Int. J. Anal. Appl. (2025), 23:37

International Journal of Analysis and Applications

Analysis of Nonlinear Hadamard Fractional Differential Inclusions via Measure of Noncompactness

Habib Djourdem[1], **Mohammed Ali Alghamdi**[2,*], **Elsayed Mohamed Elsayed**[2,3]

[1]*Department of Mathematics, Faculty of Science and Technology, Laboratory of Analysis and Geometry and Its Applications (W1421000), Relizane University, Relizane, Algeria*

[2]*Mathematics Department, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah, Saudi Arabia*

[3]*Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*

[*]*Corresponding author:* mamalgamdi@kau.edu.sa

Abstract. The goal of this paper is to consider a class of Hadamard fractional differential inclusions with three point integral boundary conditions. The proof is based on the set-valued analog of Mönch fixed point theorem combined with the technique of measures of noncompactness in order to establish the existence of at least one solution and an illustrative example is given to show the applicability of this obtained result. We also investigate some Filippov's type results for this problem.

1. Introduction

Fractional differential equations have received considerable attention due to their description of many physical phenomena in various fields of science and engineering, including viscoelasticity, physics, mechanics, aerodynamics, control theory, signal and image processing, biology, environmental science, materials, economics, and fluid dynamics (see [1, 14, 18] and their references).

Boundary value problems of fractional differential equations implicit variuos types of fractional derivatives as Riemann-Liouville-type, Caputo-type, Hadamard-type, Caputo-Hadamard-type and Hilfer-Hadamard-type fractional derivative with different kinds of boundary conditions have studied by many researchers (see [2, 6, 20, 22, 24]).

Integro-differential inclusions arise in the mathematical modeling of various problems in economics, optimal control, and stochastic analysis, see for instance ([15, 19, 25]). Some interesting

Received: Oct. 8, 2024.

2020 *Mathematics Subject Classification.* 34A08, 34A60, 47H08.

Key words and phrases. differential inclusion; Mönch fixed point theorem; Hadamard fractional integral; Filippov's theorem.

https://doi.org/10.28924/2291-8639-23-2025-37

ISSN: 2291-8639

results about initial and boundary value problems of fractional differential inclusions can be found in [3,10,21].

In the works mentioned above, compactness and Lipschitz condition are satisfied, if not these techniques cannot be used. Hence, there have been many published papers, which are devoted to the existence of solutions of nonlinear integro-differential equations by using the technique of a suitable measure of noncompactness in Banach algebras. We refer the readers to [9,17] and references therein.

Filippov's solutions for various classes of integer or fractional order differential inclusions have been considered in the literature; see for instance [8,13,16].

The main goal of this work is to study the following problem of fractional differential inclusion with nonlocal fractional integro-differential boundary conditions of the form

$$\begin{cases} \mathfrak{D}^{\varrho}\vartheta(\xi) \in \Pi(\xi,\vartheta(\xi)), \ \xi \in J = [1,\Gamma], \ 2 < \varrho \leq 3 \\ \vartheta(1) = 0, \ \mathfrak{D}^{\varrho-1}\vartheta(1) = 0, \ \vartheta(\Gamma) = \kappa(I^{z}\vartheta)(\ell) \end{cases} \tag{1.1}$$

where $\mathfrak{D}^{\varrho}$ denotes the Hadamard fractional derivative of order $2 < \varrho \leq 3$. I^{z} is the Hadamard fractional integral of order $z > 0$, $\Pi : [1,e] \times \mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, κ, ℓ are two real parameters with $\kappa > 0$, $1 < \ell < \Gamma$ and $\frac{\kappa\Gamma(\varrho-1)}{\Gamma(\varrho+z-1)}(\log \ell)^{\varrho+z-2} \neq 1$.

The rest of this paper is divided into two sections. In Section 2, some required concepts are presented that will be used in the sequel. Our main results is established in Section 3 which is divided into two susections.

2. Basic results

In this section, we give a collection of auxiliary facts which will be needed in the proof of the main results. Let $C(I,\mathbb{R})$ be the Banach space of all continuous functions from I into $\mathbb{R}$ with the norm

$$\|\vartheta\|_{\infty} = \sup\left\{\left|\vartheta(\xi)\right| : \ \xi \in I\right\}$$

$L^{1}(J,\mathbb{R})$ refers to the Banach space of measurable functions $\chi : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable; it is normed by

$$\|\vartheta\|_{L^1} = \int_{1}^{e} \left|\vartheta(\xi)\right| d\xi$$

and $AC^{i}(J,\mathbb{R})$ be the space of functions $u : [1,\Gamma] \longrightarrow \mathbb{R}$ i-differentiable and whose ith derivative, $\vartheta^{(i)}$ is absolutely continuous.

We begin by defining Hadamard fractional integrals and derivatives, and we introduce some properties that can be used thereafter.

Definition 2.1. *[18] The Hadamard fractional integral of order $\varrho \in \mathbb{R}^{+}$ for a function $\varphi \in C[a,b]$, $0 \leq a \leq \xi \leq b \leq \infty$, is defined as*

$$I^{\varrho}\varphi(\xi) = \frac{1}{\Gamma(\varrho)} \int_{a}^{\xi} \left(\log\frac{\xi}{\nu}\right)^{\varrho-1} \frac{\varphi(\nu)}{\nu} d\nu,$$

where $\Gamma(.)$ *is the Gamma function and* $\log(.) = \log_{\Gamma}(.)$

Definition 2.2. *[18] Let* $0 < a < b < \infty$ *and* $\delta = \xi \frac{d}{dt}$*. The Hadamard derivative of fractional order* $\varrho \in \mathbb{R}^{+}$ *for a function* $\varphi \in C^{n-1}([a,b], \mathbb{R})$ *is defined as*

$$\mathfrak{D}^{\varrho}\varphi(\xi) = \delta^{n}(I^{n-\varrho})(\xi) = \frac{1}{\Gamma(n-\varrho)}\left(\xi\frac{d}{dt}\right)^{n}\int_{a}^{\xi}\left(\log\frac{\xi}{\nu}\right)^{n-\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu,$$

where $n-1 < \varrho \leq n \in \mathbb{R}^{+}$, $n = [\varrho] + 1$ *denotes the integer part of the real number* ϱ.

Lemma 2.1. *([18], Property 2.24) If* $a, \varrho, \zeta > 0$*, then*

$$\left(\mathfrak{D}^{\varrho}\left(\log\frac{\xi}{a}\right)^{\zeta-1}\right)(\xi) = \frac{\Gamma(\zeta)}{\Gamma(\zeta-\varrho)}\left(\log\frac{\xi}{a}\right)^{\zeta-\varrho-1},$$

$$\left(I^{\varrho}\left(\log\frac{\xi}{a}\right)^{\zeta-1}\right)(\xi) = \frac{\Gamma(\zeta)}{\Gamma(\zeta+\alpha)}\left(\log\frac{\xi}{a}\right)^{\zeta+\varrho-1}.$$

$$\left(\mathfrak{D}^{\varrho}\left(\log\frac{\xi}{a}\right)^{\varrho-j}\right)(\xi) = 0,\ \ for\ j = 1, ..., [\varrho] + 1.$$

Lemma 2.2. *([18]) Let* $\varrho > 0$ *and* $\vartheta \in [1,\infty) \cap L^{1}[1,\infty)$*. Then the solution of Hadamard fractional differential equation* $D^{\varrho}\vartheta(\xi) = 0$ *is given by*

$$\vartheta(\xi) = \sum_{i=1}^{n} c_i (\log \xi)^{\varrho-i},$$

and the following formula holds:

$$I^{\varrho}\mathfrak{D}^{\varrho}\vartheta(\xi) = \vartheta(\xi) + \sum_{i=1}^{n} c_i (\log \xi)^{\varrho-i},$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$*, where* $n = [\varrho] + 1$.

For a separable Banach space $(\Upsilon, \|.\|)$, let $\mathcal{P}(\Upsilon) = \{\mathfrak{A} \in \Upsilon : \mathfrak{A} \neq \emptyset\}$, $\mathcal{P}_b(\Upsilon) = \{\mathfrak{A} \in \mathcal{P}(\Upsilon) : \mathfrak{A}$ *is bounded*$\}$, $\mathcal{P}_{cl}(\Upsilon) = \{\mathfrak{A} \in \mathcal{P}(\Upsilon) : \mathfrak{A}$ *is closed*$\}$, $\mathcal{P}_{cp}(\Upsilon) = \{\mathfrak{A} \in \mathcal{P}(\Upsilon) : \mathfrak{A}$ *is compact*$\}$, $\mathcal{P}_{c}(\Upsilon) = \{\mathfrak{A} \in \mathcal{P}(\Upsilon) : \mathfrak{A}$ *is convex*$\}$ and $P_{cp,cv}(\Upsilon) = P_{cp}(\Upsilon) \cap P_{cv}(\Upsilon)$.

Let Φ, Ψ be two sets, $\varphi : \Phi \longrightarrow \mathcal{P}(\Psi)$ a set-valued map, and $A \subset \Psi$. We define

$$graph(\varphi) = \{(\vartheta, v) : \vartheta \in \Phi, v \in \Psi\}.$$

Let $\Lambda > 0$ and let

$$B = \{\vartheta \in \Upsilon : |\vartheta| \leq \Lambda\}$$

and

$$U = \{\vartheta \in C(J, \Upsilon) : \|\vartheta\| < \Lambda\}.$$

Clearly $\overline{U} = C(J, B)$.

Definition 2.3. *A multivalued map $\Pi : J \times \Upsilon \longrightarrow P_{cl}(\Upsilon)$ is said to be measurable if for every $\chi \in \Upsilon$ the function*

$$\xi \longmapsto d(\chi, \Pi(\xi)) = \inf\{\|\chi - z\| : z \in \Pi(\xi)\},$$

is measurable.

Definition 2.4. *A multivalued map $\varphi : J \times \Upsilon \longrightarrow P(\Upsilon)$ is called L^1-Caratheodory if*
(i) $\xi \longmapsto \Pi(\xi, \vartheta)$ is measurable for all $\vartheta \in \mathbb{R}$,
(ii) $\tau \longmapsto \Pi(\xi, \vartheta)$ is upper semi-continuous for almost all $\xi \in [1, e]$, and
(iii) for each $\varsigma > 0$, there exists $f_\varsigma \in L^1(I, \mathbb{R}^+)$ such that

$$\left\|\Pi(\xi, \vartheta)\right\| = \sup\{|\omega|, \omega \in \Pi(\xi, \vartheta)\} \leq f_\varsigma(\xi),$$

for all $|\vartheta| \leq \varsigma$ and for a.e. $\xi \in J$.
The multivalued map Π is said to be Caratheodory if it satisfies (i) and (ii).

For each $\varphi \in C(J, \Upsilon)$ define the set of selections of Π by

$$S_{\Pi\circ\varphi} = \left\{v \in L^1(J) : \varphi(\xi) \in \Pi(\xi, \vartheta(\xi)) \ a.\Upsilon. \ \xi \in J\right\}.$$

Let (Υ, d) be a metric space induced from the normed space $(\Upsilon, \|.\|)$. The function $H_d : \mathcal{P}(\Upsilon) \times \mathcal{P}(\Upsilon) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by:

$$H_d(\mathfrak{A}, \mathfrak{B}) = \max\left\{\sup_{a\in\mathfrak{A}} d(a, \mathfrak{B}), \sup_{b\in\mathfrak{B}} d(b, \mathfrak{A})\right\},$$

is referred to as the Hausdorff-Pompeiu metric (see [15]).

Now, recall the definition of the Kuratowski measure of noncompactness, and summarize the main properties of this measure.

Definition 2.5. *[4, 5] Let Γ be a Banach space and let Ω_Γ be the family of bounded subsets of Γ. The Kuratowski measure of noncompactness is the map defined by*

$$\alpha(D) = \inf\left\{\epsilon > 0 : D \subset \cup_{i=1}^{m} D_i \, diam(D_i) \leq \epsilon\right\}$$

where $D \in \Omega_\Gamma$.

Properties 2.1. *1) $\alpha(D) = 0 \Longleftrightarrow \overline{D}$ is compact (D is relatively compact).*
2) $\alpha(D) = \alpha\left(\overline{D}\right)$.
3)$C \subset D \Longrightarrow \alpha(C) \leq \alpha(D)$.
4)$\alpha(C + D) \leq \alpha(C) + \alpha(D)$.
5) $\alpha(bD) = b\alpha(D)$, $b \in \mathbb{R}$.
6) $\alpha(conv\, D) = \alpha(D)$.
Here $\overline{D}$ and $conv(D)$ denote the closure and the convex hull of the bounded set D, respectively.

Theorem 2.1. *[12]*

Let Γ be a Banach space and $C \subset L^1(J, \Gamma)$ countable with $|\vartheta(\xi)| \leq h(\xi)$ for a.e. $\xi \in J$, and every $\vartheta \in C$ where $h \in L^1(J, \mathbb{R}_+)$. Then the function $\varphi(\xi) = \alpha(C(\xi))$ belongs to $L^1(J, \mathbb{R}_+$ and satisfies

$$\alpha\left(\left\{\int_1^e \vartheta\left(\nu\right)\nu : \ \vartheta \in C\right\}\right) \leq 2\int_1^e \alpha\left(C\left(\nu\right)\right)\nu.$$

Lemma 2.3. *([11], Theorem 19.7) Let X be a separable metric space and G a multi-valued map with nonempty closed values. Then, G has a measurable selection.*

we put

$$\Omega = 1 - \frac{\kappa\Gamma\left(\varrho - 1\right)}{\Gamma\left(\varrho + z - 1\right)}\left(\log \ell\right)^{\varrho+z-2}. \tag{2.1}$$

Lemma 2.4. *For given $\chi(.) \in C(J, \mathbb{R})$, the unique solution of the problem*

$$\begin{cases} \mathfrak{D}^{\varrho}\vartheta\left(\xi\right) = \chi(\xi),\ 1 < \xi < e,\ 2 < \varrho \leq 3 \\ \vartheta\left(1\right) = 0,\ D^{\varrho-1}\vartheta\left(1\right) = 0,\ \vartheta\left(e\right) = \kappa\left(I^{z}\vartheta\right)\left(\ell\right) \end{cases} \tag{2.2}$$

is

$$\begin{aligned} \vartheta\left(\xi\right) &= \frac{1}{\Gamma\left(\varrho\right)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\chi\left(\nu\right)}{\nu}\nu \\ &\quad + \frac{\left(\log \xi\right)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma\left(\varrho+z\right)} \in \xi_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{p+z-1}\frac{\chi\left(\nu\right)}{\nu}\nu\right. \\ &\quad \left. - \frac{1}{\Gamma\left(\varrho\right)} \in \xi_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\chi\left(\nu\right)}{\nu}\nu\right) \end{aligned} \tag{2.3}$$

Proof. Let ϑ_0 be a solution for equation $\mathfrak{D}^{\varrho}\vartheta\left(\xi\right) = \chi(\xi)$, for $\xi \in [1, e]$ and $2 < \varrho \leq 3$. By virtue of the lemma 2.2, there exist constants c_1, c_2, $c_3 \in \mathbb{R}$ provided that

$$\begin{aligned} \vartheta_0\left(\xi\right) &= \frac{1}{\Gamma\left(\varrho\right)} \in \xi_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\hbar\left(\nu\right)}{\nu}\nu \\ &\quad + c_1\left(\log \xi\right)^{\varrho-1} + c_2\left(\log \xi\right)^{\varrho-2} + c_3\left(\log \xi\right)^{\varrho-3}, \end{aligned} \tag{2.4}$$

The conditions $\vartheta\left(1\right) = 0$, $\mathfrak{D}^{\varrho-1}\vartheta(1) = 0$ imply that $c_1 = c_3 = 0$. For $z > 0$ and applying the Hadamard integral operator I^z to (2.4) and using Lemma 2.1, we have

$$\begin{aligned} (I^{z}\vartheta)(\xi) &= \frac{1}{\Gamma\left(\varrho+z\right)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho+z-1}\frac{\hbar\left(\nu\right)}{\nu}\nu \\ &\quad + c_2\frac{\Gamma\left(\varrho-1\right)}{\Gamma\left(\varrho+z-1\right)}\left(\log \xi\right)^{\varrho+z-2}. \end{aligned}$$

By using the condition $\vartheta(e) = \kappa\left(I^{z}\vartheta\right)\left(\ell\right)$, we get

$$c_2 = \frac{1}{\omega}\left(\frac{\kappa}{\Gamma\left(\varrho+z\right)} \in \xi_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\hbar\left(\nu\right)}{\nu}\nu - \frac{1}{\Gamma\left(\varrho\right)} \in \xi_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\hbar\left(\nu\right)}{\nu}\nu\right).$$

By inserting the values c_i for $i = 1, 2, 3$ in (2.4), we get

$$\vartheta_0(\xi) = \frac{1}{\Gamma(\varrho)} \int_1^{\xi} \left(\log \frac{\xi}{\nu}\right)^{\varrho-1} \frac{\hbar(\nu)}{\nu} \nu + \frac{(\log \xi)^{\varrho-2}}{\omega} \left(\frac{\kappa}{\Gamma(\varrho+z)} \int_1^{\ell} \left(\log \frac{\ell}{\nu}\right)^{\varrho+z-1} \frac{\hbar(\nu)}{\nu} \nu - \frac{1}{\Gamma(\varrho)} \int_1^{e} \left(\log \frac{e}{\nu}\right)^{\varrho-1} \frac{\hbar(\nu)}{\nu} \nu \right).$$

This means that ϑ_0 is a solution for integral equation (2.3). Conversely, one can easily see that ϑ_0 is a solution of problem (2.2) whenever ϑ_0 is a solution of the equation (2.3). □

3. Main result

3.1. **Existence of solution.** Useful Hypotheses will given in the following.

(H_1) $\Pi : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cp,cv}(\mathbb{R})$ is a Carathéodory multi-valued map.

$(H2)$ For each $R > 0$ there exists a function $p \in L^1(J, \mathbb{R}_+)$ such that

$$\left\| \Pi(\xi, \chi) \right\| = \sup \{ |\chi| : \chi(\xi) \in \Pi(\xi, \chi) \} \le p(\xi),$$

for each $(\xi, \chi) \in J \times \mathbb{R}$ with $|\chi| \le R$, and

$$\lim_{z \to +\infty} \inf \frac{\omega \int_1^e p(\xi)\, dt}{z} = \rho < \infty$$

where

$$\omega = \frac{1 + |\omega|}{|\omega| \Gamma(\varrho)} + \frac{\kappa}{|\omega| \Gamma(\varrho+z)} (\log \ell)^{\varrho+z}.$$

$(H3)$ There exists a Carathéodory function $\psi : J \times [0, 2R] \longrightarrow \mathbb{R}_+$ such that

$$\alpha(\Pi(\xi, M(\xi))) \le \psi(\xi, \alpha(M(\xi))), \text{ for all } \xi \in J, \text{ and each } M \subset B,$$

and the unique solution $\mathfrak{N} \in C([0, 2R])$ of the inequality

$$\mathfrak{N}(\xi) \le \frac{1}{\Gamma(\varrho)} \int_1^{\xi} \left(\log \frac{\xi}{\nu}\right)^{\varrho-1} \frac{\psi(\nu, \alpha(M(\nu)))}{\nu} \nu + \frac{(\log \xi)^{\varrho-2}}{\omega} \left(\frac{\kappa}{\Gamma(\varrho+z)} \int_1^{\ell} \left(\log \frac{\ell}{\nu}\right)^{p+z-1} \frac{\psi(\nu, \alpha(M(\nu)))}{\nu} \nu - \frac{1}{\Gamma(\varrho)} \int_1^{e} \left(\log \frac{e}{\nu}\right)^{\varrho-1} \frac{\psi(\nu, \alpha(M(\nu)))}{\nu} \nu \right), \ \xi \in J$$

is $\mathfrak{N} \equiv 0$.

The main tools that will be used to establish our results are respectively the set-valued analog of Monch fixed point theorem and an another Lemma.

Theorem 3.1. *[23] Let K be a bounded, closed and convex subset of a Banach space Γ, $\mathcal{U}$ a relatively open subset of K, and $\mathfrak{F} : \overline{\mathcal{U}} \longrightarrow \mathcal{P}_{cv}(K)$. Assume that graph$(\mathfrak{F})$ is closed, $\mathfrak{F}$ maps compact sets into relatively compact sets, and that for some $x_0 \in \mathcal{U}$ the following two conditions are satisfied:*

$$\begin{cases} M \subset \overline{\mathcal{U}},\ M \subset conv(x_0 \cup \mathfrak{F}(M)) \\ and\ \overline{M} = \overline{C}\ with\ C \subset M\ countable \end{cases} \longrightarrow \overline{M}\ compact, \tag{3.1}$$

$$x \notin (1-\lambda)\, x_0 + \lambda \mathfrak{F}(x)\,,\ \forall x \in \overline{\mathcal{U}} \setminus \mathcal{U},\ \lambda \in (0,1)\,. \tag{3.2}$$

Then there exists $x \in \overline{\mathcal{U}}$ with $x \in \mathfrak{F}(x)$.

Lemma 3.1. *Let J be a compact real interval. Let Π be a multivalued map satisfying (H_1) and let Θ be a linear continuous map from $L^1(J,\Gamma) \longrightarrow C(J,\Gamma)$. Then the operator*

$$\Theta \circ S_{\Pi,\chi} :\ C(J,\Gamma) \longrightarrow \mathcal{P}_{cp,cv}(C(J,\Gamma))\,,\ \chi \longmapsto (\Theta \circ S_{\Pi,\chi})(\chi) = \Theta(S_{\Pi,\chi})$$

is a closed graph operator in $C(J,\Gamma) \times C(J,\Gamma)$.

Theorem 3.2. *Assume that $\frac{\kappa\Gamma(\varrho-1)}{\Gamma(\varrho+z-1)} \neq 1$ that (H_1)–(H_3) hold. Then problem (1.1) has at least one solution on $C(J,B)$, provided that*

$$\frac{\omega}{R}\int_1^e p(\nu)\,\nu \leq 1. \tag{3.3}$$

Proof. Transform the problem (1.1) into a fixed point problem. Consider the multivalued operator

$$Q(\chi) = \left\{ \begin{array}{l} \tilde{\mathfrak{m}} \in C(J,\mathbb{R})\,,\ \text{there exists } \varphi \in S_{\Pi,\chi} \text{ such that} \\ \tilde{\mathfrak{m}}(\xi) = \left\{ \begin{array}{l} \frac{1}{\Gamma(\varrho)}\int_1^\xi \left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}\nu \\ +\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^\ell\left(\log\frac{\ell}{\nu}\right)^{p+z-1}\frac{\varphi(\nu)}{\nu}\nu - \frac{1}{\Gamma(\varrho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}\nu\right) \end{array}\right. \end{array} \right\}.$$

We shall show that ϱ satisfies the assumptions of the set-valued analog of Monch's fixed point theorem. We divide the proof in five steps.

Step 1. $Q(\vartheta)$ is convex for each $\chi \in C(J,\mathbb{R})$.

$$\begin{aligned} \tilde{\mathfrak{m}}_i(\xi) = {} & \frac{1}{\Gamma(\varrho)}\int_1^\xi \left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_i(\nu)}{\nu}\nu \\ & + \frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^\ell\left(\log\frac{\ell}{\nu}\right)^{p+z-1}\frac{\varphi_i(\nu)}{\nu}\nu \right. \\ & \left. -\frac{1}{\Gamma(\varrho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_i(\nu)}{\nu}\nu\right),\ i = 1,2. \end{aligned}$$

Let $\zeta \in [0,1]$. Then, for each $\xi \in J$, we have

$$\begin{aligned} [\zeta\tilde{\mathfrak{m}}_1 + (1-\zeta)\,\tilde{\mathfrak{m}}_2](\xi) = {} & \frac{1}{\Gamma(\varrho)}\int_1^\xi \left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{[\zeta\varphi_1 + (1-\zeta)\,\varphi_2](\nu)}{\nu}\nu \\ & + \frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^\ell\left(\log\frac{\ell}{\nu}\right)^{z+\varrho-1}\frac{[\zeta\varphi_1 + (1-\zeta)\,\varphi_2](\nu)}{\nu}\nu \right. \\ & \left. -\frac{1}{\Gamma(\varrho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{[\zeta\varphi_1 + (1-\zeta)\,\varphi_2](\nu)}{\nu}\nu\right). \end{aligned}$$

Since φ has convex values, $S_{\Pi,\chi}$ is convex, so $\zeta\tilde{\mathfrak{m}}_1 + (1-\zeta)\,\tilde{\mathfrak{m}}_2 \in Q(\chi)$.

Step 2. $Q(M)$ is relatively compact for each compact $M \subset \overline{\mathcal{U}}$.

Let $M \subset \overline{\mathcal{U}}$ be a compact set and let $\{\tilde{\mathfrak{m}}_n\}$ be any sequence of elements of $Q(M)$. We show that

$\{\tilde{\mathfrak{m}}_n\}$ has a convergent subsequence by using the Ascoli-Arzela criterion of compactness in $C(J,\mathbb{R})$. Since $\tilde{\mathfrak{m}}_n \in Q(M)$, there exist $\{\chi_n\} \in M$ and $\varphi_n \in S_{\Pi,\chi_n}$ such that

$$\begin{aligned}\tilde{\mathfrak{m}}_n(\xi) &= \frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu \\ &\quad+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right. \\ &\quad\left.-\frac{1}{\Gamma(q)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right).\end{aligned}$$

Using Theorem 2.1 and the properties of the measure of noncompactness of Kuratowski α, we have

$$\begin{aligned}\alpha(\{\tilde{\mathfrak{m}}_n(\xi)\}) &\le \frac{1}{\Gamma(\varrho)}\int_1^{\xi}\alpha\left(\left\{\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu \\ &\quad+\frac{(\log\xi)^{\varrho-2}}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\alpha\left(\left\{\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu\right. \\ &\quad\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\alpha\left(\left\{\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu\right).\end{aligned} \tag{3.4}$$

On the other hand, since $M(\nu)$ is compact in $C(J,\mathbb{R})$, the set $\{\varphi_n(\nu) : n \ge 1\}$ is compact. Consequently, $\alpha(\{\varphi_n(\nu) : n \ge 1\}) = 0$ for all $\nu \in J$. Furthermore,

$\alpha\left(\left\{\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu} : n \ge 1\right\}\right) = \frac{1}{\nu}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\alpha(\{\varphi_n(\nu) : n \ge 1\}) = 0,$

$\alpha\left(\left\{\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu} : n \ge 1\right\}\right) = \frac{1}{\nu}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\alpha(\{\varphi_n(\nu) : n \ge 1\}) = 0,$

$\alpha\left(\left\{\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu} : n \ge 1\right\}\right) = \frac{1}{\nu}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\alpha(\{\varphi_n(\nu) : n \ge 1\}) = 0,$

for all $\xi, \nu \in J$. Then (3.4) implies that $\{\varphi_n(\nu) : n \ge 1\}$ is relatively compact in $C(J,\mathbb{R})$ for each $\xi \in J$.

In addition, for each ξ_1 and t_2 from $C(J,\mathbb{R})$ $\xi_1 < \xi_2$ we have

$$\begin{aligned}\left|\tilde{\mathfrak{m}}_n(\xi_2) - \tilde{\mathfrak{m}}_n(\xi_1)\right| &= \left|\frac{1}{\Gamma(\varrho)}\int_1^{\xi_1}\left[\left(\log\frac{\xi_2}{\nu}\right)^{\varrho-1} - \left(\log\frac{\xi_1}{\nu}\right)^{\varrho-1}\right]\frac{\varphi_n(\nu)}{\nu}d\nu\right. \\ &\quad+\frac{1}{\Gamma(\varrho)}\int_{\xi_1}^{\xi_2}\left[\left(\log\frac{\xi_2}{\nu}\right)^{\varrho-1}\right]\frac{\varphi_n(\nu)}{\nu}d\nu \\ &\quad+\frac{(\log\xi_2)^{\varrho-2} - (\log\xi_1)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right. \\ &\quad\left.\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right)\right| \\ &\le \frac{p(\xi)}{\Gamma(\varrho)}\int_1^{\xi_1}\left[\left(\log\frac{\xi_2}{\nu}\right)^{\varrho-1} - \left(\log\frac{\xi_1}{\nu}\right)^{\varrho-1}\right]\frac{1}{\nu}d\nu \\ &\quad+\frac{p(\xi)}{\Gamma(\varrho)}\int_{\xi_1}^{\xi_2}\left[\left(\log\frac{\xi_2}{\nu}\right)^{\varrho-1}\right]\frac{1}{\nu}d\nu \\ &\quad+\frac{p(\xi)\left[(\log\xi_2)^{\varrho-2} - (\log\xi_1)^{\varrho-2}\right]}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{1}{\nu}d\nu\right.\end{aligned}$$

$$+\frac{1}{\Gamma(\varrho)}\int_1^e\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{1}{\nu}d\nu\Bigg)$$

$$\leq\frac{p(\xi)}{\Gamma(\varrho+1)}\Big\{\Big[(\log\xi_2)^{\varrho}-(\log\xi_1)^{\varrho}\Big]$$

$$+\frac{\left[(\log\xi_2)^{\varrho-2}-(\log\xi_1)^{\varrho-2}\right]}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z+1)}(\log\ell)^{\varrho+z}+1\right)\Bigg\}. \tag{3.5}$$

The right hand side of the above inequality tends to zero, when $\xi_1\longrightarrow\xi_2$, that means $\{\tilde{\mathfrak{m}}_n : n\geq 1\}$ is equicontinuous. Then, $\{\tilde{\mathfrak{m}}_n : n\geq 1\}$ is relatively compact in $C(J,\mathbb{R})$.

Step 3. Q has a closed graph.

Let $(\chi_n,\tilde{\mathfrak{m}}_n)\in graph(Q)$, $n\geq 1$, $\|\chi_n-\chi\|$, $\|\tilde{\mathfrak{m}}_n-h\|\longrightarrow 0$, as $n\longrightarrow\infty$. We must show that $(\chi,h)\in graph(Q)$.

$(\chi_n,\tilde{\mathfrak{m}}_n)\in graph(Q)$ means that $\tilde{\mathfrak{m}}_n\in Q(\chi_n)$,which means that there exists $\varphi_n\in S_{\varphi,\chi_n}$, such that for each $\xi\in J$

$$\tilde{\mathfrak{m}}_n(\xi)=\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu$$
$$+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right.$$
$$\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right).$$

Consider the continuous linear operator $\Theta : L^1(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ defined by

$$\varphi\longmapsto\Theta(\varphi)(\xi)=\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu$$
$$+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi(\nu)}{\nu}d\nu\quad-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu\right)$$

Clearly

$$\left|\tilde{\mathfrak{m}}_n(\xi)-h(\xi)\right|=\left|\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{(\varphi_n(\nu)-\varphi(\nu))}{\nu}d\nu\right.$$
$$+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{(\varphi_n(\nu)-\varphi(\nu))}{\nu}d\nu\right.$$
$$\left.\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{(\varphi_n(\nu)-\varphi(\nu))}{\nu}d\nu\right)\right|\to 0\ as\ n\to\infty.$$

In view, of Lemma 3.1, we state that $\Theta\circ S_{\varphi}$ is a closed graph operator. Moreover, we have

$$\tilde{\mathfrak{m}}_n\in\Theta\left(S_{\Pi,\chi_n}\right).$$

Since $\chi_n \longrightarrow \chi$, we get

$$\begin{aligned}\tilde{\mathfrak{m}}(\xi) = &\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}\nu \\ &+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi(\nu)}{\nu}d\nu\right.\\ &\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu\right)\end{aligned}$$

for some $\varphi \in S_{\varphi,\chi}$.

Step 4. Suppose $M \subset \overline{\mathcal{U}}, M \subset conv(\{0\} \cup Q(M))$, and $\overline{M} = \overline{C}$ for some countable set $C \subset M$. Using an estimation of type (3.5), we see that $Q(M)$ is equicontinuous. Then from $M \subset conv(\{0\} \cup Q(M))$, we deduce that M is equicontinuous, too. In order to apply the Arzèla-Ascoli theorem, it remains to show that $M(\xi)$ is relatively compact in $C(J,\mathbb{R})$ for each $\xi \in J$.

Since

$$C \subset M \subset onv(\{0\} \cup Q(M)) \text{ and } C \text{ is countable},$$

We can find a countable set $H = \{\tilde{\mathfrak{m}}_n : n \geq 1\} \subset Q(M)\ withC \subset conv(\{0\} \cup H)$.
Then there exist $\chi_n \in M$ and $\varphi_n \in S_{\varphi,\chi_n}$ such that

$$\begin{aligned}\tilde{\mathfrak{m}}_n(\xi) = &\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu \\ &+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right.\\ &\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}d\nu\right).\end{aligned}$$

From, $M \subset \overline{C} \subset \overline{conv}(\{0\} \cup H)$ and the properties of the measure of noncompactness, we have

$$\alpha(M(\xi)) \leq \alpha\left(\overline{C}(\xi)\right) \leq \alpha(H(\xi)) = \alpha(\{\tilde{\mathfrak{m}}_n(\xi) : n \geq 1\}).$$

By using Theorem 2.1 and inequality (3.4), we obtain

$$\begin{aligned}\alpha(\{\tilde{\mathfrak{m}}_n(\xi)\}) \leq &\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\alpha\left(\left\{\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu \\ &+\frac{(\log\xi)^{\varrho-2}}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\alpha\left(\left\{\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu\right.\\ &\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\alpha\left(\left\{\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)d\nu\right).\end{aligned}$$

Since $\varphi_n(\nu) \in \Pi(\nu, \chi_n(\nu))$ and $\chi_n \in M$, $(H3)$ ensures

$$\begin{aligned}\alpha\left(\left\{\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right) &\leq \frac{1}{\nu}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\alpha(\varphi(\nu, M(\nu))) \\ &\leq \left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\psi(\nu, \alpha(M(\nu)))\end{aligned}$$

$$\alpha\left(\left\{\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)\leq\frac{1}{\nu}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}$$
$$\leq\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\psi(\nu,M(\nu));$$

$$\alpha\left(\left\{\left(\log\frac{\Gamma}{\nu}\right)^{\varrho-1}\frac{\varphi_n(\nu)}{\nu}\right\}\right)\leq\frac{1}{\nu}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\alpha(\varphi(\nu,M(\nu)))$$
$$\leq\left(\log\frac{e}{\nu}\right)^{\varrho-1}\psi(\nu,M(\nu))$$

Then

$$\begin{aligned}\alpha(M(\xi))\leq&\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\frac{1}{\nu}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\psi(\nu,M(\nu))\,d\nu\\&+\frac{(\log\xi)^{\varrho-2}}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\frac{1}{\nu}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\psi(\nu,M(\nu))\,d\nu\right.\\&\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\frac{1}{\nu}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\psi(\nu,M(\nu))\,d\nu\right).\end{aligned}$$

Also, the function φ given by $\varphi(\xi)=\alpha(M(\xi))$ belongs to $C(J,[0,2R])$. Consequently, by $(H3)$ $\varphi\equiv0$, that is $\alpha(M(\xi))$ for all $\xi\in J$.

Then, by the Ascoli-Arzela theorem, M is relatively compact in $C(J,\Gamma)$.

Step 5. Let $h\in Q(\chi)$ with $\chi\in\overline{\mathcal{U}}$. Since $|\chi(\nu)|\leq z$ and $(H2)$ holds, we have $Q(\overline{\mathcal{U}})\subseteq\overline{\mathcal{U}}$. If it was not true, there would exist a function $\chi\in\overline{\mathcal{U}}$, but $\|Q(\chi)\|_{\mathcal{P}}>z$ and

$$\begin{aligned}\tilde{m}(\xi)=&\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu\\&+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\varphi(\nu)}{\nu}\nu-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\varphi(\nu)}{\nu}d\nu\right),\end{aligned}$$

for some $\varphi\in S_{\Pi,\chi}$. On the other hand we have

$$\begin{aligned}\Lambda&<\|Q(\chi)\|_{\mathcal{P}}\\&<\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{|\varphi(\nu)|}{\nu}d\nu\\&\quad+\frac{(\log\xi)^{\varrho-2}}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{|\varphi(\nu)|}{\nu}d\nu\right.\\&\quad\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{|\varphi(\nu)|}{\nu}d\nu\right)\\&<\frac{1}{\Gamma(\varrho)}(\log\xi)^{\varrho}\int_1^{e}p(\nu)\,d\nu\\&\quad+\frac{1}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}(\log\ell)^{\varrho+z}\int_1^{e}p(\nu)\,\nu+\frac{1}{\Gamma(\varrho)}\int_1^{e}p(\nu)\,d\nu\right)\end{aligned}$$

$$< \frac{1}{\Gamma(\varrho)} (\log \xi)^{\varrho} \int_1^e p(\nu)\, d\nu$$
$$+\frac{1}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)} (\log \ell)^{\varrho+z} \int_1^e p(\nu)\,\nu + \frac{1}{\Gamma(\varrho)} \int_1^e p(\nu)\, d\nu\right)$$
$$= \int_1^e p(\nu)\,\nu \left[\frac{1}{\Gamma(\varrho)} (\log \xi)^{\varrho}\right.$$
$$\left. + \frac{\kappa}{|\Omega|\,\Gamma(\varrho+z)} (\log \ell)^{\varrho+z} + \frac{1}{|\Omega|\,\Gamma(\varrho)}\right]$$
$$< \omega \int_1^e p(\nu)\, d\nu.$$

Dividing both sides by Λ and taking the lower limit as $\Lambda \longrightarrow \infty$, we conclude that

$$\lim_{\Lambda\longrightarrow\infty} \inf \frac{\omega}{\Lambda} \int_1^e p(\nu)\, d\nu > 1$$

which contradicts (3.3). Hence $Q\left(\overline{\mathcal{U}}\right) \subseteq \overline{\mathcal{U}}$.
As a consequence of Steps 1-5 together with Theorem 3.1, we can conclude that $\mathcal{H}$ has a fixed point $\chi \in C(J, B)$ which is a solution of the problem (1.1).

□

Example 3.1. *Consider*

$$\begin{cases} \mathfrak{D}^{\varrho}\vartheta(\xi) \in \Pi(\xi, \vartheta(\xi)),\ \xi \in [1, e] \\ \vartheta(1) = 0,\ \mathfrak{D}^{\varrho-1}\vartheta(1) = 0,\ \vartheta(e) = \kappa\,(I^z\vartheta)(\ell) \end{cases} \tag{3.6}$$

Here $\varrho = \frac{5}{2}$, $\kappa = 0{,}5$, $R = 1{,}5$, $\ell = 1{,}5$ $\Pi : [1, e] \times \mathbb{R} \longrightarrow \mathbb{R}$ *where* $\Pi(\xi, \vartheta) = \left[0, \frac{te^{-2t}}{1+|\vartheta(\xi)|}\right]$. *Thus,*

$$\left\|\Pi(\xi, \vartheta)\right\| \leq p(\xi)$$

with $p(\xi) = e^{-\xi}$. *Hence, the hypothesis* $(H2)$ *is satisfied with* $\omega \simeq 1{,}72$. *We can easily show that all requirements of Theorem 3.2 are verified. Hence, problem (3.6) has at least one solution defined on J.*

3.2. **Filippov's Theorem.** Now, we present a Filippov's result for the problem (1.1). Let $u \in AC^1(J, \mathbb{R})$ be a solution of the following problem

$$\begin{cases} \mathfrak{D}^{\varrho}\vartheta(\xi) = \mathfrak{y}(\xi),\ 1 < \xi < e,\ 2 < \varrho \leq 3 \\ \vartheta(1) = 0,\ \mathfrak{D}^{\varrho-1}\vartheta(1) = 0,\ \vartheta(e) = \kappa\,(I^z\vartheta)(\ell) \end{cases} \tag{3.7}$$

We will consider the following two assumptions:
(C_1) The function $\Pi : J \times \mathbb{R} \longrightarrow \mathcal{P}_{cl}(\mathbb{R})$ is such that
(C_{11}) for all $v \in \mathbb{R}$, the map $\xi \longrightarrow \Pi(\xi, v)$is measurable,
(C_{12}) the map $\sigma :\ \xi \longrightarrow d(\mathfrak{y}(\xi), \Pi(\xi, u(\xi)))$ is integrable.
(C_2) There exists a function $\delta \in L^{\infty}(J, \mathbb{R}_+)$ such that

$$H_d\left(\Pi(\xi, \vartheta), \Pi\left(\xi, \overline{\vartheta}\right)\right) \leq \delta(\xi)\left|\vartheta(\xi) - \overline{\vartheta}(\xi)\right|,\ \textit{for all}\ \xi \in [0, 1].$$

Theorem 3.3. *Assume that the conditions* (C_1) *and* (C_2) *hold. If*

$$\omega \|\delta\|_{L^1} < 1,$$

then the problem (1.1) has at least one solution v *satisfying, for a.e.* $\xi \in [0,1]$ *the estimates*

$$\left|\vartheta(\xi) - v(\xi)\right| \leq \phi(\xi)$$

where

$$\phi(\xi) \leq 2\omega\left(K\|\delta\|_{L^1} + \|\sigma\|_{L^1}\right),$$
$$K = \frac{\omega}{1-\omega\|\delta\|_{L^1}}\|\sigma\|_{L^1}.$$

Proof. Let $\mathfrak{y}_0 = -\mathfrak{D}^{\alpha}\vartheta$ and $v_0 = \vartheta(\xi)$ for a.e. $\xi \in [1,e]$, i.e. Then, by Lemma 2.4

$$v_0(\xi) = \frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}_0(\nu)}{\nu}d\nu$$
$$+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\mathfrak{y}_0(\nu)}{\nu}d\nu - \frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}_0(\nu)}{\nu}d\nu\right).$$

Let $\mathcal{U}_1 : [1,e] \longrightarrow \mathcal{P}(\mathbb{R})$ given by $\mathcal{U}_1(\xi) = \Pi(\xi, v_0(\xi)) \cap (\mathfrak{y}(\xi), \sigma)$. The multi-valued map $\mathcal{U}_1(\xi)$ is measurable (see Proposition III.4 in [7]), so there exists a function $\xi \longrightarrow \mathfrak{y}(\xi)$ which is a measurable selection for $\mathcal{U}_1$.
Let

$$v_1(\xi) = \frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}_1(d\nu)}{\nu}\nu$$
$$+\frac{(\log\xi)^{\varrho-2}}{\omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\mathfrak{y}_1(\nu)}{\nu}d\nu - \frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}_1(\nu)}{\nu}d\nu\right).$$

Then, we have

$$\left|v_1(\xi) - v_0(\xi)\right| \leq v_1(\xi) = \frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\sigma(\nu)}{\nu}d\nu$$
$$+\frac{(\log\xi)^{\varrho-2}}{|\omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\sigma(\nu)}{\nu}d\nu\right.$$
$$\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\sigma(\nu)}{\nu}d\nu\right)$$
$$\leq \omega\int_1^{e}\sigma(\nu)\,d\nu.$$

So, we obtain

$$\left\|v_1(\xi) - v_0(\xi)\right\| \leq \omega\|\sigma\|_{L^1}.$$

In the same way, the multi-valued map $\mathcal{U}_2(\xi) = \Pi(\xi, v_1(\xi)) \cap \left(\mathfrak{y}_1(\xi), \delta(\xi)\left|v_1(\xi) - v_0(\xi)\right|\right)$ is measurable with nonempty closed values (see [7,11]). By Lemma 2.3 (Kuratowski–Ryll–Nardzewski selection theorem), there exists a function $\mathfrak{y}_2$ which is a mesurable selection of $\mathcal{U}_2$.

Let the function

$$v_2(\xi)=\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\eta_2(\nu)}{\nu}d\nu+\frac{(\log\xi)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\eta_2(\nu)}{\nu}d\nu-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{q-1}\frac{\eta_2(\nu)}{\nu}d\nu\right).$$

Then

$$\left|v_2(\xi)-v_1(\xi)\right|\leq\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{p(\nu)\left|v_1(\nu)-v_0(\nu)\right|}{\nu}d\nu+\frac{(\log\xi)^{\varrho-2}}{|\Omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\delta(\nu)\left|v_1(\nu)-v_0(\nu)\right|}{\nu}d\nu+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\delta(\nu)\left|v_1(\nu)-v_0(\nu)\right|}{\nu}d\nu\right).$$

Hence

$$\left|v_2(\xi)-v_1(\xi)\right|\leq\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\delta(\nu)\left\|v_1(\nu)-v_0(\nu)\right\|}{\nu}d\nu+\frac{(\log\xi)^{\varrho-2}}{|\Omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\delta(\nu)\left\|v_1(\nu)-v_0(\nu)\right\|}{\nu}d\nu+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\delta(\nu)\left\|v_1(\nu)-v_0(\nu)\right\|}{\nu}d\nu\right)$$

$$\left|v_2(\xi)-v_1(\xi)\right|\leq\omega^2\left\|\delta\right\|_{L^1}.\left\|\sigma\right\|_{L^1}. \tag{3.8}$$

As above, the multi-valued map $\mathcal{U}_3(\xi)=\Pi(\xi,v_2(\xi))\cap\left(\eta_2(\xi),\delta(\xi)\left|v_2(\xi)-v_1(\xi)\right|\right)$ is measurable, so there exists a measurable selection η_3 of U_3. Consider the function

$$v_3(\xi)=\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\eta_3(\nu)}{\nu}d\nu+\frac{(\log\xi)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\eta_3(\nu)}{\nu}d\nu-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\eta_3(\nu)}{\nu}d\nu\right).$$

Then

$$\left|v_3(\xi)-v_2(\xi)\right|\leq\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{p(\nu)\left\|v_2(\nu)-v_1(\nu)\right\|}{\nu}d\nu+\frac{(\log\xi)^{\varrho-2}}{|\Omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\delta(\nu)\left\|v_2(\nu)-v_1(\nu)\right\|}{\nu}d\nu+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\delta(\nu)\left\|v_2(\nu)-v_1(\nu)\right\|}{\nu}d\nu\right)$$

$$\left|v_3(\xi)-v_2(\xi)\right|\leq\omega^3\left\|\delta\right\|_{L^1}^2.\left\|\sigma\right\|_{L^1}. \tag{3.9}$$

Repeating the process for $n = 1, 2, 3, ...$, we get

$$\left|v_n(\xi) - v_{n-1}(\xi)\right| \leq \omega \|\delta\|_{L^1}^{n-1} . \|\sigma\|_{L^1} . \tag{3.10}$$

By induction, assume that (3.9) holds for some n and check (3.10) for $n+1$. The multi-valued map $\mathcal{U}_{n+1}(\xi) = \Pi(\xi, v_n(\xi)) \cap \left(\mathfrak{y}_n(\xi), \delta(\xi)\left|v_n(\xi) - v_{n-1}(\xi)\right|\right)$. Since $\mathcal{U}_{n+1}$ is a nonempty measurable set, there exists a measurable selection $\mathfrak{y}_{n+1}(\xi) \in \mathcal{U}_{n+1}(\xi)$ which allows us to define for $n \in \mathbb{N}$

We consider

$$\begin{aligned} v_{n+1}(\xi) &= \frac{1}{\Gamma(\varrho)} \int_1^{\xi} \left(\log \frac{\xi}{\nu}\right)^{\varrho-1} \frac{\mathfrak{y}_{n+1}(\nu)}{\nu} d\nu \\ &+ \frac{(\log \xi)^{\varrho-2}}{\Omega} \left(\frac{\kappa}{\Gamma(\varrho+z)} \int_1^{\ell} \left(\log \frac{\ell}{\nu}\right)^{\varrho+z-1} \frac{\mathfrak{y}_{n+1}(\nu)}{\nu} \nu - \frac{1}{\Gamma(\varrho)} \int_1^{e} \left(\log \frac{e}{\nu}\right)^{\varrho-1} \frac{\mathfrak{y}_{n+1}(\nu)}{\nu} d\nu\right). \end{aligned}$$

Then

$$\begin{aligned} \left|v_{n+1}(\xi) - v_n(\xi)\right| \leq & \frac{1}{\Gamma(\varrho)} \in \xi_1^{\xi} \left(\log \frac{\xi}{\nu}\right)^{\varrho-1} \frac{\delta(\nu) \left\|v_n(\nu) - v_{n-1}(\nu)\right\|}{\nu} \nu \\ & + \frac{(\log \xi)^{\varrho-2}}{|\Omega|} \left(\frac{\kappa}{\Gamma(\varrho+z)} \int_1^{\ell} \left(\log \frac{\ell}{\nu}\right)^{\varrho+z-1} \frac{\delta(\nu) \left\|v_n(\nu) - v_{n-1}(\nu)\right\|}{\nu} d\nu \right. \\ & \left. + \frac{1}{\Gamma(\varrho)} \int_1^{e} \left(\log \frac{e}{\nu}\right)^{\varrho-1} \frac{\delta(\nu) \left\|v_n(\nu) - v_{n-1}(\nu)\right\|}{\nu} d\nu \right) \\ \left|v_{n+1}(\xi) - v_n(\xi)\right| \leq & \ \omega^n \|\delta\|_{L^1}^{n-1} . \left\|v_1(\nu) - v_0(\nu)\right\| . \end{aligned}$$

Since $\omega \|\delta\|_{L^1} < 1$, we deduce that $\{v_n\}$ is a Cauchy sequence in $C(J, \mathbb{R})$ converging uniformly to some $v \in C(J, \mathbb{R})$. From the definition of $\mathcal{U}_n$, $n \in \mathbb{N}$,

$$\left|\mathfrak{y}_{n+1}(\xi) - \mathfrak{y}_n(\xi)\right| \leq \delta(\xi) \left|v_n(\xi) - v_{n-1}(\xi)\right|, \ for\ a.e,\ \xi \in [1, e] . \tag{3.11}$$

Hence, for almost every $\xi \in [1, e]$, the sequence $\{\mathfrak{y}_n(\xi) : n \in \mathbb{N}\}$ is Cauchy in $\mathbb{R}$, then $\{\mathfrak{y}_n(\xi) : n \in \mathbb{N}\}$ converges almost everywhere to a measurable function $\{\mathfrak{y}(.)\}$ in $\mathbb{R}$.

Moreover, since $\mathfrak{y}_0 = \mathfrak{D}^{\alpha} \vartheta$ and by using (3.11), we obtain

$$\begin{aligned} \left|\mathfrak{y}_n(\xi) - \mathfrak{y}_0(\xi)\right| &\leq \left|\mathfrak{y}_n(\xi) - \mathfrak{y}_{n-1}(\xi)\right| + \left|\mathfrak{y}_{n-1}(\xi) - \mathfrak{y}_{n-2}(\xi)\right| + ... \\ &\quad ... + \left|\mathfrak{y}_1(\xi) - \mathfrak{y}_0(\xi)\right| \\ &\leq \sum_{k=1}^{n-1} \delta(\xi) \left|v_k(\xi) - v_{k-1}(\xi)\right| + \left|\mathfrak{y}_1(\xi) - \mathfrak{y}_0(\xi)\right| \\ &\leq \delta(\xi) \sum_{k=1}^{\infty} \omega^k \|\delta\|_{L^1}^{k-1} \|\sigma\|_{L^1} + \sigma(\xi) \\ &\leq \frac{\delta(\xi)}{1 - \omega \|\delta\|_{L^1}} \|\sigma\|_{L^1} + \sigma(\xi) \\ &= K\delta(\xi) + \sigma(\xi) \end{aligned}$$

where $K=\frac{\omega}{1-\omega\|\delta\|_{L^1}}\|\sigma\|_{L^1}$.
Then, for all $n\in\mathbb{N}$

$$\left|\mathfrak{y}_n(\xi)-\mathfrak{y}_0(\xi)\right|\leq K\delta(\xi)+\sigma(\xi). \tag{3.12}$$

By (3.12), we deduce that $\mathfrak{y}_n$ converges to $\mathfrak{y}\in L^1([0.1],\mathbb{R})$. Consequently,

$$\begin{aligned}v(\xi)&=\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}(\nu)}{\nu}d\nu\\&+\frac{(\log\xi)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\mathfrak{y}(\nu)}{\nu}d\nu-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\mathfrak{y}(\nu)}{\nu}d\nu\right)\end{aligned}$$

is a solution for the problem (1.1). Then, $v\in S_{\varphi,\vartheta}$.
Finally, we prove that the solution $v(\xi)$ verifies the estimate:

$$\left|\vartheta(\xi)-v(\xi)\right|\leq\Phi(\xi),\quad \xi\in[1,e].$$

$$\begin{aligned}\left|\vartheta(\xi)-v(\xi)\right|&=\left|\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{(\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu))}{\nu}d\nu\right.\\&+\frac{(\log\xi)^{\varrho-2}}{\Omega}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{(\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu))}{\nu}d\nu\right.\\&\left.\left.-\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{(\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu))}{\nu}d\nu\right)\right|\\&\leq\frac{1}{\Gamma(\varrho)}\int_1^{\xi}\left(\log\frac{\xi}{\nu}\right)^{\varrho-1}\frac{\left|\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu)\right|}{\nu}d\nu\\&+\frac{(\log\xi)^{\varrho-2}}{|\Omega|}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{\ell}\left(\log\frac{\ell}{\nu}\right)^{\varrho+z-1}\frac{\left|\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu)\right|}{\nu}d\nu\right.\\&\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\log\frac{e}{\nu}\right)^{\varrho-1}\frac{\left|\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu)\right|}{\nu}d\nu\right)\\&\leq\frac{1}{\Gamma(\varrho)}(\log\xi)^{\varrho}\int_1^{e}\left(\left|\mathfrak{y}_0(\nu)-\mathfrak{y}_n(\nu)\right|+\left|\mathfrak{y}(\nu)-\mathfrak{y}_n(\nu)\right|\right)d\nu\\&+\frac{(\log\xi)^{\varrho-2}}{|\Omega|}(\log\xi)^{\varrho+z}\left(\frac{\kappa}{\Gamma(\varrho+z)}\int_1^{e}\left(\left|\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu)\right|+\left|\mathfrak{y}(\nu)-\mathfrak{y}_n(\nu)\right|\right)d\nu\right.\\&\left.+\frac{1}{\Gamma(\varrho)}\int_1^{e}\left(\left|\mathfrak{y}_0(\nu)-\mathfrak{y}(\nu)\right|+\left|\mathfrak{y}(\nu)-\mathfrak{y}_n(\nu)\right|\right)d\nu\right)\\&\leq\omega\left[\int_1^{e}\left|\mathfrak{y}_0(\nu)-\mathfrak{y}_n(\nu)\right|d\nu+\int_1^{e}\left|\mathfrak{y}(\nu)-\mathfrak{y}_n(\nu)\right|d\nu\right]\end{aligned}$$

As $n\longrightarrow\infty$, we conclude that

$$\begin{aligned}\left|\vartheta(\xi)-v(\xi)\right|&\leq 2\omega\int_1^{e}(K\delta(\nu)+\sigma(\nu))\,d\nu.\\&\leq 2\omega^{*}\left(K\|\delta\|_{L^1}+\|\sigma\|_{L^1}\right).\end{aligned}$$

□

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Abbas, M. Benchohra, G.M. N'Guerekata, Advanced Fractional Differential and Integral Equations, Nova Science Publishers, New York, 2015.

[2] T. Abdeljawad, R.P. Agarwal, E. Karapınar, P.S. Kumari, Solutions of the Nonlinear Integral Equation and Fractional Differential Equation Using the Technique of a Fixed Point with a Numerical Experiment in Extended B-Metric Space, Symmetry 11 (2019), 686. https://doi.org/10.3390/sym11050686.

[3] B. Ahmad, N. Alghamdi, A. Alsaedi, S.K. Ntouyas, Boundary Value Problem for Nonlocal Multi-Point and Multi-Term Fractional Differential Inclusions, Appl. Anal. Optim. 5 (2021), 123–144.

[4] R.R. Akhmerov, M.I. Kamenskii, A.S. Potapov, A.E. Rodkina, B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhäuser, Basel, 1992. https://doi.org/10.1007/978-3-0348-5727-7.

[5] J.-P. Aubin, A. Cellina, Differential Inclusions: Set-Valued Maps and Viability Theory, Springer, Berlin, 1984. https://doi.org/10.1007/978-3-642-69512-4.

[6] M.T. Beyene, M.D. Firdi, T.T. Dufera, Analysis of Caputo-Hadamard Fractional Neutral Delay Differential Equations Involving Hadamard Integral and Unbounded Delays: Existence and Uniqueness, Res. Math. 11 (2024), 2321669. https://doi.org/10.1080/27684830.2024.2321669.

[7] C. Castaing, M. Valadier, Convex Analysis and Measurable Multifunctions, Springer, Berlin, 1977. https://doi.org/10.1007/BFb0087685.

[8] A. Cernea, Filippov Lemma for a Class of Hadamard-Type Fractional Differential Inclusions, Fract. Calc. Appl. Anal. 18 (2015), 163–171. https://doi.org/10.1515/fca-2015-0011.

[9] H. Djourdem, N. Bouteraa, Mild Solution for a Stochastic Partial Differential Equation with Noise, WSEAS Trans. Syst. 19 (2020), 246–256. https://doi.org/10.37394/23202.2020.19.29.

[10] H. Djourdem, Existence Results for a Nonlinear Fractional Differential Inclusion, Filomat 35 (2021), 927–939. https://doi.org/10.2298/FIL2103927D.

[11] L. Górniewicz, Topological Fixed Point Theory of Multivalued Mappings, Kluwer Academic Publishers, Dordrecht, 1999. https://doi.org/10.1007/1-4020-4666-9.

[12] H.-P. Heinz, On the Behaviour of Measures of Noncompactness with Respect to Differentiation and Integration of Vector-Valued Functions, Nonlinear Anal. Theory Methods Appl. 7 (1983), 1351–1371. https://doi.org/10.1016/0362-546X(83)90006-8.

[13] J. Henderson, A. Ouahab, A Filippov's Theorem, Some Existence Results and the Compactness of Solution Sets of Impulsive Fractional Order Differential Inclusions, Mediterr. J. Math. 9 (2012), 453–485. https://doi.org/10.1007/s00009-011-0141-9.

[14] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[15] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, Kluwer Academic Publishers, Dordrecht, 1997.

[16] U. Hrusheuski, Filippov Solutions to Systems of Ordinary Differential Equations with Delta Function Terms as Summands, in: M. Reissig, M. Ruzhansky (Eds.), Progress in Partial Differential Equations, Springer, Heidelberg, 2013: pp. 183–200. https://doi.org/10.1007/978-3-319-00125-8_8.

[17] E.T. Karimov, B. López, K. Sadarangani, About the Existence of Solutions for a Hybrid Nonlinear Generalized Fractional Pantograph Equation, Fract. Differ. Calc. 6 (2016), 95–110. https://doi.org/10.7153/fdc-06-06.

[18] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, North-Holland Mathematics Studies, Elsevier, Amsterdam, 2006.

[19] M. Kisielewicz, Differential Inclusions and Optimal Control, Kluwer, Dordrecht, 1991.

[20] J. Liu, K. Zhang, X.-J. Xie, The Existence of Solutions of Hadamard Fractional Differential Equations with Integral and Discrete Boundary Conditions on Infinite Interval, Electron. Res. Arch. 32 (2024), 2286–2309. https://doi.org/10.3934/era.2024104.

[21] M. Manigandan, K. Manikandan, H.A. Hammad, M. De La Sen, Applying Fixed Point Techniques to Solve Fractional Differential Inclusions under New Boundary Conditions, AIMS Math. 9 (2024), 15505–15542. https://doi.org/10.3934/math.2024750.

[22] S.A. Murad, A.S. Rafeeq, Existence of Solutions of Integro-Fractional Differential Equation When $\alpha \in (2, 3]$ through Fixed Point Theorem, J. Math. Comput. Sci 11 (2021), 6392–6402. https://doi.org/10.28919/jmcs/6272.

[23] H. Mönch, Boundary Value Problems for Nonlinear Ordinary Differential Equations of Second Order in Banach Spaces, Nonlinear Anal. Theory Methods Appl. 4 (1980), 985–999. https://doi.org/10.1016/0362-546X(80)90010-3.

[24] A. Samadi, S.K. Ntouyas, J. Tariboon, Coupled Systems of Nonlinear Sequential Proportional Hilfer-Type Fractional Differential Equations with Multi-Point Boundary Conditions, AIMS Math. 9 (2024), 12982–13005. https://doi.org/10.3934/math.2024633.

[25] G.V. Smirnov, Introduction to the Theory of Differential Inclusions, Vol. 41, Graduate Studies in Mathematics, American Mathematical Society, Providence, 2002.