S and T . Let $s \in F$ represents an additional fixed point of T i.e., $Ts = s$. Consequently,

$$\begin{aligned} \|t - s, u\| &= \|St - Ts, u\| \\ &< l[\|t - St, u\| + \|s - Ts, u\|] + p\|t - s, u\| \end{aligned} \tag{3.27}$$

which is true. Since $p < 1$, we deduce that $\|t - s, u\| = 0$. So, $s = t$ □

Corollary 3.2. *Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \emptyset$) endowed with a Graph i.e. $G = (V, E)$. If S, T satisfy:*

$$\|Ss - Tt, u\| \leq l \frac{\|s - Ss, u\|^2 + \|t - Tt, u\|^2}{\|s - Ss, u\| + \|t - Tt, u\|} + \phi(p(\|s - t, u\|)) \text{ where, } l > 0, p \geq 0 \text{ and } 2l + m < 1. \tag{3.28}$$

Thus, S and T share a unique fixed point $s \in F$.

Proof. Clearly (3.27) indicated in the case, for $u \neq 0$, implies an inequality (3.23). The inequality (3.23) is valid for $u = 0$. Finally, Theorem 3.3 directly implies this Corollary 3.2. □

Corollary 3.3. *Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \emptyset$) endowed with a graph i.e. $G = (V, E)$. If S, T satisfy:*

$$\|Ss - Tt, u\| \leq k\|s - Ss, u\| \cdot \sqrt[3]{\|s - Ss, u\| \cdot \|t - Tt, u\|} \cdot \psi(p(\|s - t, u\|)) \text{ where, } k \in (0, 1), s, t, u \in F = V(G). \tag{3.29}$$

Then S and T share a unique fixed point $s \in F$.

Proof. Theorem 3.3 and the arithmetic-geometric mean inequality merely lead to Corollary 3.3. for $1 = p = \frac{k}{3}$. □

Corollary 3.4. Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \emptyset$) endowed with a graph i.e. $G = (V, E)$ where $V(G) = F$ and $E(G) = \{(s, t) : s, t \in F \times F\}$. If the self map S, T on F is (S-N-G) if S, T preserve edges of G for any $(u, u), (s, t) \in E(G)$ and

$$\|Ss - Tt, u\| \leq l[\|s - S^q s, u\| + \|t - T^r t, u\|] + \psi(p(\|s - t, u\|)) \text{ where, } l > 0, p \geq 0 \text{ and } 2l + m < 1, \quad (3.30)$$

$\forall s, t, u \in V(G) = F$. Then S and T have a unique fixed point $\in F$.

Proof. Theorem 3.3 states that the mappings S^q and T^r share a unique fixed point s in F . An expression $S^q s = s$ i.e., $Ss = S(S^q s) = S^q(Ss)$ indicates that Ss is a fixed point of S^q . Similarly, $T^r s = s \Rightarrow Ts = T(T^r s) = T^r(Ts) \Rightarrow Ts$ is a fixed point of T^r . However, both T^r and S^q have a unique fixed point. Thus, $s = Ts = Ss$. Thus, s of F is unique fixed point of S and T . If t be any fixed point of S and t in F then T^r and S^q have a unique fixed point of T^r and S^q as T^r and S^q have a unique fixed point so $s = t$. \square

Definition 3.5. Let $(F, \|\cdot, \cdot\|)$ be linear 2- normed space ($F \neq \emptyset$) employing a graph i.e. $G = (V, E)$ where $V(G) = F$ and $E(G) = \{(s, t) : s, t \in F \times F\}$. Then the self map S, T on F is (S-N-A - G) if S, T preserve edges of G for any $(u, u), (s, t) \in E(G)$ and

$$\|Ss - Tt, u\| \leq l[\|s - Ss, u\|^2 + \|t - Tt, u\|^2] + \psi(p(\|s - t, u\|^2)) \text{ where, } l > 0, p \geq 0 \text{ and } 2l + m < 1. \quad (3.31)$$

Theorem 3.4. Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \emptyset$) endowed with a graph i.e. $G = (V, E)$. If S, T satisfy (S-N-A-G) contraction then S, T have a unique fixed point s in F .

Proof. Any given point s in F can serve as s_0 . For every $n = 0, 1, 2, \dots$, we define a sequence $\{s_n\}$ in F such that $s_{2n+1} = Ss_{2n}$, $s_{2n+2} = Ts_{2n+1}$. It is easily demonstrated that $u = s_n$ in F is a common fixed point of S and T , provided $n \geq 0$ is such that $s_{2n} = s_{n+1} = s_{n+2}$. Thus, let's assume that the sequence $\{s_n\}$ does not include three consecutive equivalent terms. Consequently,

$$\|s_{n+1} - s_n, u\| < k[\|s_n - s_{n-1}, u\|] \quad (3.32)$$

holds true for every $n \geq 1$ and every u in Q , for $k = \sqrt{\frac{l+p}{1-l}} < 1$. Moreover, an expression (3.32), which is equivalent to the Theorem 3.1 in this case, suggests that the sequence $\{s_n\}$ is a Cauchy sequence, i.e., there $\exists t \in F$ such that

$$\lim_{n \rightarrow \infty} s_n = t$$

We intend to show that $St = t$. Consequently

$$\begin{aligned} \|t - St, u\| &\leq \|t - s_{2n+2}, u\| + \|s_{2n+2} - St, u\| = \|t - s_{2n+2}, u\| + \|Ts_{2n+1} - St, u\| \\ &\leq \|t - s_{2n+2}, u\| + \sqrt{l[\|s_{2n+1} - Ts_{2n+1}, u\| + \|t - St, u\|] + \psi(p(\|t - s_{2n+1}, u\|))} \\ &< \|t - s_{2n+2}, u\| + \sqrt{l[\|s_{2n+1} - s_{2n+2}, u\| + \|t - St, u\|] + p(\|t - s_{2n+1}, u\|)} \end{aligned} \quad (3.33)$$

is valid for every $n \geq 1$ and every u in Q . The latter inequality is converted as $\|t - St, u\| < \sqrt{k}\|t - St, u\|$ for any $u \in F$. Nevertheless, $k < 1$ since for every u in Q this also implies that $\|t - St, u\| = 0$. Hence, $St = t$ meaning that S has fixed point t . Also, it is possible to demonstrate that t is a fixed point of T , meaning that t is a fixed point of both S and T . We shall demonstrate that there is just one common fixed point between S and T . Assume that s in F such that $Ts = s$. Consequently from (3.29) we deduce that,

$$\|t - St, u\|^2 \leq p\|t - s, u\|^2 \Rightarrow s = t.$$

□

Corollary 3.5. Let $(F, \|\cdot, \cdot\|)$ be 2 - Banach space ($F \neq \emptyset$) endowed with a graph i.e. $G = (V, E)$. If the self map S^q, T^r on F satisfies:

$$\|S^q s - T^r t, u\| \leq l[\|s - T^r t, u\|^2 + \|t - S^q t, u\|^2] + \psi(p\|s - t, u\|^2) \text{ where, } l > 0, p \geq 0 \text{ and } 2l + m < 1 \quad (3.34)$$

Then $S = t = Tt$, which is unique as well $\in F$.

Proof. The proof is the same as Corollary 3.4.

□

4. CONCLUSION

Numerous generalizations of well-known fixed point conclusions and theorems for single map as well as common fixed points of mapping to 2 - Banach space are demonstrated in this study. Considering the significance of fixed point theory and its implementation, it is reasonable to wonder what other outcomes can be applied universally. Here, we outline the our outcomes of our future objectives:

- (1) To determine fixed point outcomes in linear n - normed space.
- (2) Determine the outcomes for the mappings of 2 - contraction and ψ contraction in linear 2 -normed space.

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