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Solution of Prey–Predator System by ADM

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Abstract. A prey-predator system with an abundance of nutrients is considered. Utilizing Adomian decomposition method to numerate and approximate the solution of that governing system. Providing many examples to obtain some numerical simulation solutions and plot the results for the prey and predator populations versus time.

1. Introduction

The prey-predator system had been studied in many researchers [1-12]. Abundance of nutrients is assumed, and the prey have been enough nutrition to consume. The prey-predator system is taken the form:

$$\begin{aligned} \dot{M} &= aM(t) - bM(t)N(t), \\ \dot{N} &= cM(t)N(t) - kN(t). \end{aligned} \tag{1.1}$$

where, $M(t)$ and $N(t)$ are the constraint of the prey and the predator at time t, respectively. Otherwise, a,b,c and k are constants and dot refer to the derivative with respect to time t. On this problem, the predator has been discreated and inverse proportional to e^{kt}, in absent of the prey. Moreover, the prey has been increased and proportional to e^{at} in non-existence of predator. Otherwise, they are living together when $k < c$ and $a < b$.

2. Adomian Decomposition Method

In [13-16] ADM had been addressed and had been employed. The solution of equation (1.1) is considered to be as infinite series, as usual in ADM:

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$$M(t) = \sum_{n=0}^{\infty} M_n(t), \quad N(t) = \sum_{n=0}^{\infty} N_n(t). \tag{2.1}$$

Define $M_0 = M(0)$, and $N_0 = N(0)$, and apply ADM at (1.1), to obtain the following scheme:

$$M_{n+1}(t) = a \int_0^t M_n(t)\, dt - b \int_0^t G_n(t)\, dt. \tag{2.2}$$

$$N_{n+1}(t) = c \int_0^t G_n(t)\, dt - k \int_0^t N_n(t)\, dt. \tag{2.3}$$

where,

$$G_n(t) = \frac{1}{n!} \frac{d^n}{d\beta^n} \left(\sum_{n=0}^{\infty} \beta^n M_n(t) \sum_{n=0}^{\infty} \beta^n N_n(t) \right) \Bigg|_{\beta=0}. \tag{2.4}$$

which is called Adomian's polynomial for all $0 \leq n$ and it is the approximation of the nonlinear term $M(t)N(t)$, it means that

$$M(t)N(t) = \sum_{n=0}^{\infty} G_n(M_0, M_1, \ldots, M_n, N_0, N_1, \ldots, N_n). \tag{2.5}$$

To compute $G_n(t)$, we find that

$$G_n(t) = \sum_{i=0}^{n} M_{n-i}(t) N_i(t). \tag{2.6}$$

Utilizing Python programming to obtain recursively the partial sum of the solution as following:

$$M(t) = \lim_{p \to \infty} \sum_{n=0}^{p} M_n(t). \tag{2.7}$$

$$N(t) = \lim_{p \to \infty} \sum_{n=0}^{p} N_n(t). \tag{2.8}$$

3. Numerical Results

The polynomials of the decomposition series partial sums of fourth degree are derived by ADM according to the parameter's values $a, b, c, k, M(0) = m_0, N(0) = n_0$. The general approximate expressions for $M(t)$ and $N(t)$ are taken the form:

$$
\begin{aligned}
M(t) \approx\ & m_0 + t(am_0 - bm_0n_0) \\
& + t^2\left(a\left(\frac{am_0}{2} - \frac{bm_0n_0}{2}\right) - b\left(\frac{1}{2}am_0n_0 - \frac{1}{2}bm_0n_0^2 + \frac{1}{2}cm_0^2n_0 - \frac{1}{2}km_0n_0\right)\right) \\
& + t^3\left(a\left(\frac{1}{6}a^2m_0 - \frac{1}{3}abm_0n_0 + \frac{1}{6}b^2m_0n_0^2 - \frac{1}{6}bcm_0^2n_0 + \frac{1}{6}bkm_0n_0\right)\right. \\
& -b\left(\frac{1}{6}a^2m_0n_0 - \frac{1}{3}abm_0n_0^2 + \frac{1}{2}acm_0^2n_0 - \frac{1}{3}akm_0n_0 + \frac{1}{6}b^2m_0n_0^3\right) \\
& \left.-\frac{2}{3}bcm_0^2n_0^2 + \frac{1}{2}bkm_0n_0^2 + \frac{1}{6}c^2m_0^3n_0 - \frac{1}{3}ckm_0^2n_0 + \frac{1}{6}k^2m_0n_0\right)\Big) \\
& + t^4\left(a\left(\frac{1}{24}a^3m_0 - \frac{1}{8}a^2bm_0n_0 + \frac{1}{8}ab^2m_0n_0^2 - \frac{1}{6}abcm_0^2n_0 + \frac{1}{8}abkm_0n_0\right.\right. \\
& \left.-\frac{1}{24}b^3m_0n_0^3 + \frac{1}{6}b^2cm_0^2n_0^2 - \frac{1}{8}b^2km_0n_0^2 - \frac{1}{24}bc^2m_0^3n_0 + \frac{1}{12}bckm_0^2n_0 - \frac{1}{24}bk^2m_0n_0\right) \\
& -b\left(\frac{1}{24}a^3m_0n_0 - \frac{1}{8}a^2bm_0n_0^2 + \frac{7}{24}a^2cm_0^2n_0 - \frac{1}{8}a^2km_0n_0 + \frac{1}{8}ab^2m_0n_0^3\right. \\
& -\frac{11}{12}abcm_0^2n_0^2 + \frac{13}{24}abkm_0n_0^2 + \frac{5}{12}ac^2m_0^3n_0 - \frac{13}{24}ackm_0^2n_0 + \frac{1}{8}ak^2m_0n_0 \\
& -\frac{1}{24}b^3m_0n_0^4 + \frac{5}{8}b^2cm_0^2n_0^3 - \frac{5}{12}b^2km_0n_0^3 - \frac{5}{8}bc^2m_0^3n_0^2 + \frac{11}{12}bckm_0^2n_0^2 - \frac{7}{24}bk^2m_0n_0^2 \\
& \left.\left.+\frac{1}{24}c^3m_0^4n_0 - \frac{1}{8}c^2km_0^3n_0 + \frac{1}{8}ck^2m_0^2n_0 - \frac{1}{24}k^3m_0n_0\right)\right)
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
N(t) \approx\ & n_0 + t\left(cm_0n_0 - kn_0\right) \\
& + t^2\left(c\left(\frac{1}{2}am_0n_0 - \frac{1}{2}bm_0n_0^2 + \frac{1}{2}cm_0^2n_0 - \frac{1}{2}km_0n_0\right) - k\left(\frac{cm_0n_0}{2} - \frac{kn_0}{2}\right)\right) \\
& + t^3\left(c\left(\frac{1}{6}a^2m_0n_0 - \frac{1}{3}abm_0n_0^2 + \frac{1}{2}acm_0^2n_0 - \frac{1}{3}akm_0n_0\right.\right. \\
& \left.+\frac{1}{6}b^2m_0n_0^3 - \frac{2}{3}bcm_0^2n_0^2 + \frac{1}{2}bkm_0n_0^2 + \frac{1}{6}c^2m_0^3n_0 - \frac{1}{3}ckm_0^2n_0 + \frac{1}{6}k^2m_0n_0\right) \\
& \left.-k\left(\frac{1}{6}acm_0n_0 - \frac{1}{6}bcm_0n_0^2 + \frac{1}{6}c^2m_0^2n_0 - \frac{1}{3}ckm_0n_0 + \frac{1}{6}k^2n_0\right)\right) \\
& + t^4\left(c\left(\frac{1}{24}a^3m_0n_0 - \frac{1}{8}a^2bm_0n_0^2 + \frac{7}{24}a^2cm_0^2n_0 - \frac{1}{8}a^2km_0n_0\right.\right. \\
& +\frac{1}{8}ab^2m_0n_0^3 - \frac{11}{12}abcm_0^2n_0^2 + \frac{13}{24}abkm_0n_0^2 + \frac{5}{12}ac^2m_0^3n_0 - \frac{13}{24}ackm_0^2n_0 + \frac{1}{8}ak^2m_0n_0 \\
& -\frac{1}{24}b^3m_0n_0^4 + \frac{5}{8}b^2cm_0^2n_0^3 - \frac{5}{12}b^2km_0n_0^3 - \frac{5}{8}bc^2m_0^3n_0^2 + \frac{11}{12}bckm_0^2n_0^2 \\
& \left.-\frac{7}{24}bk^2m_0n_0^2 + \frac{1}{24}c^3m_0^4n_0 - \frac{1}{8}c^2km_0^3n_0 + \frac{1}{8}ck^2m_0^2n_0 - \frac{1}{24}k^3m_0n_0\right) \\
& -k\left(\frac{1}{24}a^2cm_0n_0 - \frac{1}{12}abcm_0n_0^2 + \frac{1}{8}ac^2m_0^2n_0 - \frac{1}{8}ackm_0n_0 + \frac{1}{24}b^2cm_0n_0^3\right. \\
& \left.\left.-\frac{1}{6}bc^2m_0^2n_0^2 + \frac{1}{6}bckm_0n_0^2 + \frac{1}{24}c^3m_0^3n_0 - \frac{1}{8}c^2km_0^2n_0 + \frac{1}{8}ck^2m_0n_0 - \frac{1}{24}k^3n_0\right)\right)
\end{aligned}
\tag{3.2}
$$

Cases 1-3 demonstrate the relation between the concentration of prey and predator population versus time.

Case 1: The resulting series partial sums of 9^{th} degree equations of prey and predator for the values: $a = 0.4, b = 0.8, c = 0.5, k = 0.2, M(0) = 0.5, N(0) = 0.3$ are given by

$$\begin{aligned} M(t) \approx\ & 0.5 + 0.08t + 0.0034t^2 - 0.0009t^3 - 0.0002t^4 \\ & - 1.6417 \times 10^{-5}t^5 + 1.0624 \times 10^{-6}t^6 \\ & + 8.1348 \times 10^{-7}t^7 + 2.5879 \times 10^{-7}t^8 + 7.3847 \times 10^{-8}t^9. \\ N(t) \approx\ & 0.3 + 0.015t + 0.0063t^2 + 0.0004t^3 + 4.0003 \times 10^{-5}t^4 \\ & - 1.3878 \times 10^{-6}t^5 - 1.3018 \times 10^{-6}t^6 \\ & - 4.3329 \times 10^{-7}t^7 - 1.2549 \times 10^{-7}t^8 - 3.6177 \times 10^{-8}t^9. \end{aligned} \tag{3.3}$$

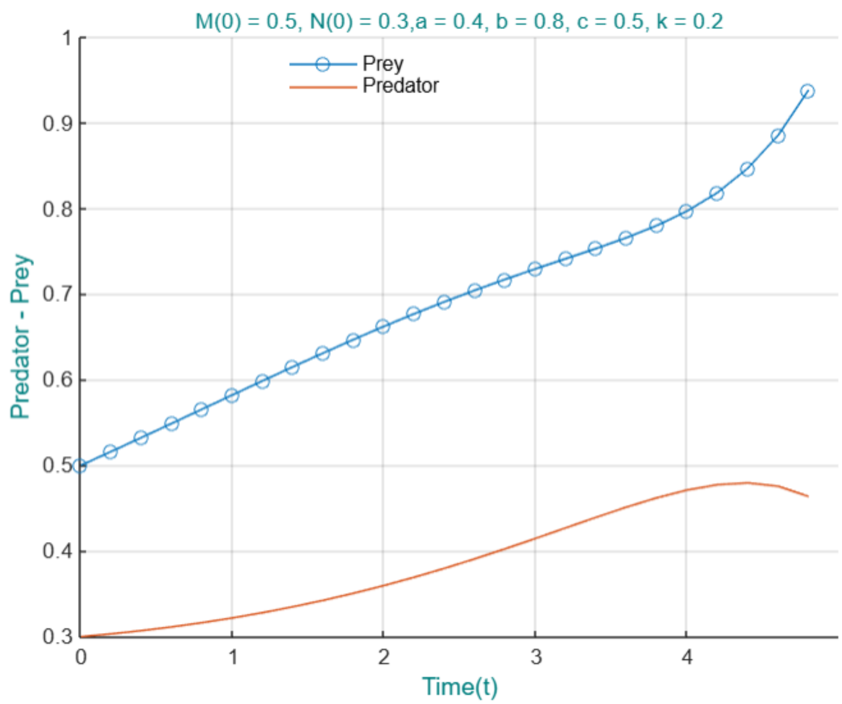

Figure 1. r=9

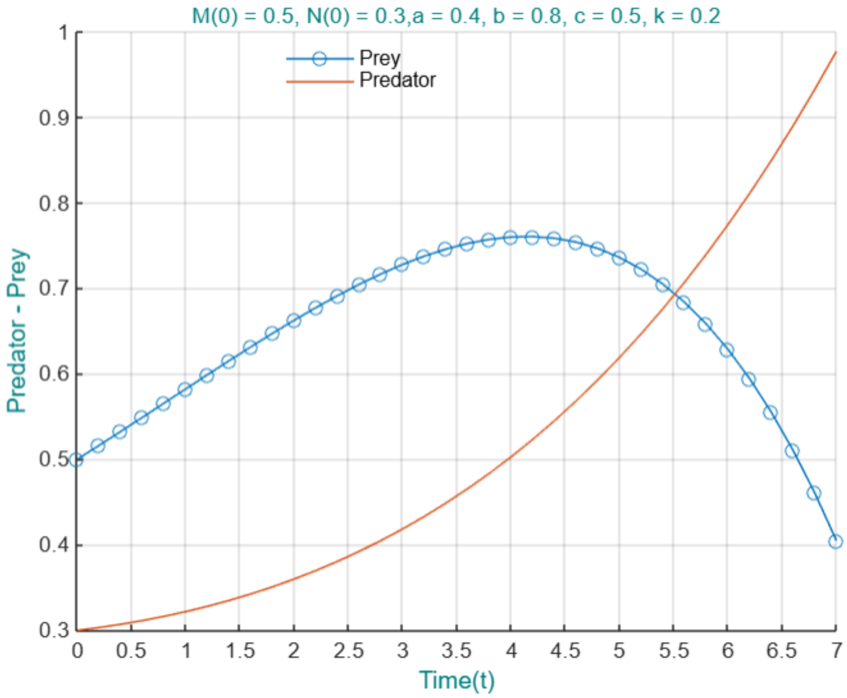

Figure 2. r=4

In Figure 1 the prey and predator have no equivalent concentration at any time. However in Figure 2 the two curves have equivalent concentration at the approximate point $t_1 = 5.5125$, where $N(t_1) = M(t_1) \approx 0.6931254$.

Case 2: The 9^{th} degree of series partial sums for the two populations at parameter values $a = 0.4, b = 0.8, c = 0.5, k = 0.2, M(0) = 0.3, N(0) = 0.6$ are obtained as follow:

$$\begin{aligned}
M(t) \approx\ & 0.3 - 0.0240t + 0.0046t^2 - 8.5600\times 10^{-5}t^3 \\
& + 9.6620\times 10^{-6}t^4 + 3.1071\times 10^{-7}t^5 \\
& - 1.6200\times 10^{-6}t^6 + 1.1238\times 10^{-6}t^7 \\
& - 9.4829\times 10^{-7}t^8 + 1.0853\times 10^{-6}t^9. \\
N(t) \approx\ & 0.6 - 0.0300t - 0.0029t^2 + 0.0006t^3. \\
& - 4.2564\times 10^{-5}t^4 + 1.9915\times 10^{-6}t^5 \\
& + 9.5909\times 10^{-7}t^6 - 7.8761\times 10^{-7}t^7 \\
& + 6.4749\times 10^{-7}t^8 - 7.1902\times 10^{-7}t^9. \qquad (3.4)
\end{aligned}$$

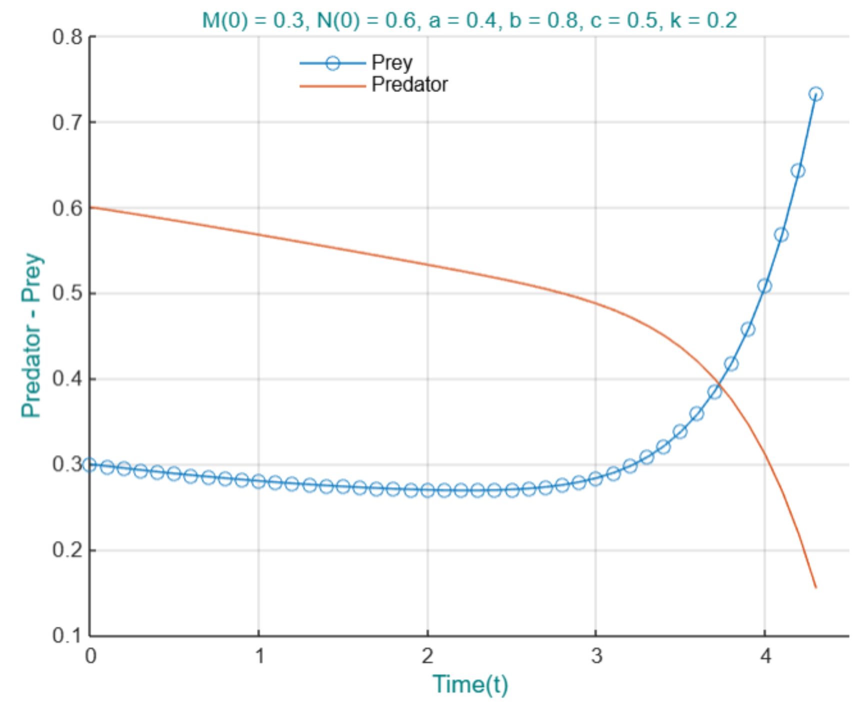

FIGURE 3. r=9

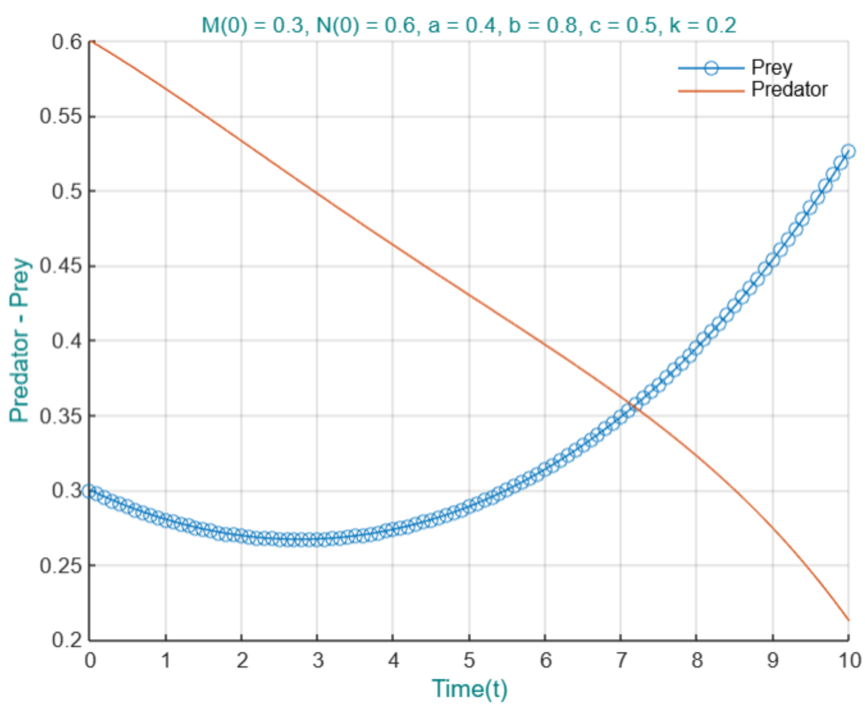

FIGURE 4. r=4

In Figure 3 the prey concentration is parity to the predator concentration at $t_2 \approx 3.72596$, with $M(t_2) = N(t_2) \approx 0.39339$. Moreover the two curves intersect at the approximate point $t_3 = 7.162345$ where $N(t_3) = M(t_3) \approx 0.3560$ in Figure 4.

Case 3: We have secured that the 6^{th} degree of series partial sums of $M(t)$ and $N(t)$ for the parameter of values $a = 0.11, b = 0.12, c = 0.15, k = 0.14, M(0) = 0.1, N(0) = 0.2.$

$$\begin{aligned}
M(t) &\approx 0.1 + 8.4 \times 10^{-3}t + 5.153 \times 10^{-4}t^2 \\
&+ 1.6212 \times 10^{-5}t^3 + 4.9414 \times 10^{-7}t^4 \\
&+ 1.8518 \times 10^{-8}t^5 - 3.2254 \times 10^{-11}t^6 \\
N(t) &\approx 0.2 - 2.5 \times 10^{-2}t + 1.6885 \times 10^{-3}t^2 \\
&- 7.5701 \times 10^{-5}t^3 + 2.5938 \times 10^{-6}t^4 \\
&- 8.1450 \times 10^{-8}t^5 + 2.3294 \times 10^{-9}t^6
\end{aligned}$$

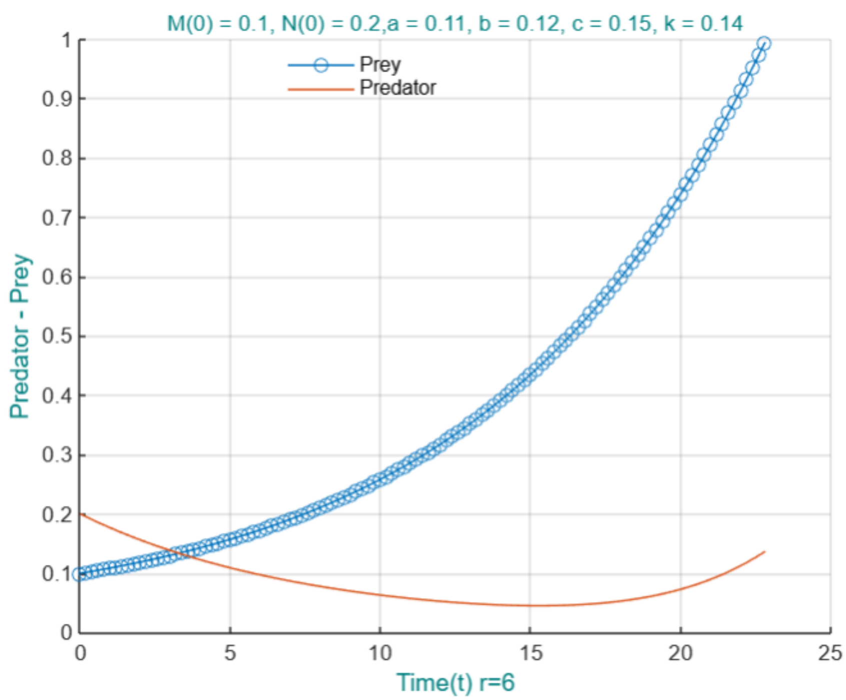

FIGURE 5. r=6

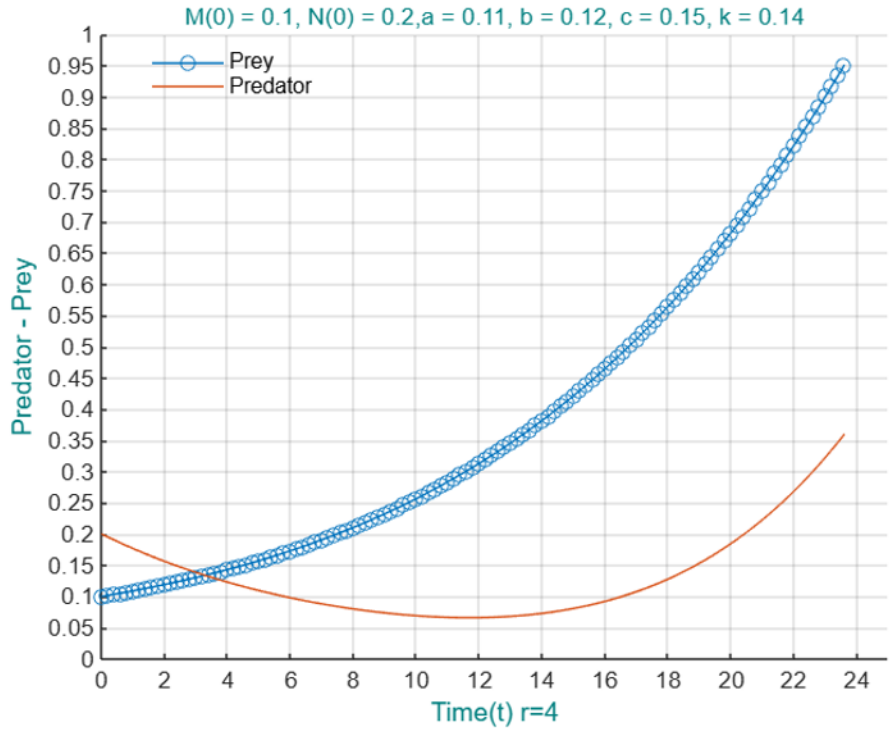

Figure 6. r=4

In Figure 5 the intersection point of the two curves is at $t_4 \approx 3.281210$, where $M(t_4) = N(t_4) \approx=$ 0.1337473. Furthermore in Figure 6 an equivalent concentration point of the two populations is at $t_5 \approx 3.282475$, where $M(t_5) = N(t_5) \approx 0.1337556$.

Conclusion

The general form of equations (3.1) and (3.2) are represented by the 4^{th} degree polynomial ($r = 4$) of time (t) for the two concentrations of prey and predator, respectively. These equations depend on the zero-time concentration (m_0, n_0), the consumption coefficients (a, b, c), and the death coefficients of the predator (k). Cases 1-3 have three pairs of graphs; each pair of graphs is computed for distinct values of r (which is more accurate for large r) and has the same initial values of the parameters. The first pair of graphs, the stable-predator extinction at $r = 4$, will be transformed to the stable-prey extinction at $r = 9$. However, in the second pair of graphs, a surviving-prey population is discovered with differences in lifetime and equivalence concentration points. Eventually, more biological sense is illustrated in case 3 because the parameter values are so close. An optimized stable-coexistence is manifested in the third pair of graphs at $r = 4$ and $r = 6$ depending on their mentioned initial values.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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