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An Innovative Technique for Various Types of Tri Ideals in b-Semirings

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Abstract. This paper explores the structural complexities of tri-ideals and quasi-ideals within b-semirings. We provide characterizations of tri-ideals in b-semirings and define their significant algebraic properties. We investigate the properties of S tri-ideals and their implications in algebra. By emphasizing the algebraic coherence of these structures, we demonstrate how the intersection of 1-left tri-ideals and right ideals can be used to generate a 1-left tri-ideal. We also provide relevant examples for better understanding. Additionally, we rigorously establish key theorems for various scenarios involving tri-ideal, highlighting their theoretical foundations. The main motivation for this study is to emphasize the growing importance of tri-ideal categories over b-semirings in the real algebraic structures.

1. Introduction

Semigroups are fundamental algebraic structures that have been widely used in theoretical computer science, graph theory, optimization theory, as well as in the study of automata, coding theory, and formal languages. The concept of ideals, initially introduced by Dedekind for algebraic numbers, was later generalized by E. Noether for associative rings. One and two-sided ideals are fundamental notions in ring theory. It is well-established that the concept of a one-sided ideal in any algebraic structure represents a broader notion than that of an ideal. Quasi-ideals, which extend the concepts of left and right ideals, and bi-ideals, which further generalize quasi-ideals, play pivotal roles in this theoretical framework, underpinning various algebraic investigations and offering essential tools for understanding the structural properties of algebraic systems.

Arulmozhi [1] provided an introductory explanation of the fundamentals of the algebraic theory of semigroups and semirings. Semigroups and semirings are essential algebraic structures with

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applications in various areas. Arulmozhi [2] also presented a novel approach to studying various ideals in ternary semirings. Ideals are subsets of a ring or semiring that satisfy certain properties, and understanding them is crucial in algebraic structure theory. Daddi and Pawar [3] delved into the properties and characteristics of completely regular ternary semirings. Completely regular semirings have additional properties that make them particularly interesting objects of study in algebraic theory. Dubey and Anuradha [4] discussed prime quasi-ideals and their significance in ternary semirings. Prime quasi-ideals are special subsets of a semiring that possess important algebraic properties and play an important role in the structure and behavior of the ternary semirings. Dutta and Kar [5] explored the properties and structures of regular ternary semirings. Regular semirings are those that satisfy certain regularity conditions, and understanding their properties is essential in algebraic structure theory.

Suguna et al. [6] provided a comprehensive generalization of m-bi-ideals in b-semirings, presenting an extension that characterizes their structural properties. In another study, they introduced a novel approach to examining various types of bi-quasi ideals in b-semirings, enriching the theoretical framework and applications of these algebraic structures [7]. Salahuddin et al. [8] analyzed the convergence results of differential variational inequality problems, contributing valuable insights into mathematical optimization and variational inequality theory. Additionally, Suguna et al. [9] investigated different types of almost ideals in b-semirings, offering new perspectives and extending existing theories to enhance understanding of these algebraic constructs.

Henriksen [10] explored properties and characteristics of ideals in semigroups, contributing to the understanding of ideal theory. Studying ideals in semigroups with commutative addition provides insights into the structure and behavior of these algebraic structures. Iseki [11] contributed to the ideal theory of semirings, expanding the theoretical framework for studying algebraic structures. Ideal theory is a fundamental aspect of algebraic structure theory, and contributions in this area enhance our understanding of semirings. Izuka [12] discussed the Jacobson radical of a semigroup, offering insights into the structure and properties of semigroups. The Jacobson radical is a fundamental concept in the study of semigroups, and understanding its properties provides insights into the structure of semigroups. Jagatap and Pawar [13] investigates properties of quasi-ideals and minimal quasi-ideals in semirings, contributing to the understanding of ideal structures. Quasi-ideals and minimal quasi-ideals are subsets of a semiring with certain properties, and studying their properties enhances our understanding of semirings.

Jagatap and Pawar [14] explored properties and characteristics of bi-ideals in semirings, expanding the study of algebraic structures. Bi-ideals are subsets of a semiring with certain properties, and understanding their properties enriches our understanding of semirings. Lajos [15] studied the properties and characteristics of bi-ideals in semirings, contributing to the understanding of their algebraic properties. Bi-ideals are subsets of a semiring with certain algebraic properties, and understanding their properties enhances our understanding of semirings. Rao [16] defines tri-ideals as subsets that satisfy specific closure properties and explores their structural role within semirings. By analyzing conditions for their formation and their relationship to traditional ideals, Rao demonstrates how tri-ideals contribute to the decomposition and internal symmetry of semirings, with potential applications in fields like automata theory and formal languages. Suguna et al. [17] introduces innovative methodologies for studying various types of bi-quasi ideals, which are subsets of b-semirings possessing quasi-ideal properties. b-semirings, extending semiring concepts by relaxing certain properties, hold significance in abstract algebra and find applications across mathematics and computer science. The paper likely categorizes bi-quasi ideals into different types based on their distinct properties, aiming to provide a comprehensive understanding of their characteristics within b-semirings. Moreover, it extends these concepts to broader contexts or related algebraic structures, enhancing their applicability and relevance.

This paper is dedicated to exploring several seminal results in tri ideals, aiming to extend the concept within the framework of b-semirings. Organized into six distinct sections, each section delves into specific aspects of tri ideals within this extended context. The structured organization begins by laying out foundational principles and gradually progresses towards more intricate analyses and applications. Throughout these sections, classical results are revisited, elucidated, and extended to accommodate the nuances inherent in b-semirings, thereby enriching our understanding of the interplay between tri ideals and this broader algebraic structure. Through this systematic arrangement, the paper offers a comprehensive treatment of tri ideals, showcasing their relevance and applicability within the real m of b-semirings and providing a valuable resource for further research and exploration in this area. First section is referred to as an introduction. There is a brief description of an ordered b-semiring, Type-1 Tri-ideals, quasi-ideals, bi ideals and related information is provided in Section 2. Section 3 discusses some theorems of Type-1 (Type-2) Tri-ideals with numerical examples. Section 4 discusses theorems of Type-2 Tri-ideals, with a valid example. A combined results are discussed in the Section 5. Finally, a conclusion is provided in Section 6. The objective of this paper:

(1) To demonstrate the implications based on Type-1(T_1) and Type-2(T_2) Tri-ideals.
(2) We are going to demonstrate some theorems in Type-1(T_1) and Type-2(T_2) Tri-ideals in quasi and bi-quasi ordered systems.
(3) To give suitable examples which will define the implications are not true in the reverse implication in case of a b-semiring.

2. Preliminaries

In this paper, $\diamond_1$and $\diamond_2$ represents min-max-product and max-min-product respectively.

Definition 2.1. *Let A and B be the subsets of* $(S, \diamond_1, \diamond_2)$. *Then the* $\diamond_1$*product and* $\diamond_2$*product of A and B, denoted by* $A\diamond_1 B$ *and* $A\diamond_2 B$ *respectively are defined as follows:* $A\diamond_1 B = \Big\{a\diamond_1 b | a \in A \text{ and } b \in B\Big\}$ *and* $A\diamond_2 B = \Big\{a\diamond_2 b | a \in A \text{ and } b \in B\Big\}$.

Definition 2.2. *The subset of A of S is known as weak-1 right ideal (weak-1 left ideal) of S if* $a_1\diamond_1 a_2 \in A$ *and* $a_1\diamond_2 s \in A$ $(s\diamond_2 a_1 \in A)$ *for all* $a_1, a_2 \in A$ *and* $s \in S$.

Definition 2.3. *The subset of A of S is known as weak-1 ideal of S if it is both weak-1 right ideal and weak-1 left ideal of S.*

Definition 2.4. *The subset of A of S is known as weak-2 right ideal of S (weak-2 left ideal) if* $a_1\diamond_2 a_2 \in A$ *and* $a_1\diamond_1 s \in A$ $(s\diamond_1 a_1 \in A)$ *for all* $a_1, a_2 \in A$ *and* $s \in S$.

Definition 2.5. *The subset of A of S is known as weak-2 ideal of S if it is both weak-2 right ideal and weak-2 left ideal of S.*

Definition 2.6. *The subset of A of S is known as right (left) ideal of S if it satisfies both weak-1 right(left) ideal and weak-2 right(left) ideal of S.*

Definition 2.7. *The subset of A of S is known as ideal of S if it is both right ideal and left ideal of S.*

3. T_1-Tri-ideals of b-semiring

In this section, we lay the groundwork for understanding the concept of 1-tri-ideals within the framework of b-semirings, focusing specifically on their properties. Within this context, where S denotes a b-semiring, we investigate the intersection of a 1-left (or right) tri-ideal with a weak-1-right (or left) ideal. To address this inquiry, we introduce the notion of a 1-left tri-ideal. By delineating the characteristics and behavior of 1-tri-ideals within b-semirings, we provide a systematic approach to understanding their intersection with weak-1-right (or left) ideals, thereby offering insights into the interplay between these algebraic structures and paving the way for further exploration into their properties and applications.

Definition 3.1. *A non-empty subset B of b-semiring S is said to be* 1*-right tri-ideal of S if B is a sub b-semiring of S and* $B\diamond_2 B\diamond_2 S\diamond_2 B \subseteq B$.

Definition 3.2. *A non-empty subset B of b-semiring S is said to be* 1*-left tri-ideal of S if B is a sub b-semiring of S and* $B\diamond_2 S\diamond_2 B\diamond_2 B \subseteq B$.

Definition 3.3. *A non-empty subset B of a b-semiring S is said to be* 1*-tri-ideal of S if B is a sub b-semiring of S and B is a* 1*-left tri-ideal and a* 1*-right tri-ideal of S.*

Theorem 3.1. *Every weak-*1*-left ideal is* 1*-tri-ideal of S.*

Proof. Let Q be weak-1-left ideal. Now, $Q\diamond_2 Q\diamond_2 S\diamond_2 Q \subseteq S\diamond_2 S\diamond_2 S\diamond_2 Q \subseteq S\diamond_2 Q \subseteq Q$ and $Q\diamond_2 S\diamond_2 Q\diamond_2 Q \subseteq S\diamond_2 S\diamond_2 S\diamond_2 Q \subseteq S\diamond_2 Q \subseteq Q$. Therefore, Q is a 1-tri-ideal. □

Remark 3.1. *The Converse is not necessarily true by the Example 3.1.*

Example 3.1. *Consider (S,$\diamond_2$) be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 \\ m_2 & m_3 & 0 & 0 \\ m_4 & m_5 & m_6 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$Let\ Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}.$$

$$Now,\ Q\diamond_2 Q\diamond_2 S\diamond_2 Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

$$Q\diamond_2 S\diamond_2 Q\diamond_2 Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

Therefore Q is a 1*-tri-ideal but not weak-*1*-left ideal of S by*

$$(S\diamond_2 Q) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\} \quad \not\subset B.$$

$$(Q\diamond_2 S) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ o_1 & 0 & 0 & 0 \end{pmatrix} \middle| o_i^{'s} \in Z^* \right\} \quad \not\subset B.$$

Corollary 3.1. *Every weak-*1*-right ideal is* 1*-tri-ideal of S.*

Theorem 3.2. *Every* 1*-quasi-ideal is* 1*-tri-ideal of S.*

Proof. Given that, $(S\diamond_2 B) \cap (B\diamond_2 S) \subseteq B$. Now, $B\diamond_2 B\diamond_2 S\diamond_2 B \subseteq B\diamond_2 S\diamond_2 S\diamond_2 S \subseteq B\diamond_2 S$ and $B\diamond_2 B\diamond_2 S\diamond_2 B \subseteq S\diamond_2 S\diamond_2 S\diamond_2 B \subseteq S\diamond_2 B$. Therefore, $B\diamond_2 B\diamond_2 S\diamond_2 B \subseteq (S\diamond_2 B) \cap (B\diamond_2 S) \subseteq B$. □

Remark 3.2. *Converse is not true by the following Example 3.2.*

Example 3.2. *Consider* $(S,\diamond_2)$ *be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 \\ m_2 & m_3 & 0 & 0 \\ m_4 & m_5 & m_6 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$Let\ B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}.$$

$$Now,\ B\diamond_2 B\diamond_2 S\diamond_2 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

$$B\diamond_2 S\diamond_2 B\diamond_2 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

Thus B is a 1*-tri-ideal but not* 1*-quasi ideal of S by*

$$(S\diamond_2 B) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ n_1 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\}.$$

$$(B\diamond_2 S) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ o_1 & 0 & 0 & 0 \end{pmatrix} \middle| o_i^{'s} \in Z^* \right\}.$$

$$(S\diamond_2 B)\cap(B\diamond_2 S)=\left\{\left.\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\p_1&0&0&0\end{pmatrix}\right| p_i^{'s}\in Z^*\right\}\quad\not\subset B.$$

Theorem 3.3. *Let L be a weak-1-left ideal and R is a weak-1-right ideal of S, then $B=R\cap L$ is a 1-tri-ideal of S. Converse is not true by the following Example 3.3.*

Proof. Given that L is weak-1-left ideal of S and R is weak-1-right ideal of S. To Prove that, $R\cap L$ is a 1-tri-ideal. Now, $(R\cap L)\diamond_2(R\cap L)\diamond_2 S\diamond_2(R\cap L)\subseteq R\diamond_2 R\diamond_2 S\diamond_2 R\subseteq R\diamond_2 S\diamond_2 S\diamond_2 S\subseteq R\diamond_2 S\subseteq R$ and $(R\cap L)\diamond_2(R\cap L)\diamond_2 S\diamond_2(R\cap L)\subseteq L\diamond_2 L\diamond_2 S\diamond_2 L\subseteq S\diamond_2 S\diamond_2 S\diamond_2 L\subseteq S\diamond_2 L\subseteq L$. Therefore, $(R\cap L)\diamond_2(R\cap L)\diamond_2 S\diamond_2(R\cap L)\subseteq R\cap L$ and $(R\cap L)M(R\cap L)(R\cap L)\subseteq R\diamond_2 S\diamond_2 R\diamond_2 R\subseteq R\diamond_2 S\diamond_2 S\diamond_2 S\subseteq R\diamond_2 S\subseteq R$ and $(R\cap L)\diamond_2 S\diamond_2(R\cap L)\diamond_2(R\cap L)\subseteq L\diamond_2 S\diamond_2 L\diamond_2 L\subseteq S\diamond_2 S\diamond_2 S\diamond_2 L\subseteq S\diamond_2 L\subseteq L$. Therefore, $(R\cap L)\diamond_2 S\diamond_2(R\cap L)\diamond_2(R\cap L)\subseteq R\cap L$. Then, $R\cap L$ is a 1-tri-ideal of S. □

Example 3.3. *Consider $(S,\diamond_2)$ be a b-semiring.*

$$Let\ S=\left\{\left.\begin{pmatrix}0&0&0&0&0\\m_1&0&0&0&0\\m_2&m_3&0&0&0\\m_4&m_5&m_6&0&0\\m_7&m_8&m_9&m_{10}&0\end{pmatrix}\right| m_i^{'m}\in Z^*\}\right\}.$$

$$Let\ R=\left\{\left.\begin{pmatrix}0&0&0&0&0\\a_1&0&0&0&0\\a_2&0&0&0&0\\a_3&a_4&0&0&0\\0&a_5&0&a_6&0\end{pmatrix}\right| a_i^{'s}\in Z^*\}\right\}.$$

$$Let\ L=\left\{\left.\begin{pmatrix}0&0&0&0&0\\b_1&0&0&0&0\\0&b_2&0&0&0\\b_3&b_4&0&0&0\\0&0&b_5&0&0\end{pmatrix}\right| b_i^{'s}\in Z^*\}\right\}.$$

$$Now,\ (R\cap L)=\left\{\left.\begin{pmatrix}0&0&0&0&0\\c_1&0&0&0&0\\0&0&0&0&0\\c_2&c_3&0&0&0\\0&0&0&0&0\end{pmatrix}\right| \in Z^*\right\}\quad\subseteq B.$$

Thus $(R \cap L)$ is a 1-tri-ideal but neither weak-1-left ideal L nor weak-1-right ideal R.

Theorem 3.4. *Let L be a weak-1-left ideal and R is a weak-1-right ideal of a b-semiring S, then $B = R\diamond_2 L$ is a 1-tri-ideal of S.*

Proof. Given that

$$(R\diamond_2 L)\diamond_2(R\diamond_2 L)\diamond_2 S\diamond_2(R\diamond_2 L) \subseteq R\diamond_2(L\diamond_2(R\diamond_2 L)\diamond_2 S\diamond_2(R\diamond_2 L)) \subseteq R(L\diamond_2 S\diamond_2 S\diamond_2 S\diamond_2 S\diamond_2 S) \subseteq (R\diamond_2 L)$$

and

$$(R\diamond_2 L)\diamond_2 S\diamond_2(R\diamond_2 L)\diamond_2(R\diamond_2 L) \subseteq ((R\diamond_2 L)\diamond_2 S\diamond_2(R\diamond_2 L)\diamond_2(R\diamond_2 L)) \subseteq (R\diamond_2 S\diamond_2 S\diamond_2 S\diamond_2 S\diamond_2 S)L \subseteq (R\diamond_2 L).$$

□

Theorem 3.5. *If Q is a 1-left bi-quasi ideal of b-semiring S, then Q is 1-tri-ideal of S.*

Proof. Suppose Q is a 1-left bi-quasi ideal of the Q-semiring S. Then $Q\diamond_2 S\diamond_2 Q \subseteq S\diamond_2 Q$. We have $Q\diamond_2 S\diamond_2 Q\diamond_2 Q \subseteq Q\diamond_2 S\diamond_2 Q$. Therefore $Q\diamond_2 S\diamond_2 Q\diamond_2 Q \subseteq Q\diamond_2 S \subseteq S\diamond_2 Q\diamond_2 Q \subseteq Q$. Thus Q is a 1-left tri-ideal of S. Similarly, we can show that Q is a 1-right tri-ideal of S. Thus Q is a 1-tri-ideal of S. □

Theorem 3.6. *Let S be a b-semiring and Q be a sub b-semiring of S and $Q = Q\diamond_2 Q$. Then Q is a 1-left tri-ideal of S if and only if there exist weak-1-left ideal L and a weak-1-right ideal R such that $R\diamond_2 L \subseteq Q \subseteq R \cap L$.*

Proof. . Suppose Q is a 1-tri-ideal of b-semiring S, then $Q\diamond_2 S\diamond_2 Q\diamond_2 Q \subseteq Q$. Let $R = Q\diamond_2 S$ and $L = S\diamond_2 Q$. Then R and L are a weak-1-right ideal and a weak-1-left ideal of S respectively. Therefore $R\diamond_2 L \subseteq Q \subseteq R \cap L$. Conversely, suppose that there exist R and L are a weak-1-right ideal and a weak-1-left ideal of S respectively such that $R\diamond_2 L \subseteq Q \subseteq R \cap L$. Then $Q\diamond_2 S\diamond_2 Q\diamond_2 Q \subseteq (R \cap L)\diamond_2 S\diamond_2(R \cap L)\diamond_2(R \cap L) \subseteq R\diamond_2 L \subseteq Q$. Thus Q is a 1-left tri-ideal of S. □

Theorem 3.7. *The intersection of a 1-left tri-ideal B of a b-semiring S and a weak-1-left ideal A of S is always a 1-left tri-ideal of S.*

Proof. Suppose $C = B \cap A$. Then $C\diamond_2 S\diamond_2 C\diamond_2 C \subseteq B\diamond_2 S\diamond_2 B\diamond_2 B \subseteq B$ and $C\diamond_2 S\diamond_2 C\diamond_2 C \subseteq A\diamond_2 S\diamond_2 A\diamond_2 A \subseteq A$. Since A is a weak-1-left ideal of S, we have $C\diamond_2 S\diamond_2 C\diamond_2 C \subseteq B \cap A = C$. Thus the intersection of a 1-left tri-ideal B of the b-semiring S and 1-left ideal of A of S is always a 1-left tri-ideal of S. □

Corollary 3.2. *The intersection of a 1-right tri-ideal B of a b-semiring S and a weak-1-right ideal A of S is always a 1-right tri-ideal of S. Converse is not true by the following Example 3.4.*

Example 3.4. *Consider (S,$\diamond_2$) be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ m_1 & 0 & 0 & 0 & 0 \\ m_2 & m_3 & 0 & 0 & 0 \\ m_4 & m_5 & m_6 & 0 & 0 \\ m_7 & m_8 & m_9 & m_{10} & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$Let\ A = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 \\ a_2 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & a_5 & 0 & 0 \\ 0 & a_6 & 0 & a_7 & 0 \end{pmatrix} \middle| a_i^{'s} \in Z^*\} \right\}.$$

$$Let\ B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 & 0 \\ b_3 & b_4 & 0 & 0 & 0 \\ 0 & 0 & b_5 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^*\} \right\}.$$

$$Now,\ (A \cap B) = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ c_2 & c_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \subseteq B.$$

Thus $(A \cap B)$ is a 1-right tri-ideal but neither 1-left ideal A nor right tri ideal B.

In the above section 3, we have discussed about the tri-ideals in different aspects and also proved under the operation max-min product $\diamond_2$ in 1-Tri ideals in *b*-semirings whereas the similar condition are also satisfying by the operator min-max product $\diamond_1$ in 2-Tri ideals in *b*-semirings.

4. T_2-Tri-ideals of *b*-semiring

In this section, we introduce the notion of 2-tri-ideal in *b*-semiring and study the properties of 2-tri-ideal of a *b*-semiring. Throughout this paper *S* is a *b*-semiring. What is the intersection of a 2-left(right) tri-ideal and a weak-2-left(right) ideal?. We answer the questions by introducing 2-left(right) tri-ideal.

Definition 4.1. *A non-empty subset B of a b-semiring S is said to be 2-right tri-ideal of S if B is a sub b-semiring of S and $B\diamond_1 B\diamond_1 S\diamond_1 B \subseteq B$.*

Definition 4.2. *A non-empty subset B of a b-semiring S is said to be 2-left tri-ideal of S if B is a sub b-semiring of S and $B\diamond_1 S\diamond_1 B\diamond_1 B \subseteq B$.*

Definition 4.3. *A non-empty subset B of a b-semiring S is said to be 2-tri-ideal of S if B is a sub b-semiring of S and B is a 2-left tri-ideal and a 2-right tri-ideal of S.*

Theorem 4.1. *Every weak-2-left ideal is a 2-tri-ideal of S.*

Proof. Let B be weak-2-left ideal. Now, $B\diamond_1 B\diamond_1 S\diamond_1 B \subseteq S\diamond_1 S\diamond_1 S\diamond_1 B \subseteq S\diamond_1 B \subseteq B$ and $B\diamond_1 S\diamond_1 B\diamond_1 B \subseteq S\diamond_1 S\diamond_1 S\diamond_1 B \subseteq S\diamond_1 B \subseteq B$. Therefore, B is a 2-tri-ideal. □

Remark 4.1. *Converse is not true by the following Example 4.1.*

Example 4.1. *Consider $(S,\diamond_1)$ be a b-semiring.*

$$\text{Let } S = \left\{ \begin{pmatrix} 0 & m_1 & m_2 & m_3 \\ 0 & 0 & m_4 & m_5 \\ 0 & 0 & 0 & m_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$\text{Let } B = \left\{ \begin{pmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}.$$

$$\text{Now, } B\diamond_1 B\diamond_1 S\diamond_1 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \subseteq B.$$

$$B\diamond_1 S\diamond_1 B\diamond_1 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \subseteq B.$$

Thus B is a 2-tri-ideal but not weak-2-left ideal of S by

$$(S\diamond_1 B) = \left\{ \begin{pmatrix} 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\} \quad \not\subset B.$$

$$(B\diamond_1 S) = \left\{ \begin{pmatrix} 0 & 0 & 0 & o_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| o_i^{'s} \in Z^* \right\} \quad \not\subset B.$$

Corollary 4.1. *Every weak-2-right ideal is a 2-tri-ideal of S.*

Theorem 4.2. *Every 2-quasi-ideal is a 2-tri-ideal of S.*

Proof. Given that $(S\diamond_1 B) \cap (B\diamond_1 S) \subseteq B$. Now, $B\diamond_1 B\diamond_1 S\diamond_1 B \subseteq B\diamond_1 S\diamond_1 S\diamond_1 S \subseteq B\diamond_1 S$ and $B\diamond_1 B\diamond_1 S\diamond_1 B \subseteq S\diamond_1 S\diamond_1 S\diamond_1 B \subseteq S\diamond_1 B$. Therefore, $B\diamond_1 B\diamond_1 S\diamond_1 B \subseteq (S\diamond_1 B) \cap (B\diamond_1 S) \subseteq B$. □

Remark 4.2. *Converse is not true by the following Example 4.2.*

Example 4.2. *Consider (S,$\diamond_1$) be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & m_1 & m_2 & m_3 \\ 0 & 0 & m_4 & m_5 \\ 0 & 0 & 0 & m_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^* \} \right\}.$$

$$Let\ B = \left\{ \begin{pmatrix} 0 & 0 & b_1 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\}.$$

$$Now,\ B\diamond_1 B\diamond_1 S\diamond_1 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

$$B\diamond_1 S\diamond_1 B\diamond_1 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

Thus B is a 2-tri-ideal but not 2-quasi ideal of S by

$$(S\diamond_1 B) = \left\{ \begin{pmatrix} 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\}.$$

$$(B\diamond_1 S) = \left\{ \begin{pmatrix} 0 & 0 & 0 & o_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| o_i^{'s} \in Z^* \right\}.$$

$$(S\diamond_1 B) \cap (B\diamond_1 S) = \left\{ \begin{pmatrix} 0 & 0 & 0 & p_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| p_i^{'s} \in Z^* \right\} \quad \not\subseteq B.$$

Theorem 4.3. *If L is a weak-2-left ideal and R is a weak-2-right ideal of S, then $B = R \cap L$ is a 2-tri-ideal of S. Converse is not true by the following Example 4.3*

Proof. Given that L is weak-2-left ideal of S and R is weak-2-right ideal of S. To Prove that, $R \cap L$ is a 2-tri-ideal. Now, $(R\cap L)\diamond_1(R\cap L)\diamond_1 S\diamond_1(R\cap L) \subseteq R\diamond_1 R\diamond_1 S\diamond_1 R \subseteq R\diamond_1 S\diamond_1 S\diamond_1 S \subseteq R\diamond_1 S \subseteq R$ and $(R\cap L)\diamond_1(R\cap L)\diamond_1 S\diamond_1(R\cap L) \subseteq L\diamond_1 L\diamond_1 S\diamond_1 L \subseteq S\diamond_1 S\diamond_1 S\diamond_1 L \subseteq S\diamond_1 L \subseteq L$. Therefore, $(R\cap L)\diamond_1(R\cap L)\diamond_1 S\diamond_1(R\cap L) \subseteq R\cap L$ and $(R\cap L)\diamond_1 S\diamond_1(R\cap L)\diamond_1(R\cap L) \subseteq R\diamond_1 S\diamond_1 R\diamond_1 R \subseteq R\diamond_1 S\diamond_1 S\diamond_1 S \subseteq R\diamond_1 S \subseteq R$ and $(R\cap L)\diamond_1 S\diamond_1(R\cap L)\diamond_1(R\cap L) \subseteq L\diamond_1 S\diamond_1 L\diamond_1 L \subseteq S\diamond_1 S\diamond_1 S\diamond_1 L \subseteq S\diamond_1 L \subseteq L$. Therefore, $(R\cap L)\diamond_1 S\diamond_1(R\cap L)\diamond_1(R\cap L) \subseteq R\cap L$. Then, $R\cap L$ is a 2-tri-ideal of S. □

Example 4.3. *Consider $(S,\diamond_1)$ be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & m_1 & m_2 & m_3 & m_4 \\ 0 & 0 & m_5 & m_6 & m_7 \\ 0 & 0 & 0 & m_8 & m_9 \\ 0 & 0 & 0 & 0 & m_{10} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$Let\ L = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| a_i^{'s} \in Z^*\} \right\}.$$

$$Let\ R = \left\{ \begin{pmatrix} 0 & b_1 & 0 & b_2 & 0 \\ 0 & 0 & b_3 & b_4 & 0 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^*\} \right\}.$$

$$Now,\ (R \cap L) = \left\{ \begin{pmatrix} 0 & c_1 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

Thus $(R \cap L)$ *is a* 2*-tri-ideal but neither weak-*2*-left ideal L nor weak-*2*-right ideal R.*

Theorem 4.4. *If L is a weak-*2*-left ideal and R is a weak-*2*-right ideal of a b-semiring S, then* $B = R\diamond_1 L$ *is a* 2*-tri-ideal of S.*

Proof. Given that

$$(R\diamond_1L)\diamond_1(R\diamond_1L)\diamond_1S\diamond_1(R\diamond_1L) \subseteq R\diamond_1(L\diamond_1(R\diamond_1L)\diamond_1S\diamond_1(R\diamond_1L)) \subseteq R(L\diamond_1S\diamond_1S\diamond_1S\diamond_1S\diamond_1S) \subseteq (R\diamond_1L)$$

and

$$(R\diamond_1L)\diamond_1S\diamond_1(R\diamond_1L)\diamond_1(R\diamond_1L) \subseteq ((R\diamond_1L)\diamond_1S\diamond_1(R\diamond_1L)\diamond_1(R\diamond_1L)) \subseteq (R\diamond_1S\diamond_1S\diamond_1S\diamond_1S\diamond_1S)L \subseteq (R\diamond_1L).$$

□

Theorem 4.5. *If B is a* 2*-left bi-quasi ideal of as b-semiring S, then B is* 2*-tri-ideal of S.*

Proof. Suppose B is a 2-left bi-quasi ideal of the b-semiring S. Then $B\diamond_1S\diamond_1B \subseteq S\diamond_1B$. We have $B\diamond_1S\diamond_1B\diamond_1B \subseteq B\diamond_1S\diamond_1B$. Therefore $B\diamond_1S\diamond_1B\diamond_1B \subseteq B\diamond_1S \subseteq S\diamond_1B\diamond_1B \subseteq B$. Thus B is a 2-left tri-ideal of S. Similarly we can show that B is a 2-right tri-ideal of S. Thus B is a 2-tri-ideal of S. □

Theorem 4.6. *Let S be a b-semiring and B be a sub b-semiring of S and* $B = B\diamond_1B$*. Then B is a* 2*-left tri-ideal of S if and only if there exist weak-*2*-left ideal L and a weak-*2*-right ideal R such that* $R\diamond_1L \subseteq B \subseteq R \cap L$*.*

Proof. Suppose B is a 2-tri-ideal of the b-semiring S, then $B\diamond_1S\diamond_1B\diamond_1B \subseteq B$. Let $R = B\diamond_1S$ and $L = S\diamond_1B$. Then R and L are a weak-2-right ideal and a weak-2-left ideal of S respectively. Therefore $R\diamond_1L \subseteq B \subseteq R \cap L$. Conversely, suppose that there exist R and L are a weak-2-right ideal and a weak-2-left ideal of S respectively such that $R\diamond_1L \subseteq B \subseteq R \cap L$. Then $B\diamond_1S\diamond_1B\diamond_1B \subseteq (R \cap L)\diamond_1S\diamond_1(R \cap L)\diamond_1(R \cap L) \subseteq R\diamond_1L \subseteq B$. Thus B is a 2-left tri-ideal of S. □

Theorem 4.7. *The intersection of a* 2*-left tri-ideal B of a b-semiring S and a weak-*2*-left ideal A of S is always a* 2*-left tri-ideal of S.*

Proof. Suppose $C = B \cap A$. Then $C\diamond_1 S\diamond_1 C\diamond_1 C \subseteq B\diamond_1 S\diamond_1 B\diamond_1 B \subseteq B$ and $C\diamond_1 S\diamond_1 C\diamond_1 C \subseteq A\diamond_1 S\diamond_1 A\diamond_1 A \subseteq A$. Since A is a weak-2-left ideal of S, we have $C\diamond_1 S\diamond_1 C\diamond_1 C \subseteq B \cap A = C$. Thus the intersection of a 2-left tri-ideal B of the b-semiring S and 2-left ideal of A of S is always a 2-left tri-ideal of S. □

Corollary 4.2. *The intersection of a 2-right tri-ideal B of a b-semiring S and a weak-2-right ideal A of S is always a 2-right tri-ideal of S. Converse is not true by the following Example 4.4.*

Example 4.4. *Consider $(S,\diamond_1)$ be a b-semiring.*

$$Let\ S = \left\{ \begin{pmatrix} 0 & m_1 & m_2 & m_3 & m_4 \\ 0 & 0 & m_5 & m_6 & m_7 \\ 0 & 0 & 0 & m_8 & m_9 \\ 0 & 0 & 0 & 0 & m_{10} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'m} \in Z^*\} \right\}.$$

$$Let\ L = \left\{ \begin{pmatrix} 0 & a_1 & a_2 & a_3 & 0 \\ 0 & 0 & 0 & a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| a_i^{'s} \in Z^*\} \right\}.$$

$$Let\ R = \left\{ \begin{pmatrix} 0 & b_1 & 0 & b_2 & 0 \\ 0 & 0 & b_3 & b_4 & 0 \\ 0 & 0 & 0 & 0 & b_5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^*\} \right\}.$$

$$Now,\ (R \cap L) = \left\{ \begin{pmatrix} 0 & c_1 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \in Z^* \right\} \quad \subseteq B.$$

Thus $(A \cap B)$ is a 2-right tri-ideal but neither weak-2-left ideal A nor right tri ideal B.

5. Tri-ideals in b-semirings

Theorem 5.1. *Every left ideal is a tri-ideal of S.*

Proof. The Proof follows from Theorem 3.1 and Theorem 4.1. □

Theorem 5.2. *Every quasi-ideal is a tri-ideal of S.*

Proof. The Proof follows from Theorem 3.2 and Theorem 4.2. □

Theorem 5.3. *If L is a left ideal and R is a right ideal of S then* $B = R \cap L$ *is a tri-ideal of S.*

Proof. The Proof follows from Theorem 3.3 and Theorem 4.3. □

Theorem 5.4. *If L is a left ideal and R is a right ideal of a b-semiring S then* $B = R * L$ *is a tri-ideal of S.*

Proof. The Proof follows from Theorem 3.4 and Theorem 4.4. □

Theorem 5.5. *Let B be a left bi-quasi ideal of b-semiring S, then B is tri-ideal of S.*

Proof. The Proof follows from Theorem 3.5 and Theorem 4.5. □

Theorem 5.6. *Let S be a b-semiring and B be a sub b-semiring of S and* $B = B * B$*. Then B is a left tri-ideal of S if and only if there exist left ideal L and a right ideal R such that* $R * L \subseteq B \subseteq R \cap L$*.*

Proof. The Proof follows from Theorem 3.6 and Theorem 4.6. □

Theorem 5.7. *The intersection of a left tri-ideal B of a b-semiring S and a left ideal A of S is always a left tri-ideal of S.*

Proof. The Proof follows from Theorem 3.7 and Theorem 4.7. □

6. Conclusion

This article delves into the analysis of Type-1 and Type-2 tri-ideals within ordered *b*-semirings, scrutinizing their behavior in relation to the operations $\diamond_2$ and $\diamond_1$, along with providing characterizations of these quasi tri-ideals. Investigations into various properties of tri-ideals are conducted, alongside elucidating methods for generating diverse tri-ideals within an ordered *b*-semiring utilizing individual elements and subsets thereof. Future endeavors are outlined to further characterize additional classes of tri-ideals within *b*-semirings, incorporating the exploration of maximal and minimal tri-ideals, and their implications within the context of ordered *b*-semirings, considering diverse tri ideals and their generation by elements and subsets. The discussion also extends to examining the relationship between tri-ideals and quasi-ideals, with plans to delve into other types of prime tri-ideals in subsequent research. Additionally, forthcoming studies will explore the application of tri-ideals and tri-quasi-ideals in the analysis of hyper *b*-semirings, thereby enriching the understanding of these algebraic structures and their implications across various mathematical domains.

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