Int. J. Anal. Appl. (2025), 23:120

International Journal of Analysis and Applications

Decision-Making on a Common Equilibrium Marketing Fixed-Point Theorem for Two and Three Meir-Keeler Condensing Supply Mappings in Banach Space

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Abstract. This paper is aimed to prove common equilibrium marketing fixed-point theorem for two and three mappings in Banach space by the use of measure of non-compactness on Meir Keeler condensing supply operators. We attempt to show the existence of common equilibrium marketing fixed-point theorem for two and three commuting supply maps in this paper.

1. Introduction and preliminaries

There is a wide range use of Compactness in fixed point theory, many authors like G. Darbo in 1955 (see [14]) studied fixed point theory by using non compactness and non compact operators ,Schauder used compactness in fixed point theory. The non compact operators are studied mainly to develop a new class of operators which converts the boundedness to compactness in sets via Meir Keeler condensing operators.

Received: Mar. 19, 2025.

2020 *Mathematics Subject Classification.* 47H08, 37C25, 54H25, 55M20, 58C30.

Key words and phrases. Schaurder fixed point theorem; Darbo's fixed point theorem; common equilibrium marketing; Banach space; Meir Keeler condensing supply operators; measure of non-compactness.

https://doi.org/10.28924/2291-8639-23-2025-120

ISSN: 2291-8639

Definition 1.1: Let M be the subset of metric space N and then Kuratowski's measure of non compactness is defined as

$$\theta(M) = \inf\Big\{\gamma > 0 : M = \cup_{j=1}^{n} V_j \text{ for some } V_j \text{ with } \operatorname{diam}(V_j) \le \gamma, 1 \le j \le n < \infty\Big\}.$$

Here diam $(V_j) = \sup\{d(a,b) : a, b \in V_j\}$ (see [15]).

Definition 1.2: Let M be a non-void bounded subset of M_B,where M_B denotes the non-void class of subsets of Banach space B , then the map $\alpha : M_B \to [0,\infty)$ is said to be Hausdorff measure of non compactness of $M \subseteq M_B$ such that

$$\alpha(M) = \inf\Big\{\delta > 0 : M \text{ has a finite } \delta-\text{net in B}\Big\} \text{ (see [11]).}$$

Definition 1.3: Let M be a non-void bounded subset of M_N, where M_N denotes the class of non-void subsets of complete metric space N then the function $\alpha : M_N \to [o,\infty)$ is a Hausdorff measure of non compactness of $M \subseteq M_N$ such that

$$\alpha(M) = \inf\Big\{\delta > 0 : M \subseteq \cup_{i=1}^{n} B'(a_i, b_i), a_i \in N, b_i < \delta \;\; i = 1,2,3,....,n\Big\} \text{ (see [11]).}$$

Theorem 1.4: Schauder's fixed point theorem: Let Θ be a non-empty bounded subsets of Banach space B then for every continuous function $A : \Theta \to \Theta$ has atleast one fixed point (see [11]). We abbreviate Schauder's fixed point theorem as S. f. t throughout this paper.

Definition 1.5: Let ν_B is the class of all non void bounded subsets of Banach space B then the function $m* : \nu_B \longrightarrow \mathbf{R}_+$ is the measure of non compactness in the Banach space B if following axioms holds

1. The class Ker $m* = \{F \in \nu_B : m^*(F) = 0\}$ is non void and Ker $m^* \subseteq \nu_B$, where ν_B is the class of non void relatively compact subsets of B.
2. $F_1 \subseteq F_2 \Rightarrow m^*(F_1) \le m^*(F_2)$
3. $m^*(F) = m^*(\overline{F})$
4. $m^*(F) = m^*(conv(F))$
5. $m^*\{\lambda F_1 + (1-\lambda)(F_2)\} \le \lambda m^*(F_1) + (1-\lambda)m^*(F_2), \forall \lambda \in [0,1]$
6. If $\{F_n\}$ is the sequence of closed sets from ν_B s.t $F_{n+1} \subseteq F_n$ for $n = 1,2,3,....$ and $\lim\limits_{n\to\infty} m^*(F_n) = 0$, then the intersection $F_\infty = \cap_{i=1}^{\infty} F_n$ is non void (see [11]).

Definition 1.6: Let Θ be a non void closed,bounded and convex subset of Banach space B. A map $P : \Theta \to \Theta$ is called a m^* contractive map if for some constant $k \in (0,1)$ s.t $m^*(P(F)) \le km^*(F)$, for non void subset F of Θ.

Theorem 1.7: Darbo's fixed point theorem: Let Θ be a non-void,closed and bounded subsets of Banach space B, let $T : \Theta \to \Theta$ be a continuous map and if T is $m*$ contraction then T has atleast one fixed point (see [4]).

Definition 1.8: Meir Keeler Contraction: A self map T from metric space N into N is a Meir Keeler contraction if for given $\rho > 0$, there exists $\sigma > 0$ such that

$$\rho \le d(a,b) < \rho + \sigma \Rightarrow d(T_a, T_b) < \rho, \quad \text{for all } a, b \in M.$$

Definition 1.9:Meir Keeler Condensing Operator: Let Θ be a non-empty subset of Banach space B. Then the operator $T : \Theta \to \Theta$ is said to be a Meir Keeler condensing operator if for given $\rho > 0$, there exists $\sigma > 0$ s.t

$$\rho \leq m^*(F) < \rho + \sigma \Rightarrow m^*(T(F)) < \rho$$

where m^* is arbitrary measure of non compactness on F and F is the bounded subset of Θ. For more details about Meier Keeler condensing operator and measure of non compactness we may refer to ([1–3, 5, 6, 8–10, 13, 16]) and references therein.

The primary goal of this manuscript is to show the existence of equilibrium marketing fixed-point theorem by using measure of non compactness for two and three commuting Meir Keeler condensing operators in Banach space.

2. Common fixed point theorem via two continuous linear operators on Banach Space B

Theorem 2.1 : *Let B be the Banach space and Θ be a non void closed,bounded and convex subset of the Banach space B.Let J and K be two self operators on Θ are continuous and K is Meir Keeler condensing operator such that $K(J(F)) \subseteq J(F), F \subseteq \Theta$.We have $m^*(J(F)) \leq \xi\{max\{m^*(F), m^*(K(F)\}\}$, where m^* defined on ν_B is a measure of non compactness and ξ from R_+ to R_+ is a increasing function, such that $\xi(x) < x$, for $x \geq 0$,*

$$\lim_{n\to\infty} \xi^n(x) = 0,$$

then J and K have a common equilibrium marketing fixed-point.

Proof. Let F be a non void subset of Θ and m^* the measure of non compactness defined on ν_B.
Now we consider a sequence of subsets $\{\Theta_n\}$ of B as $\Theta = \Theta_0, \Theta_n = convJ(\Theta_{n-1}), n \geq 1$.
Then $K(\Theta_n) \subset \Theta_n$ and $\Theta_n \subseteq \Theta_{n-1}$ (2.1)
Clearly $K(\Theta_1) \subseteq Conv(KJ(\Theta_0)) \subseteq Conv(J(\Theta_0)) = \Theta_1$ and $\Theta_1 \subset \Theta_0$. Therefore $K(\Theta_1) \subseteq \Theta_1$ and so equation (2.1) is true for n=1. Let it be true for $n \geq 1$ then $\Theta_{n+1} = Conv(J(\Theta_n)) \subseteq Conv(J(\Theta_{n-1})) = \Theta_n$, as $\Theta_n \subseteq \Theta_{n-1}$, thus $\Theta_{n+1} \subseteq \Theta_n$ and

$$K(\Theta_{n+1}) = K(Conv(J(\Theta_n))) \subseteq Conv(KJ(\Theta_n)) \subseteq ConvJ(\Theta_n) = \Theta_{n+1},$$

hence $K(\Theta_{n+1}) \subseteq \Theta_{n+1}$ and this implies that $\Theta_0 \supset \Theta_1 \supset \Theta_2$...
If $m^*(\Theta_n) = 0$ for some $n \geq 0$ then Θ_n is relatively compact and since $J(\Theta_n) \subseteq ConvJ(\Theta_n) = \Theta_{n+1} \subseteq \Theta_n$, thus by S. f. t J has fixed point.
Now, suppose $m^*(\Theta_n) \neq 0, n \geq 0$. Define $\rho_n = m^*(\Theta_n)$ and $\sigma_n = \sigma(\rho_n) > 0$.

Now by definition of Θ_n and $\rho_n < \rho_n + \sigma_n$,

$$\begin{aligned}
\rho_{n+1} &= m^*(\Theta_{n+1}) \\
&= m^*(convJ(\Theta_n)) \\
&= m^*(J(\Theta_n)) \\
&\leq \xi\{max\{m^*(\Theta_n), m^*(K(\Theta_n)\}\} \\
&\leq \xi(m^*(J(\Theta_n))) \\
&\leq (m^*(J(\Theta_n)) \\
&\leq (m^*(\Theta_n)) = \rho_n.
\end{aligned}$$

Therefore, $\rho_{n+1} \leq \rho_n$ implies that $\{\rho_n\}$ is the non-increasing sequence of positive real numbers and for some $r \geq 0$ such that $\rho_n \to r$ as $n \to \infty$. We will show the case for $r = 0$, let us assume that $r \neq 0$ then for some n_0 such that $n > n_0$ implies $r \leq \rho_n < r + \sigma(r)$. Hence by definition of Meir Keeler condensing operator we have $\rho_{n+1} < r$, which is not true, so $r = 0$. Hence, we have $\lim_{n\to\infty} m^*(\Theta_n) \to 0$.
Therefore, $\Theta_{n+1} \subseteq \Theta_n$ and condition 6 of measure of non compactness implies that $\Theta_\infty = \cup_{n=1}^{\infty}\Theta_n$ is non void closed and convex set with $\Theta_\infty \subset \Theta$. Also the operator J keeps Θ_∞ invariant and lies in $Kerm^*$, thus J has a fixed point in Θ by S.f.t .
Now, Consider $G_J = \{t \in \Theta : J(t) = t\}$.
Then by continuity of J, clearly G_J is closed and given K is Meir Keeler operator , we have $K(G_J) \subseteq G_J$. So $K(t)$ is the fixed point of J for any $t \in G_J$ and

$$\begin{aligned}
m^*(G_J) &= m^*(J(G_J)) \\
&\leq \xi(max\{m^*(G_J), m^*(Q(G_J))\}) \\
&= \xi(m^*(G_J)) \\
&\leq m^*(G_J).
\end{aligned}$$

Therefore, G_J is compact and as $m^*(G_J) = 0$ then K has fixed point by S.f.t and by continuity of K, the set $G_K = \{t \in \Theta : K(t) = t\}$ is closed . Also by S. f. t, $J(t)$ is the fixed point of K as $K(G_J) \subseteq G_J$, for all $t \in G_K$. Since $J, K : G_J \cap G_K \to G_J \cap G_K$ are continuous self maps and $G_J \cap G_K \subseteq G_J \subseteq \Theta$ is compact subset so , J & K have a common equilibrium marketing fixed-point in Θ by S. f. t . □

Corrolary 2.2 *: Let Θ be a non empty closed,bounded and convex subset of the Banach space B. Suppose J and K be commutative continuous self operators on Θ and K is Meir Keller condensing operator such that $K(J(F)) \subseteq J(F)$, $F \subseteq \Theta$. We have*

$$m^*(J(F)) \leq \xi(max\{m^*(F),\ m^*(K(F))\}),$$

where m^* is any measure of non compactness defined on ν_B, ξ is increasing functions from R_+ to R_+ such that $\xi(x) < x$, for $x \geq 0$ and F is non empty subset of Θ,

$$\lim_{n\to\infty} \xi^n(x) = 0,$$

then maps J and K have a common equilibrium marketing fixed-point fixed point.

Proof. We omit the details as the proof is straightforward. □

Remark: *In next theorem we use three continuous Commuting Meir Keeler operators on Banach space and prove common equilibrium marketing fixed-point theorem for them.*

Theorem 2.3: *Let B be a Banach space and Θ be a non void bounded, closed and convex subset of B. Suppose J, K & L be three continuous linear operators from Θ into Θ and K, L are Meir Keeler condensing operator such that*

1. *J, K & L are mutually commutative*
2. *K & L are Meir Keeler condensing operators and $K(J(F)) \subseteq J(F)$, $L(J(F)) \subseteq J(F)$*
3. *For any subset $F \subseteq \Theta$, We have*

$$m^*(L(F)) \leq \xi(\{m^*K(F) - \xi(m^*(K(convJ(F))))\}),$$

where m^* is arbitrary measure of non compactness defined on ν_B and ξ is a increasing functions from R_+ to R_+ such that $\xi(x) < x$, for $x \geq 0$,

$$\lim_{n\to\infty} \xi^n(x) = 0,$$

then there exists a common equilibrium marketing fixed-point of J, K and L.

Proof. Let m^* is a measure of non compactness defined on ν_B and F be a non empty subset of B. Now we consider a sequence of subsets $\{\Theta_n\}$ of B as $\Theta = \Theta_0, \Theta_n = convJ\Theta_{n-1}, n \geq 1$. such that $\Theta_n \subseteq \Theta_{n-1}$ and $K(\Theta_n) \subset \Theta_n$ (2.2)
Clearly $\Theta_1 \subset \Theta_0$ and $K(\Theta_1) \subseteq Conv(KJ(\Theta_0)) \subseteq Conv(J(\Theta_0)) = \Theta_1$. Therefore $K(\Theta_1) \subseteq \Theta_1$ and thus equation (2.2) is true for n = 1. Let it be true for $n \geq 1$ then $(\Theta_{n+1}) = Conv(J(\Theta_n)) \subseteq Conv(J(\Theta_{n-1})) = \Theta_n$, as $\Theta_n \subseteq \Theta_{n-1}$, thus $\Theta_{n+1} \subseteq \Theta_n$ and

$$K(\Theta_{n+1}) = K(Conv(J(\Theta_n))) \subseteq Conv(KJ(\Theta_n)) \subseteq ConvJ(\Theta_n) = \Theta_{n+1},$$

hence $K(\Theta_{n+1}) \subseteq \Theta_{n+1}$ and this implies that $\Theta_0 \supset \Theta_1 \supset \Theta_2 ...$
If $m^*(\Theta_n) = 0$ for some $n \geq 0$ then Θ_n is relatively compact and thus by S. f. t J has fixed point as $J(\Theta_n) \subseteq ConvJ(\Theta_n) = \Theta_{n+1} \subseteq \Theta_n$.
Now, we assume that $m^*(\Theta_n) \neq 0, n \geq 0$. Define $\rho_n = \xi(m^*K(\Theta_n))$ and $\sigma_n = \sigma(\rho_n) > 0$.
Now by definition of Θ_n and $\rho_n < \rho_n + \sigma_n$, then given inequality implies $\xi(m^*K(\Theta_n)) -$

$\xi(m^*(K(convJ(\Theta_n)))) \geq 0$, for all n ≥ 0. Thus, we have

$$\begin{aligned} \rho_{n+1} &= \xi(m^*K(\Theta_n)) \\ &= \xi(m^*(K(convJ(\Theta_n)))) \\ &\leq \xi(m^*(K(\Theta_n))) \\ &= \rho_n, \end{aligned}$$

for all $n \geq 0$. Therefore, $\rho_{n+1} \leq \rho_n$ implies that $\{\rho_n\}$ is the non increasing sequence of positive real numbers and for some $r \geq 0$ such that $\rho_n \to r$ as $n \to \infty$. We will show the case for $r = 0$, let us assume that $r \neq 0$ then for some n_0, $n > n_0$ implies $r \leq \rho_n < r + \sigma(r)$. Thus by Meir Keeler condensing operator definition we get $\rho_{n+1} < r$, which is not true as per our supposition , so $r = 0$. Hence, we have $\lim\limits_{n\to\infty} \xi\{ m^*K(\Theta_n)\} \to 0 \Rightarrow \lim\limits_{n\to\infty} m^*K(\Theta_n) \to 0$. Also $m^*L(\Theta_n) \leq \xi(m^*K(\Theta_n)) - \xi(m^*K(convJ\Theta_n)) = \rho_n - \rho_{n+1}$. So, $\lim\limits_{n\to\infty} m^*L(\Theta_n) = 0$.

Now if we fix $\Theta'_n = m^*(\overline{L(\Theta_n)})$ then by definition of measure of non compactness, $m^*(\Theta'_n) = m^*(\overline{L(\Theta_n)}) = m^*(L(\Theta_n))$. Hence, $\lim\limits_{n\to\infty} m^*(\Theta'_n) = \lim\limits_{n\to\infty} m^*L(\Theta_n) \to 0$. Since $\{\Theta_n\}$ is a nested sequence ,so $\Theta'_{n+1} \subseteq \Theta'_n, \forall n \in N$. Thus $m^*(\Theta'_\infty) \leq m^*(\Theta'_n), \forall n \in N$ and $\Theta'_\infty = \cap_{i=1}^{\infty} \Theta'_n$ is non empty. Therefore, $m^*(\Theta'_\infty) = 0 \quad as \quad n \to \infty$, so Θ'_∞ is closed, compact and convex as L is Meir Keeler condensing operator.

Also $K(\Theta_n) \subseteq (\Theta_n)$ and $J(\Theta_n) \subseteq (\Theta_n)$, so we obtain

$$K(\Theta'_n) = K(\overline{L(\Theta_n)} \subseteq \overline{K(L(\Theta_n))} \subseteq \overline{L(K(\Theta_n)} \subseteq \overline{L(\Theta_n)} = \Theta'_n$$

$$J(\Theta'_n) = J(\overline{L(\Theta_n)} \subseteq \overline{J(L(\Theta_n))} \subseteq \overline{L(J(\Theta_n)} \subseteq \overline{L(\Theta_n)} = \Theta'_n$$

and similarly

$$L(\Theta'_n) = L(\overline{L(\Theta_n)} \subseteq \overline{L(L(\Theta_n))} \subseteq \overline{L(L(\Theta_n)} \subseteq \overline{L(\Theta_n)} = \Theta'_n.$$

Hence the set Θ'_∞ is invariant under the operators J, K, L belongs to $Kerm^*$,so by S.f.t J, K, L have fixed point in Θ. Now, Suppose $G_L = \{t \in \Theta : J(t) = t\}$. Then obviously G_L is closed because of continuity of L and by assumption $K\&L$ are Meir Keeler condensing operator,so $K(G_L) \subseteq G_L, L(G_L) \subseteq G_L$. So $K(t)$ is the fixed point of L for any $t \in G_L$ and

$$\begin{aligned} m^*(G_L) &= m^*(L(G_L)) \\ &\leq \xi(m^*K(G_L)) - \xi(m^*(K(convJ(G_L)))\}) \\ &\leq \xi(m^*K(G_L)) \\ &\leq m^*K(G_L) \\ &\leq m^*(G_L). \end{aligned}$$

Therefore G_L is compact as $m^*(G_L) = 0$. Thus K has fixed point and thus K and L have common equilibrium marketing fixed-point and set $G = \{t \in \Theta : L(t) = K(t) = t\}$ is convex and closed subset of Θ and $J(G) \subseteq G$. Hence a fixed point of J exists in G and so a common fixed point exists in Θ of J, K and L. $\square$

Authors contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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