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Nodecness of Soft Generalized Topological Spaces

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Abstract. In this work, we define a new class of soft generalized topological spaces, namely strongly soft nodec, with the use of strongly soft nowhere dense sets. Then, we study the basic properties of these spaces and show that if the product of two soft generalized topological spaces is a strongly soft nodec space, then each one is a strongly soft nodec space. Then, we extend these notions to T_0-strongly soft nodec generalized topological spaces by using the soft quotient functions and discussing their main properties. We also show the inverse of a surjective soft quotient function preserves the soft closure and soft interior of a soft subset of a codomain soft set in soft generalized topological space. Further, we use soft quasi-homeomorphism and soft quotient functions to make comparisons and connections between these spaces with the support of appropriate counterexamples. Then, we successfully determine a condition under which the soft generalized topological space is a soft weak Baire space and hence a strongly soft second category.

1. Introduction

Soft topology [32] is an extension of classical topology that has various benefits as it allows for greater flexibility in defining open sets by a collection of parameters or functions that give each point a certain amount of openness. It merges soft set theory, the theory given by Molodtsov [29] as an approach to deal with uncertainty, with general topology. Several subclasses of soft topological spaces were suggested, including soft separation axioms [21], soft nodec spaces [7], soft submaximal spaces [4], soft separable spaces [21], congruence representations via soft ideals [13], soft compact spaces [19], soft Lindelof spaces [3], soft paracompact [26], soft connected spaces [26], soft simple extended space [9], and soft extremely disconnected spaces [17]. Furthermore, different generalized soft (open) sets in soft topological spaces were also proposed such as soft sets of the

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first or second Baire category [12]. The literature reviews in [1, 2, 8, 14, 16] include numerous research papers on soft continuity, supra soft continuity and its characterizations. Recently, soft topologies generated by some special soft operators or formulas were studied in [5, 11, 15, 18, 20].

In 2002, the notion of generalized topology was introduced by Császár [22] who studied some of its basic properties. After that the structure of soft generalized topology was established by Thomas and John [33]. They provided the basic notions and concepts of soft generalized topological spaces such as soft basis, subspace soft generalized topology, soft interior, soft closure, soft neighborhood, soft limit point, soft boundary, soft exterior and soft continuity of soft functions.

In 2013, Korczak-Kubiak et al. [25] defined strongly nowhere dense sets, and studied their properties. They also gave a relation between nowhere dense and strongly nowhere dense sets in generalized topological space. In [34], they analyzed properties of strongly nowhere dense sets in generalized topological spaces. In this paper, we introduce the concept of strongly nowhere dense soft sets in soft generalized topological spaces to study a comparable version of soft nodecness of soft generalized topological space.

The structure of this work is designed as follows: In Section 2, we provide basic concepts and results that make the paper more accessible to the reader. In Section 3, we present the definition of a strongly soft nodec generalized topological space, followed by some properties. We also prove that the product of two soft generalized topological spaces is a strongly soft nodec, then each one is a strongly soft nodec space. In Section 4, we extend the notions of strongly soft nodec generalized topological space to T_0-strongly soft nodec generalized topological space by using the soft quotient function and discussing its main properties. Additionally, we show the inverse of a surjective soft quotient function preserves soft closure and soft interior of a soft subset of a codomain soft set. In Section 5, we use soft quasi-homeomorphism and soft quotient functions to make comparisons and connections among the strongly soft nodec and the T_0-strongly soft nodec with the support of appropriate counterexamples. Then, we show that the image and the inverse-image of a strongly soft nodec is a strongly soft nodec under a bijective soft quasi homeomorphism in a soft generalized topological space. We finalize this work by successfully determining the conditions under which the soft generalized topological space is a soft weak Baire space and hence a strongly soft second category.

2. Fundamentals of Soft Sets and Soft Generalized Topologies

In this section, we present the basic definitions and results of soft-sets and soft generalized topologies.

Definition 2.1. *[29] Assume X and $\mathfrak{P}$ are respectively the initial universal set and a set of parameters. Let $K : \rho \to 2^X$ be a set-valued function, whereas $\rho \subseteq \mathfrak{P}$ and 2^X is the power set of X. An ordered pair $(K, \rho) = \{(r, K(r)) : r \in \rho\}$ is stated to be the soft set over X.*

Remark 2.1. *We can extend a soft set (K, ρ) to the soft set $(K, \mathfrak{P})$ by assuming $K(r) = \emptyset$ for any $r \in \mathfrak{P} - \rho$.*

Definition 2.2. *[31] The soft complement $(K,\rho)^c$ of a soft set (K,ρ) is a soft set (K^c,ρ) such that $K^c : \rho \to 2^X$ is a mapping having the property that $K^c(r) = X - K(r)$ for all $r \in \rho$.*

Definition 2.3. *[6,23] A soft set (K,ρ) over X is called:*

(1) *A null soft set with respect to ρ, denoted by Φ_ρ, if $K(r) = \emptyset$ for all $r \in \rho$. The (full) null set is denoted by $\Phi_{\mathfrak{P}}$.*
(2) *An absolute soft set with respect to ρ, denoted by X_ρ, if $K(r) = X$ for all $r \in \rho$. The (full) absolute set is denoted by $X_{\mathfrak{P}}$.*
(3) *Finite (resp. countable) if $K(r)$ is finite (resp. countable) for each $r \in \rho$. Otherwise, it is called infinite (resp. uncountable).*

$SS(X_{\mathfrak{P}})$ (resp. $SS(X_\rho)$ refers to the class of all soft sets over X linked with $\mathfrak{P}$ (resp. ρ).

Definition 2.4. *[6,27] For an index set I, let $\{(K_i,\rho) : i \in I\}$ be a family of soft sets over X.*

(1) *The soft intersection of (K_i,ρ), for $i \in I$, is a soft set (K,ρ) such that $K(r) = \bigcap_{i\in I} K_i(r)$ for all $r \in \rho$ and is denoted by $(K,\rho) = \widetilde{\bigcap}_{i\in I}(K_i,\rho)$.*
(2) *The soft union of (K_i,ρ), for $i \in I$, is a soft set (K,ρ) such that $K(r) = \bigcup_{i\in I} K_i(r)$ for all $r \in \rho$ and is denoted by $(K,\rho) = \widetilde{\bigcup}_{i\in I}(K_i,\rho)$.*

Definition 2.5. *[6,10] Let $(K,\rho),(L,\rho) \in SS(X_\rho)$. Then the soft difference (K,ρ) and (L,ρ) is defined to be the soft set $(H,\rho) = (K,\rho) - (L,\rho)$ such that $H(r) = K(r) - L(r)$ for all $r \in \rho$.*

Definition 2.6. *[27,31] Let $\rho_1,\rho_2 \subseteq \mathfrak{P}$. It is said that (K_1,ρ_1) is a soft subset of (K_2,ρ_2) (denoted by $(K_1,\rho_1)\widetilde{\subseteq}(K_2,\rho_2)$) if $\rho_1 \subseteq \rho_2$ and $K_1(r) \subseteq K_2(r)$ for all $r \in \rho_1$. Moreover, (K_1,ρ_1) is soft equal to (K_2,ρ_2), written by $(K_1,\rho_1) = (K_2,\rho_2)$, if $(K_1,\rho_1)\widetilde{\subseteq}(K_2,\rho_2)$ and $(K_2,\rho_2)\widetilde{\subseteq}(K_1,\rho_1)$.*

Definition 2.7. *[35] A soft point is a soft set (K,ρ) over X, denoted by x_r, provided that $K(r) = \{x\}$ for some $r \in \rho$ and $K(s) = \emptyset$ for all $s \in \rho$ with $r \neq s$, where $r \in \rho$ and $x \in X$. By a statement $x_r \in (K,\rho)$, we shall mean $x \in K(r)$. We denote the set of all soft points over X along with ρ by $SP(X_\rho)$.*

If x_r, y_s are two different soft points, then either $x \neq y$ or $r \neq s$ and by two disjoint soft sets $(K,\rho),(L,\rho)$ over X, we mean $(K,\rho)\widetilde{\cap}(L,\rho) = \Phi_\rho$.

Definition 2.8. *[30] Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$. Then, (K,ρ) is called a soft neighborhood of $x_r \in SP(X,\rho)$ if there exists $(W,\rho) \in \mu(x_r)$ such that $x_r \in (W,\rho)\widetilde{\subseteq}(K,\rho)$, where $\mu(x_r)$ is the family of all elements of μ that contain x_r.*

Definition 2.9. *[24,36] Let $SS(X_{\rho_1}), SS(Y_{\rho_2})$ be collections of soft sets, and let $p : X \to Y, q : \rho_1 \to \rho_2$ be mappings. The image of a soft set $(F,\rho_1) \in SS(X_{\rho_1})$ under $h : SS(X_{\rho_1}) \to SS(Y_{\rho_2})$ is a soft set $h(F,\rho_1) = (h(F), q(\rho_1))$ in $SS(Y_{\rho_2})$ which is given by*

$$h(F)(r_2) = \begin{cases} \bigcup\limits_{r_1 \in q^{-1}(r_2)\cap\rho_1} p\left(F(r_1)\right), & q^{-1}(r_2)\cap\rho_1 \neq \emptyset \\ \emptyset, & \textit{otherwise,} \end{cases}$$

for each $r_2 \in \rho_2$.

The inverse image of a soft set $(G, \rho_2) \in SS(Y_{\rho_2})$ *under h is a soft subset* $h^{-1}(G, \rho_2) = (h^{-1}(G), q^{-1}(\rho_2))$ *such that*

$$(h^{-1}(G)(r_1) = \begin{cases} p^{-1}\left(G(q(r_1))\right), & q(r_1) \in \rho_2 \\ \emptyset, & otherwise, \end{cases}$$

for each $r_1 \in \rho_1$.

The soft mapping h is injective (resp. surjective, bijective) if both p and q are injective (resp. surjective, bijective).

Definition 2.10. *[33] A soft generalized topology over X is a subfamily* $\mu \subseteq SS(X_\rho)$ *containing* Φ_ρ *and satisfying the condition that the soft union of the arbitrary number of soft sets of* μ *is in* μ.

The triple (X, μ, ρ) is called a soft generalized topological space, where every member of μ is called a soft g_μ-open set.

Definition 2.11. *[33] A strong soft generalized topology space* (X, μ, ρ) *is a generalized topology space such that* $X_\rho \in \mu$.

Definition 2.12. *[33] Let* (X, μ, ρ) *be a soft generalized topological space and* $(F, \rho)\widetilde{\subseteq}(X, \rho)$. *Then* (F, ρ) *is called a soft* g_μ*-closed set if its soft complement* $(F, \rho)^c$ *is a soft* g_μ*-open set. The family of all soft* g_μ*-open sets are denoted by* μ^c

Definition 2.13. *[33] Let* $(Y, \rho) \neq \Phi_\rho$ *be a soft subset of a soft generalized topological space* (X, μ, ρ). *Then* $\mu_{(Y,\rho)} = \{(L, \rho)\widetilde{\cap}(Y, \rho) : (L, \rho) \in \mu\}$ *is called a relative soft generalized topology over Y and* $(Y, \mu_{(Y,\rho)}, \rho)$ *is a soft subspace of* (X, μ, ρ).

Definition 2.14. *[33] Let* $(L, \rho)\widetilde{\in}(X, \mu, \rho)$. *Then* (L, ρ) *is called a soft neighborhood of* $x_r \in SP(X_\rho)$ *if there exists* $(W, \rho) \in \mu(x_r)$ *such that* $x_r \in (W, \rho)\widetilde{\subseteq}(L, \rho)$, *where* $\mu(x_r)$ *is the family of all elements of* μ *that contain* x_r.

Definition 2.15. *[33] Let* (K, ρ) *be a soft subset of a soft generalized topological space* (X, μ, ρ).

(1) *The soft closure of* (K, ρ), *denoted by* $cl_{g\mu}(K, \rho)$, *is the smallest soft closed set containing* (K, ρ).
(2) *The soft interior of* (K, ρ), *denoted by* $int_{g\mu}(K, \rho)$, *is the largest soft open set that is contained in* (K, ρ).

Lemma 2.1. *[33] Let* (K, ρ) *be a soft subset of a soft generalized topological space* (X, μ, ρ). *Then*

$$int_{g\mu}((K, \rho)^c) = (cl_{g\mu}(K, \rho))^c \text{ and } cl_{g\mu}((K, \rho)^c) = (int_{g\mu}(K, \rho))^c.$$

Definition 2.16. *[33] Let* (K, ρ) *be a soft subset of a soft generalized topological space* (X, μ, ρ). *The soft boundary of* (K, ρ) *is given by* $b_{g\mu}(K, \rho) = cl_{g\mu}(K, \rho) - int_{g\mu}(K, \rho)$.

Definition 2.17. *[33] Let (L, ρ) be a soft subset of a soft generalized topological space (X, μ, ρ). A soft point $x_\rho \in SP(X_\rho)$ is called a soft limit point of (L, ρ) if $(K, \rho)\widetilde{\cap}((L, \rho) - \{x_\rho\}) \neq \Phi_\rho$ for all $(K, \rho) \in \mu(x_\rho)$. The family of all soft limit points is denoted by $\mathcal{D}_{g\mu}\ (L, \rho)$.*

Definition 2.18. *[33] Let $(K, \rho), (L, \rho), (H, \rho)$ be soft subsets of a soft generalized topological space (X, μ, ρ). Then (K, ρ) is called*

1. *soft regular open if $int_{g\mu}(cl_{g\mu}(K, \rho)) = (K, \rho)$.*
2. *soft α-open if $(K, \rho)\widetilde{\subseteq}int_{g\mu}(cl_{g\mu}(int_{g\mu}(K, \rho)))$.*
3. *soft α-closed if $(K, \rho)\widetilde{\subseteq}cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K, \rho)))$.*
4. *soft g_μ-nowhere dense if $int_{g\mu}(cl_{g\mu}(K, \rho)) = \Phi_\rho$.*
5. *soft g_μ-dense in (L, ρ) if $(L, \rho)\widetilde{\subseteq}cl_{g\mu}(K, \rho)$.*
6. *soft g_μ-codense if $int_{g\mu}(K, \rho) = \Phi_\rho$.*

Definition 2.19. *[33] Let (X, μ_1, ρ_1) and (Y, μ_2, ρ_2) be two soft generalized topological spaces and $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ be a soft function, Then h is called a*

1. *soft continuous function if $h^{-1}(K, \rho_2) \in \mu_1$ for each $(K, \rho_2) \in \mu_2$.*
2. *soft open function if $h(K, \rho_1) \in \mu_2$ for each $(K, \rho_1) \in \mu_1$.*
3. *soft closed function if $h(K, \rho_1)$ is a soft closed subset of (Y, ρ_2) for each soft closed subset (K, ρ_1) of (X, ρ_1).*

Lemma 2.2. *[19] The soft projection mapping $\pi_i : (\prod X_i, \prod \mu_i, \rho)_{i\in I} \to (X_i, \mu_i, \rho)$ is soft open for each i.*

Definition 2.20. *[19] Let $\{(X_i, \mu_i, \rho) : i \in I\}$ be a family of soft topological space with a fixed parametric set ρ. The product soft topology μ on $X = \prod_{i\in I} X_i$ is the initial soft topology over X generated by the family $\{\pi_i : i \in I\}$, where π_i is the soft projection mapping from $SS(X_\rho)$ to $SS(X_{\rho_i})$ for each $i \in I$.*

Definition 2.21. *[19] Let (K, ρ_1) and (H, ρ_2) be soft sets over (X, μ_1, ρ_1) and (Y, μ_2, ρ_2) respectively. Then the Cartesian product of (K, ρ_1) and (H, ρ_2) is denoted by $(K\times H)_{\rho_1\times\rho_2}$ and is defined as $(K\times H)(x_{\rho_1}, x_{\rho_2}) = K(x_{\rho_1}) \times H(x_{\rho_2})$ for each $(x_{\rho_1}, x_{\rho_2}) \in (\rho_1 \times \rho_2)$.*

3. Strongly Soft Nodec Spaces

In this section, we define strongly soft nodec of a soft generalized topological space and present its master properties. We also prove that if the product of two soft generalized topological spaces is strongly soft nodec, then each one is a strongly soft nodec space.

Definition 3.1. *Let $(K, \rho), (L, \rho), (H, \rho)$ be soft subsets of soft generalized topological space (X, μ, ρ). Then (K, ρ) is called*

1. *strongly soft g_μ-nowhere dense if for any non-null soft open set (L, ρ), there exists a non-null soft open set (H, ρ), such that $(H, \rho)\widetilde{\subseteq}(L, \rho)$ and $(H, \rho)\widetilde{\cap}(K, \rho) = \Phi_\rho$.*
2. *strongly soft g_μ-first category if $(K, \rho) = \widetilde{\bigcup}_{n=1}^{\infty}(K_n, \rho)$, where each (K_n, ρ) is soft strongly g_μ-nowhere dense for every $n \in \mathbb{N}$. Otherwise, (K, ρ) is strongly soft g_μ-second category.*

We shall remark that every strongly soft nowhere dense (resp. strongly soft g_μ-first category) is soft nowhere dense (resp. soft g_μ-first category). However, the converse may not be the case generally.

Example 3.1. *Taking $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $\rho = \{p_1, p_2\}$ as a universal set and a set of parameters, respectively. Let $(K_1, \rho), (K_2, \rho), \ldots, (K_{10}, \rho)$ be soft sets over X given as follows:*
$(K_1, \rho) = \{(p_1, \{x_1, x_4\}), (p_2, \{x_1, x_3\})\}$,
$(K_2, \rho) = \{(p_1, \{x_1, x_5\}), (p_2, \{x_1\})\}$,
$(K_3, \rho) = \{(p_1, \{x_2, x_5\}), (p_2, \{x_3\})\}$,
$(K_4, \rho) = \{(p_1, \{x_1, x_4, x_5\}), (p_2, \{x_5\})\}$,
$(K_5, \rho) = \{(p_1, \{x_1, x_2, x_4, x_5\}), (p_2, \{x_1, x_3, x_5\})\}$,
$(K_6, \rho) = \{(p_1, \{x_1, x_4, x_5\}), (p_2, \{x_1, x_3\})\}$,
$(K_7, \rho) = \{(p_1, \{x_1, x_2, x_4, x_5\}), (p_2, \{x_1, x_3\})\}$,
$(K_8, \rho) = \{(p_1, \{x_1, x_2, x_5\}), (p_2, \{x_1, x_3\})\}$,
$(K_9, \rho) = \{(p_1, \{x_1, x_4, x_5\}), (p_2, \{x_1, x_5\})\}$, *and*
$(K_{10}, \rho) = \{(p_1, \{x_1, x_2, x_4, x_5\}), (p_2, \{x_3, x_5\})\}$
Then the family $\mu = \{(K_i, \rho) : i = 1, 2, \cdots, 10\}\widetilde{\cup}\{\Phi_\rho\}$ forms a soft generalized topology over X. The soft set $\{(p_1, \{x_3\}), (p_2, \{x_3\})\}$ is a non-null soft nowhere dense set but not strongly soft nowhere dense. The rest case is clear.

Throughout this work, soft nowhere dense (resp. strongly soft nowhere dense, strongly soft first category, strongly second category, soft open, soft closed and etc.) means soft g_μ-nowhere dense (resp. strongly soft g_μ-nowhere dense, strongly soft g_μ-first category, strongly g_μ-second category, soft g_μ-open, soft g_μ-closed and etc.) when no confusion exists.
It is easy to prove the following Lemma:

Lemma 3.1. *Let (X, μ, ρ) be a soft generalized topological space. Then*

(1) *Every soft subset of a strongly soft nowhere dense set is strongly soft nowhere dense.*
(2) *Every soft subset of a strongly soft first category set is a strongly soft first category set.*
(3) *If $(K, \rho)\widetilde{\subseteq}(X, \mu, \rho)$ is a strongly soft nowhere dense set, then $cl_{g\mu}(K, \rho)$ is a strongly soft nowhere dense set.*
(4) *For every $(K, \rho), (L, \rho)\widetilde{\subseteq}(X, \mu, \rho)$. If (K, ρ) is non-null soft opn set and $(K, \rho)\widetilde{\cap}(L, \rho) = \Phi_\rho$, then $(K, \rho)\widetilde{\cap}cl_{g\mu}(L, \rho) = \Phi_\rho$.*
(5) *For every strongly soft nowhere dense $(K, \rho)\widetilde{\subseteq}(X, \mu, \rho)$, (K, ρ) is soft codense.*
(6) *For every $(K, \rho)\widetilde{\subseteq}(X, \mu, \rho)$, $int_{g\mu}[cl_{g\mu}((K, \rho) \setminus (K, \rho))] = \Phi_\rho$.*

Definition 3.2. *A soft generalized topological space (X, μ, ρ) is called a strongly soft nodec space if each non-null strongly soft nowhere dense subset of (X, μ, ρ) is soft closed.*

Theorem 3.1. *Let (X, μ, ρ) be a strong soft generalized topology space. Then X_ρ is a strongly soft nodec space if any one of the following holds:*

(1) *Every soft α-closed set is a soft closed set.*

(2) *Every soft α-open set is a soft open set.*
(3) $(cl_{g\mu}(K,\rho) - (K,\rho))\widetilde{\subseteq} cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho)))$ *for every* $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$.
(4) $cl_{g\mu}(K,\rho) = (K,\rho)\widetilde{\cup}[cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho)))]$ *for every* $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$.
(5) $int_{g\mu}(K,\rho) = (K,\rho)\widetilde{\cap}(int_{g\mu}(cl_{g\mu}(int_{g\mu}(K,\rho))))$ *for every* $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$.

Proof. (1) Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ be a non-null strongly soft nowhere dense. Then, (K,ρ) is a non-null soft nowhere dense set and so $int_{g\mu}(cl_{g\mu}(K,\rho)) = \Phi_\rho$. Since X_ρ is a strong soft generalized topology space, $cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho))) = \Phi_\rho$, then $cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho)))\widetilde{\subseteq}(K,\rho)$ which implies (K,ρ) is a soft α-closed set; hence (K,ρ) is a soft closed set in X_ρ. Thus, X_ρ is a strongly soft nodec space.

(2) By following the same strategy as in part (1), we can say that X_ρ is a strongly soft nodec space.

(3) Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ be a non-null strongly soft nowhere dense. Then, by Lemma 3.1, (K,ρ) is a non-null soft nowhere dense set and so $int_{g\mu}(cl_{g\mu}(K,\rho)) = \Phi_\rho$. Since X_ρ is a strong soft generalized topology space, then $cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho))) = \Phi_\rho$. As $(cl_{g\mu}(K,\rho) - (K,\rho))\widetilde{\subseteq} cl_{g\mu}(int_{g\mu}(cl_{g\mu}(K,\rho))) = \Phi_\rho$, then $cl_{g\mu}(K,\rho) = (K,\rho)$. Thus, (K,ρ) is a soft closed set in X_ρ. Therefore, X_ρ is a strongly soft nodec space.

(4) It follows from [7, Theorem 3.3].

(5) It follows from [7, Theorem 3.4]. □

Lemma 3.2. *Let (X,μ,ρ) be a soft generalized topological space. Then the following statements hold:*

(1) *For every strongly soft nowhere dense $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$, $b_{g\mu}(K,\rho)$ is a strongly soft nowhere dense set.*
(2) *For every $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ and $b_{g\mu}(K,\rho)$ is a strongly soft nowhere dense set, then $b_{g\mu}(int_{g\mu}(K,\rho))$, $b_{g\mu}(cl_{g\mu}(K,\rho))$ are strongly soft nowhere dense sets.*
(3) *For every strongly soft nowhere dense $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$, $b_{g\mu}((K,\rho)\widetilde{\cap}(L,\rho))$ is a strongly soft nowhere dense set for any soft set $(L,\rho)\widetilde{\subseteq}(X,\mu,\rho)$.*

Proof. (1) Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ be a strongly soft nowhere dense. Let (L,ρ) be a non-null soft open set. Then there exists a non-null soft open set (H,ρ) such that $(H,\rho)\widetilde{\subseteq}(L,\rho)$ and $(H,\rho)\widetilde{\cap}(K,\rho) = \Phi_\rho$. By Lemma 3.1, $(H,\rho)\widetilde{\cap} cl_{g\mu}(K,\rho) = \Phi_\rho$. Since $b_{g\mu}(K,\rho)\widetilde{\subseteq} cl_{g\mu}(K,\rho)$, then $(H,\rho)\widetilde{\cap} b_{g\mu}(K,\rho) = \Phi_\rho$. Therefore, $b_{g\mu}(K,\rho)$ is a strongly soft nowhere dens set in X_ρ.

(2) The proof is straightforward.

(3) Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ be a strongly soft nowhere dense. Since $(K,\rho)\widetilde{\cap}(L,\rho)\widetilde{\subseteq}(K,\rho)$ and a subset of a strongly soft nowhere dense set is strongly soft nowhere dense, and by part (1), $b_{g\mu}((K,\rho)\widetilde{\cap}(L,\rho))$ is a strongly soft nowhere dense set for any soft set $(L,\rho)\widetilde{\subseteq}(X,\mu,\rho)$.

□

Lemma 3.3. *Let (X,μ,ρ) be a soft generalized topological space. Then every strongly soft nowhere dense in X_ρ does not contain a non-null soft open set in X_ρ.*

Proof. Let $(K,\rho)\widetilde{\subseteq}(X,\mu,\rho)$ be a strongly soft nowhere dense set. Suppose that there exists a non-null soft set (H,ρ) such that $(H,\rho)\widetilde{\subseteq}(K,\rho)$. Then there is no soft set $(L,\rho) \neq \Phi_\rho$ such that $(L,\rho)\widetilde{\subseteq}(H,\rho)$

and $(L,\rho)\widetilde{\cap}(K,\rho)=\Phi_\rho$ which is a contradiction to that (K,ρ) is a strongly soft nowhere dense set in X_ρ. Hence, $(H,\rho)=\Phi_\rho$. Therefore, (K,ρ) does not contain a non-null soft open set in X_ρ. □

Definition 3.3. *A soft generalized topological space (X,μ,ρ) is called soft nodec if each non-null soft nowhere dense subset of X_ρ is soft closed.*

Obviously, every strongly soft nodec generalized topological space is soft nodec. The reverse may not always true, and the counterexample can be concluded from Example 3.1.

Theorem 3.2. *Let (Y,μ_Y,ρ) be a soft dense subspace of a soft generalized topological space (X,μ,ρ). If every non-null strongly soft nowhere dense set in Y_ρ is a soft closed set in X_ρ, and every non-null strongly soft nowhere dense set in X_ρ meets Y_ρ, then X_ρ is a strongly soft nodec space.*

Proof. Let (K,ρ) be a non-null strongly soft nowhere dense set in (X,μ,ρ). By hypothesis, $(K,\rho)\cap(Y,\rho)\neq\Phi_\rho$. Let (L,ρ) be a non-null soft open set in Y. Then, $(L,\rho)=(H,\rho)\widetilde{\cap}(Y,\rho)$ where (H,ρ) is a non-null soft open set in X_ρ. Since (K,ρ) is a strongly soft nowhere dense set in X_ρ, then there exists a non-null soft open set (F,ρ) in X_ρ such that $(F,\rho)\widetilde{\subseteq}(H,\rho)$ and $(F,\rho)\widetilde{\cap}(K,\rho)=\Phi_\rho$. Since Y is soft dense subspace of X, then $(F,\rho)\widetilde{\cap}(Y,\rho)$ is a non-null soft open set in Y. Take $(G,\rho)=(F,\rho)\widetilde{\cap}(Y,\rho)$. Hence, there exists a non-null soft open set in Y such that $(G,\rho)\widetilde{\subseteq}(L,\rho)$ and $(G,\rho)\widetilde{\cap}(K,\rho)=\Phi_\rho$. Therefore, (K,ρ) is a non-null strongly soft nowhere dense in Y. By hypothesis, (K,ρ) is a soft closed set in X_ρ. Hence, X_ρ is a strongly soft nodec space. □

Lemma 3.4. *Let (X,μ_1,ρ), (Y,μ_2,ρ) be soft generalized topological spaces. Then the following statements hold:*

(1) *If $(K,\rho),(H,\rho)$ are strongly soft nowhere dense subsets of X_ρ, Y_ρ, respectively, then $(K,\rho)\times(H,\rho)$ is a strongly soft nowhere dense set in $X_\rho\times Y_\rho$.*
(2) *If $(K,\rho)\times(H,\rho)$ is a strongly soft nowhere dense set in $X_\rho\times Y_\rho$, then (K,ρ) is a strongly soft nowhere dense set in X_ρ or (H,ρ) is a strongly soft nowhere dense set in Y_ρ.*

Proof. (1) Suppose that $(K,\rho),(H,\rho)$ are strongly soft nowhere dense subsets of X_ρ, Y_ρ, respectively. Let $(F,\rho)\times(G,\rho)$ is a non-null soft set in $X_\rho\times Y_\rho$. Then, (F,ρ) is a non-null soft set in X_ρ and (G,ρ) is a non-null soft set in Y_ρ. By hypothesis, there exists a non-null soft sets $(F_1,\rho),(G_1,\rho)$ in X_ρ, Y_ρ respectively, such that $(F_1,\rho)\widetilde{\subseteq}(F,\rho),(G_1,\rho)\widetilde{\subseteq}(G,\rho)$ and $(F_1,\rho)\widetilde{\cap}(K,\rho)=\Phi_\rho$, $(G_1,\rho)\widetilde{\cap}(H,\rho)=\Phi_\rho$. Hence, there exists $(F_1,\rho)\times(G_1,\rho)$ is a non-null soft in $X_\rho\times Y_\rho$ such that $(F_1,\rho)\times(G_1,\rho)\widetilde{\subseteq}(F,\rho)\times(G,\rho)$ and $(F_1,\rho)\times(G_1,\rho)\widetilde{\cap}(K,\rho)\times(H,\rho)=\Phi_\rho$. Therefore, $(K,\rho)\times(H,\rho)$ is a strongly soft nowhere dense set in $X_\rho\times Y_\rho$.

(2) Suppose that $(K,\rho)\times(H,\rho)$ is a strongly soft nowhere dense set in $X_\rho\times Y_\rho$. Let $(F,\rho),(G,\rho)$ be non-null soft subsets of X_ρ, Y respectively. By hypothesis, there exists a non-null soft set $(F_1,\rho)\times(G_1,\rho)$ in $X_\rho\times Y_\rho$ such that $(F_1,\rho)\times(G_1,\rho)\widetilde{\subseteq}(F,\rho)\times(G,\rho)$ and $(F_1,\rho)\times(G_1,\rho)\widetilde{\cap}(K,\rho)\times(H,\rho)=\Phi_\rho$. Since $(F_1,\rho)\times(G_1,\rho)\neq\Phi_\rho$ and $(F_1,\rho)\times(G_1,\rho)\widetilde{\subseteq}(F,\rho)\times(G,\rho)$, then $(F_1,\rho)\widetilde{\subseteq}(F,\rho)$ and $(G_1,\rho)\widetilde{\subseteq}(G,\rho)$. Since $(F_1,\rho)\times(G_1,\rho)\widetilde{\cap}(K,\rho)\times(H,\rho)=\Phi_\rho$, then $(F_1,\rho)\widetilde{\cap}(K,\rho)=\Phi_\rho$ or

$(G_1, \rho)\widetilde{\cap}(H, \rho) = \Phi_\rho$. Thus, (K, ρ) is a strongly soft nowhere dense set in X_ρ or (H, ρ) is a strongly soft nowhere dense set in Y_ρ. □

Theorem 3.3. *If the product of two soft generalized topological spaces is a strongly soft nodec space, then each one is a strongly soft nodec space.*

Proof. Suppose that (X, μ_1, ρ), (Y, μ_2, ρ) be soft generalized topological spaces and $X_\rho \times Y_\rho$ is a strongly soft nodec space. Let $(K, \rho), (H, \rho)$ be non-null strongly soft nowhere dense subsets of X_ρ, Y_ρ respectively. Then, by Lemma 3.4, $(K, \rho) \times (H, \rho)$ is a non-null strongly soft nowhere dense subset of $X_\rho \times Y_\rho$. By hypothesis, $(K, \rho) \times (H, \rho)$ is a soft closed set in $X_\rho \times Y_\rho$. Hence, (K, ρ) is a soft closed set in X_ρ and (H, ρ) is a soft closed set in Y_ρ. Therefore, X_ρ and Y_ρ are strongly soft nodec spaces. □

4. T_0-Strongly Soft Nodec Spaces

Let (X, μ, ρ) be a soft generalized topological space. A binary relation $\sim$ on X_ρ is defined by $(x_r \sim x_t)$ if and only if $cl_{g\mu}\{x_r\} = cl_{g\mu}\{x_t\}$, where $x_r, x_t \widetilde{\in} SP(X_\rho)$, see [28]. The $\sim$ is an equivalence relation on X_ρ and the resulting soft quotient space $T_0(X_\rho) = X_\rho / \sim$ is the soft T_0-reflection of X_ρ and the soft generalized quotient topology over $T_0(X)$ is defined to be $\mu_\pi = \{(K, \rho)\widetilde{\subseteq} T_0(X_\rho) : h^{-1}((K, \rho))\widetilde{\in}\mu\}$ where π is a soft quotient function from X_ρ into $T_0(X_\rho)$ by setting $x_r \widetilde{\in} SP(X_\rho)$ to its soft equivalence class $[x_r]$ in $T_0(X_\rho)$. Then, the triple $(T_0(X), \mu_\pi, \rho)$ is called the soft generalized soft quotient topological space of X_ρ. A soft function $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ is said to be a soft quasi-homeomorphism if $(K, \rho_1) \to h^{-1}(K, \rho_1)$ (resp. $(F, \rho_1) \to h^{-1}(F, \rho_1)$) defines a bijection function $\mu_1 \to \mu_2$ (resp. $\mu_1^c \to \mu_2^c$). Equivalently, a soft function $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ is said to be a soft quasi-homeomorphism if for each $(K, \rho_1) \in \mu_1$, there exists a unique $(L, \rho_2) \in \mu_2$ such that $(K, \rho_1) = h^{-1}(L, \rho_2)$ (equivalently, for each soft closed subset (K, ρ_1) of X_{ρ_1}, there exists a unique soft closed subset (L, ρ_2) of Y_{ρ_2} such that $(K, \rho_1) = h^{-1}(L, \rho_2)$).

Proposition 4.1. *Let $(X, \mu_1, \rho_1), (Y, \mu_2, \rho_2)$ be soft generalized topological spaces. If $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ is a surjective soft quasi-homeomorphism, then h is a soft open (resp. soft closed) function.*

Proof. Let (K, ρ_1) be a soft open (resp. soft closed) set in X_{ρ_1}. Since h is a soft quasi-homeomorphism, then there exists a unique soft open (resp. closed) set (L, ρ_2) in Y_{ρ_2} such that $(K, \rho_1) = h^{-1}((L, \rho_2))$. Since h is a surjective function, then $h((K, \rho_1)) = h(h^{-1}((L, \rho_2))) = (L, \rho_2)$. Hence, $h((K, \rho_1))$ is a soft open (resp. soft closed) set in Y_{ρ_2}. Therefore, h is a soft open (resp. soft closed) function. □

Notation 4.1. *Let (X, μ, ρ) be a soft generalized topological space and let $x_r, x_t \widetilde{\in} SP(X_\rho)$. Then,*

(1) $H_0(x_r) = \{x_t \widetilde{\in} SP(X_\rho) : cl_{g\mu}\{x_r\} = cl_{g\mu}\{x_t\}\}$.
(2) $H_0((K, \rho)) = \widetilde{\bigcup}\{H_0(x_r) : x_r \widetilde{\in} (K, \rho)\}$.

Definition 4.1. *A soft generalized topological space (X, μ, ρ) is called a T_0-strongly soft nodec space if its soft T_0-reflection is strongly soft nodec.*

Example 4.1. *Taking $X = \{x_1, x_2, x_3, x_4\}$ and $\rho = \{p\}$ as a universal set and a set of parameters, respectively. Let $(K_1, \rho), (K_2, \rho), (K_3, \rho), (K_4, \rho), (K_5, \rho), (K_6, \rho)$ be soft sets over X given as follows:*
$(K_1, \rho) = \{(p, \{x_1\})\}$
$(K_2, \rho) = \{(p, \{x_2, x_3\})\}$
$(K_3, \rho) = \{(p, \{x_1, x_2\})\}$
$(K_4, \rho) = \{(p, \{x_1, x_2, x_3\})\}$
$(K_5, \rho) = \{(p, \{x_1, x_2, x_4\})\}$
$(K_6, \rho) = \{(p, \{x_1, x_3, x_4\})\}$
Then the family $\mu = \{\Phi_\rho, (K_1, \rho), (K_2, \rho), (K_3, \rho), (K_4, \rho), (K_5, \rho), (K_6, \rho), X_\rho\}$ forms a strongly soft generalized topology over X. Define a soft function $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ by assigning each $x_r \widetilde{\in} SP(X_\rho)$ to its soft equivalence class $[x_r]$ in $T_0(X_\rho)$. Hence, $\mu_\pi = \{\Phi_\rho, (G, \rho)\}$, where $(G, \rho) = \{(p, \{[x_1]\})\}$ and $\pi^{-1}((G, \rho)) = \{(p, \{x_1\})\} \widetilde{\in} \mu$. Thus, every non-null strongly soft nowhere dense set in $T_0(X_\rho)$ is a soft closed which implies that $T_0(X_\rho)$ is a strongly soft nodec space. Therefore, X_ρ is a T_0-strongly soft nodec space.

Lemma 4.1. *Let (X, μ, ρ) be a soft generalized topological space and let $(K, \rho) \widetilde{\subseteq} (X, \mu, \rho)$. If $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ is a surjective soft quotient function, then the following statements hold:*

(1) *The function π is a soft quasi-homoeomorphism.*
(2) *The function π is soft open and soft closed.*
(3) *$(K, \rho) \widetilde{\subseteq} H_0(K, \rho) \widetilde{\subseteq} cl_{g\mu}(K, \rho)$ and consequently $cl_{g\mu}(H_0(K, \rho)) = cl_{g\mu}(K, \rho)$.*
(4) *If (K, ρ) is a soft closed set, then $H_0(K, \rho) = (K, \rho)$.*
(5) *$H_0(K, \rho) = \pi^{-1}(\pi(K, \rho))$.*
(6) *If $\{(K, \rho)_n\}_{n \in \mathbb{N}}$ is a family of soft subsets of X, then $H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n) = \bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n)$.*

Proof. (1) It follows from the definition of μ_π and the surjectivity of h.

(2) It follows from Proposition 4.1.

(3) Evidently, $(K, \rho) \widetilde{\subseteq} H_0(K, \rho)$. Let $x_r \widetilde{\in} H_0(K, \rho)$. Then, $x_r \widetilde{\in} H_0(x_t)$ and $cl_{g\mu}\{x_r\} = cl_{g\mu}\{x_t\}$ for some $x_t \widetilde{\in} (K, \rho)$. Thus, $x_r \widetilde{\in} cl_{g\mu}\{x_t\} \widetilde{\subseteq} cl_{g\mu}(K, \rho)$ which implies that $x_r \widetilde{\in} cl_{g\mu}(K, \rho)$. Hence, $H_0(K, \rho) \widetilde{\subseteq} cl_{g\mu}(K, \rho)$. Therefore, $(K, \rho) \widetilde{\subseteq} H_0(K, \rho) \widetilde{\subseteq} cl_{g\mu}(K, \rho)$; hence $cl_{g\mu}(H_0(K, \rho)) = cl_{g\mu}(K, \rho)$.

(4) It follows from 3.

(5)It follows from the definition of $H_0(K, \rho)$ and a soft quotient function π.

(6) Let $x_t \widetilde{\in} H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n)$. Then, $x_t \widetilde{\in} \bigcup H_0(x_r)$ for all $x_r \widetilde{\in} \bigcup_{n \in \mathbb{N}} (K, \rho)_n$. This implies that $cl_{g\mu}\{x_r\} = cl_{g\mu}\{x_t\}$ for some $x_r \widetilde{\in} (K, \rho)_i$ for some $i \in \mathbb{N}$. Hence, $x_t \widetilde{\in} \bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n)$. Therefore, $H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n) \widetilde{\subseteq} \bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n)$.

Conversely, let $x_t \widetilde{\in} \bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n)$. Then, $x_t \widetilde{\in} (K, \rho)_i$ which implies that $cl_{g\mu}\{x_{r_i}\} = cl_{g\mu}\{x_t\}$ for some $x_{r_i} \widetilde{\in} (K, \rho)_i$ and $i \in \mathbb{N}$, which implies that $x_t \widetilde{\in} H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n)$. Hence, $\bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n) \widetilde{\subseteq} H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n)$. Therefore, $H_0(\bigcup_{n \in \mathbb{N}} (K, \rho)_n) = \bigcup_{n \in \mathbb{N}} H_0((K, \rho)_n)$. □

Theorem 4.1. *Let (X, μ, ρ) be a soft generalized topological space and let $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ be a surjective soft quotient function. Then, the following statements are equivalent.*

(1) *X_ρ is a T_0-strongly soft nodec space.*

(2) *Any non-null strongly soft nowhere dense subset (K,ρ) of X_ρ, $H_0((K,\rho))$ is a soft closed set.*

Proof. (1) $\Rightarrow$ (2) Suppose that X_ρ is a T_0-strongly soft nodec generalized topological space. Let (K,ρ) be a non-null strongly soft nowhere dense subset of X. By Lemma 3.1, $cl_{g\mu}(K,\rho)$ is a non-null strongly soft nowhere dense subset of X. Assume that $cl_{g\mu_\pi}(\pi(K,\rho))$ is not strongly soft nowhere dense set in $T_0(X_\rho)$. Then, by Lemma 3.3, there is a non-null soft open set (L,ρ) of subset of $T_0(X_\rho)$ such that $(L,\rho)\widetilde{\subseteq} cl_{g\mu_\pi}(\pi(K,\rho))$. Since π is a soft continuous function, then $\pi^{-1}((L,\rho))$ is a non-null soft open set in X_ρ and $\pi^{-1}((L,\rho))\widetilde{\subseteq}\pi^{-1}(cl_{g\mu_\pi}(\pi(K,\rho)))$. Since $cl_{g\mu}(K,\rho)$ is a soft closed set in X_ρ, $(K,\rho)\widetilde{\subseteq} cl_{g\mu}(K,\rho)$, then $\pi(K,\rho)\widetilde{\subseteq}\pi(cl_{g\mu}(K,\rho))$; hence $cl_{g\mu_\pi}(\pi(K,\rho))\widetilde{\subseteq} cl_{g\mu_\pi}(\pi(cl_{g\mu}(K,\rho))$. By Lemma 4.1 (2), π is a soft closed soft function implies that $\pi(cl_{g\mu}(K,\rho))$ is a soft closed set in $T_0(X_\rho)$. Hence, $cl_{g\mu_\pi}(\pi(K,\rho))\widetilde{\subseteq} cl_{g\mu_\pi}(\pi(cl_{g\mu}(K,\rho)) = \pi(cl_{g\mu}(K,\rho))$. Thus, $\pi^{-1}(cl_{g\mu_\pi}(\pi(K,\rho)))\widetilde{\subseteq}\pi^{-1}(\pi(cl_{g\mu}(K,\rho))) = cl_{g\mu}(K,\rho)$ which contradicts the fact that $cl_{g\mu}(K,\rho)$ is a strongly soft nowhere dense set in X_ρ. Hence, $cl_{g\mu_\pi}(\pi(K,\rho))$ is a non-null strongly soft nowhere dense set in $T_0(X_\rho)$. Thus, $\pi(K,\rho)$ is a non-null strongly soft nowhere dense set in $T_0(X_\rho)$ because a soft subset of strongly soft nowhere dense set is a strongly soft nowhere dense set. Since $T_0(X_\rho)$ is a strongly soft nodec space, then $\pi(K,\rho)$ is a soft closed set in $T_0(X_\rho)$. By soft continuity of π, $\pi^{-1}(\pi(K,\rho))$ is a soft closed set in X_ρ. By Lemma 4.1 (5), $H_0((K,\rho)) = \pi^{-1}(\pi(K,\rho))$; hence $H_0((K,\rho))$ is a soft closed set in X_ρ.

(2) $\Rightarrow$ (1) Let (K,ρ) be a non-null strongly soft nowhere dense subset of $T_0(X_\rho)$ and $(L,\rho) = \pi^{-1}(K,\rho)$. By surjectivity of π, $\pi((L,\rho)) = \pi(\pi^{-1}(K,\rho)) = (K,\rho)$. Hence, $\pi((L,\rho))$ is a non-null strongly soft nowhere dense subset of $T_0(X_\rho)$. Since π is a surjective soft function, then $\pi^{-1}(\pi((L,\rho))) = \pi^{-1}(\pi(\pi^{-1}(K,\rho))) = \pi^{-1}((K,\rho)) = (L,\rho)$. Since $\pi((L,\rho))$ is a non-null soft strongly nowhere dense subset of strongly soft nodec space $T_0(X_\rho)$ and π is a soft function, then $\pi((L,\rho))$ is a soft closed set in $T_0(X_\rho)$. By Lemma 4.1 (5), $H_0(L,\rho) = \pi^{-1}(\pi(L,\rho))$ and $\pi^{-1}(\pi((L,\rho))) = \pi^{-1}(\pi(\pi^{-1}(K,\rho))) = \pi^{-1}((K,\rho)) = (L,\rho)$. Then, $H_0(L,\rho) = (L,\rho)$. Suppose that (L,ρ) is not strongly soft nowhere dense set in X_ρ. Then, by Lemma 3.3, there is a non-null soft open subset (H,ρ) of X_ρ such that $(H,\rho)\widetilde{\subseteq}(L,\rho)$. Since π is a soft open function, by Lemma 4.1 (2), then $\pi((H,\rho))$ is a non-null soft open set in $T_0(X_\rho)$ and $\pi((H,\rho))\widetilde{\subseteq}\pi(((L,\rho))$. By Lemma 3.3, $\pi((L,\rho))$ is not strongly soft nowhere dense set in $T_0(X_\rho)$ which contradicts the fact that $\pi((L,\rho))$ is a strongly soft nowhere dense set in $T_0(X_\rho)$. Hence, (L,ρ) is a non-null strongly soft nowhere dense set in X_ρ. Since $H_0(L,\rho) = (L,\rho)$ and by hypothesis, (L,ρ) is a soft closed set in X_ρ, then $\pi((L,\rho))$ is a soft closed set in $T_0(X_\rho)$. Hence, (K,ρ) is a soft closed set in $T_0(X_\rho)$ since $(K,\rho) = \pi((L,\rho))$. Therefore, X_ρ is T_0-strongly soft nodec space. □

The following proposition shows that the inverse of a surjective soft quotient function preserves soft closure and soft interior of a soft subset of a codomain soft set.

Proposition 4.2. *Let (X,μ,ρ) be a soft generalized topological space and let $\pi : (X,\mu,\rho) \to (T_0(X),\mu_\pi,\rho)$ be a surjective soft quotient function. Then the following statements hold:*

(1) *For every $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$, $\pi^{-1}(cl_{g\mu_\pi}(K,\rho)) = cl_{g\mu}(\pi^{-1}(K,\rho))$.*

(2) *For every* $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$, $\pi^{-1}(int_{g\mu_\pi}(K,\rho)) = int_{g\mu}(\pi^{-1}(K,\rho))$.
(3) *For every* $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$, $\pi^{-1}(cl_{g\mu_\pi}(int_{g\mu_\pi}(cl_{g\mu_\pi}(K,\rho)))) = cl_{g\mu}(int_{g\mu}(cl_{g\mu}(\pi^{-1}(K,\rho))))$.

Proof. (1) Let $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$. Let x_p be a soft point in $\pi^{-1}(cl_{g\mu_\pi}(K,\rho))$. Let (L,ρ) be any soft open set in X_ρ such that $x_p\widetilde{\in}(L,\rho)$. By Lemma 4.1 (2), π is a soft open function which implies that $\pi((L,\rho))$ is a soft open set in $T_0(X_\rho)$. By surjectivity of π, we have that $\pi(x_p)\widetilde{\in}\pi((L,\rho))$. Then, there exists a soft point x_r in $T_0(X_\rho)$ such that $x_r\widetilde{\in}\pi(x_p)$, $x_r\widetilde{\in}cl_{g\mu_\pi}(K,\rho)$ and $x_r\widetilde{\in}\pi((L,\rho))$; hence $\pi((L,\rho))\widetilde{\cap}cl_{g\mu_\pi}(K,\rho)) \neq \Phi_\rho$. Thus, $\pi^{-1}(\pi((L,\rho))\widetilde{\cap}(K,\rho) \neq \Phi_\rho$ which implies that $(L,\rho)\widetilde{\cap}(K,\rho) \neq \Phi_\rho$. Therefore, $x_p\widetilde{\in}cl_{g\mu}(\pi^{-1}(K,\rho))$; hence $\pi^{-1}(cl_{g\mu_\pi}(K,\rho))\widetilde{\subseteq}cl_{g\mu}(\pi^{-1}(K,\rho))$. Since π is a soft continuous function and by [33, Theorem 4.5.], $cl_{g\mu}(\pi^{-1}(K,\rho))\widetilde{\subseteq}\pi^{-1}(cl_{g\mu_\pi}(K,\rho))$. Therefore, $\pi^{-1}(cl_{g\mu_\pi}(K,\rho)) = cl_{g\mu}(\pi^{-1}(K,\rho))$.

(2) Let $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$. Since π is a soft continuous function and π^{-1} is a soft open function. Then, $\pi^{-1}(int_{g\mu_\pi}(K,\rho)) = int_{g\mu}(\pi^{-1}(int_{g\mu_\pi}(K,\rho)))\widetilde{\subseteq}int_{g\mu}(\pi^{-1}(K,\rho))$. Hence, $\pi^{-1}(int_{g\mu_\pi}(K,\rho))\widetilde{\subseteq}int_{g\mu}(\pi^{-1}(K,\rho))$. Let x_p be a soft point in $int_{g\mu}(\pi^{-1}(K,\rho))$. Then, there exists a soft open set (L,ρ) in X_ρ such that $x_p\widetilde{\in}(L,\rho)$ and $(L,\rho)\widetilde{\subseteq}\pi^{-1}((K,\rho))$. By Lemma 4.1(1) and the definition of a soft quasi-homeomorphism, there exists a unique soft open set (H,ρ) in $T_0(X_\rho)$ such that $\pi^{-1}((H,\rho)) = (L,\rho)$. Thus, $\pi^{-1}((H,\rho))\widetilde{\subseteq}\pi^{-1}((K,\rho))$ which implies that $(H,\rho)\widetilde{\subseteq}\pi(\pi^{-1}((K,\rho))) = (K,\rho)$ since π is a surjective soft function. Hence, $\pi(x_p)\widetilde{\in}int_{g\mu}((K,\rho))$ and so $x_p\widetilde{\in}\pi^{-1}(int_{g\mu_\pi}((K,\rho)))$. Thus, $int_{g\mu}(\pi^{-1}(K,\rho))\widetilde{\subseteq}\pi^{-1}(int_{g\mu_\pi}(K,\rho))$. Therefore, $\pi^{-1}(int_{g\mu_\pi}(K,\rho)) = int_{g\mu}(\pi^{-1}(K,\rho))$.

(3) Let $(K,\rho)\widetilde{\subseteq}(T_0(X),\mu_\pi,\rho)$. By part (1),

$$\pi^{-1}(cl_{g\mu_\pi}(int_{g\mu_\pi}(cl_{g\mu_\pi}(K,\rho)))) = cl_{g\mu}(\pi^{-1}(int_{g\mu_\pi}(cl_{g\mu_\pi}(K,\rho)))),$$

and by part (2),

$$cl_{g\mu}(\pi^{-1}(int_{g\mu_\pi}(cl_{g\mu_\pi}(K,\rho)))) = cl_{g\mu}(int_{g\mu}(\pi^{-1}(cl_{g\mu_\pi}(K,\rho)))).$$

Hence,

$$cl_{g\mu}(int_{g\mu}(\pi^{-1}(cl_{g\mu_\pi}(K,\rho)))) = cl_{g\mu}(int_{g\mu}(cl_{g\mu}(\pi^{-1}(K,\rho)))).$$

Therefore,

$$\pi^{-1}(cl_{g\mu_\pi}(int_{g\mu_\pi}(cl_{g\mu_\pi}(K,\rho)))) = cl_{g\mu}(int_{g\mu}(cl_{g\mu}(\pi^{-1}(K,\rho)))).$$

□

5. Comparisons and Connections

In this section, we use soft quasi-homeomorphisms and soft quotient functions to make comparisons and connections between the strongly soft nodec and the T_0-strongly soft nodec with support of appropriate counterexamples. Additionally, we show that the image and inverse image of a strongly soft nodec space is a strongly soft nodec space under a bijective soft quasi homeomorphism in soft generalized topological space. Further, we successfully determine the conditions under which the soft generalized topological space is a soft weak Baire space; hence a strongly soft second category.

Lemma 5.1. *Let (X, μ_1, ρ_1), (Y, μ_2, ρ_2) be soft generalized topological spaces and $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ be a soft quasi-homeomorphism. Then the following statements hold:*

(1) *If h is a bijective soft quasi-homeomorphism and (K, ρ_1) is a strongly soft nowhere dense set in X_{ρ_1}, then $h((K, \rho_1))$ is a strongly soft nowhere dense set in Y_{ρ_2}.*

(2) *If (L, ρ_2) is a strongly soft nowhere dense set in Y_{ρ_2}, then $h^{-1}((L, \rho_2))$ is a strongly soft nowhere dense set in X_{ρ_1}.*

(3) *If (K, ρ_1) is a strongly soft second category set in X_{ρ_1}, then $h((K, \rho_1))$ is a strongly soft second category set in Y_{ρ_2}.*

(4) *If h is a bijective soft quasi-homeomorphism and (L, ρ_2) is a strongly soft second category set in Y_{ρ_2}, then $h^{-1}((L, \rho_2))$ is a strongly soft second category set in X_{ρ_1}.*

Proof. (1) Let (K, ρ_1) be a strongly soft nowhere dense set in X_{ρ_1}. Let (L, ρ_2) be a non-null soft open set in Y. Since h is a bijective soft quasi-homeomorphism, then $h^{-1}((L, \rho_2))$ is a non-null soft open set in X_{ρ_1}. By the definition of strongly soft nowhere denseness, there exists a non-null soft open set (H, ρ_1) in X_{ρ_1} such that $(H, \rho_1)\widetilde{\subseteq}h^{-1}((L, \rho_2))$ and $(H, \rho_1)\widetilde{\cap}(K, \rho_1) = \Phi_{\rho_1}$. Since h is a quasi-homeomorphism, then there exists a unique soft open set (G, ρ_2) in Y_{ρ_2} such that $h^{-1}((G, \rho_2)) = (H, \rho_1)$. Since h is a surjective function, then (G, ρ_2) is a non-null soft open set in Y_{ρ_2} and $h((H, \rho_1)) = (G, \rho_2)$. Since $(H, \rho_1)\widetilde{\subseteq}h^{-1}((L, \rho_2))$, then $h((H, \rho_1))\widetilde{\subseteq}(L, \rho_2)$ and so $(G, \rho_2)\widetilde{\subseteq}(L, \rho_2)$. Since h is an injective function, then $h[(H, \rho_1)\widetilde{\cap}(K, \rho_1)] = h((H, \rho_1))\widetilde{\cap}h((K, \rho_1)) = \Phi_{\rho_2}$. Thus, $(G, \rho_2)\widetilde{\cap}h((K, \rho_1)) = \Phi_{\rho_2}$. Therefore, $h((K, \rho_1))$ is a strongly soft nowhere dense set in Y_{ρ_2}.

(2) Let (L, ρ_2) be a strongly soft nowhere dense set in Y_{ρ_2}. Let (K, ρ_1) be a non-null soft open set in X_{ρ_1}. Since h is a soft quasi-homeomorphism, then there exists a unique soft open set (G, ρ_2) in Y_{ρ_2} such that $h^{-1}((G, \rho_2)) = (K, \rho_1)$. Hence, (G, ρ_2) is a non-null soft open set in Y_{ρ_2}, and by hypothesis, there exists a non-null soft open set (H, ρ_2) in Y_{ρ_2} such that $(H, \rho_2)\widetilde{\subseteq}(G, \rho_2)$ and $(H, \rho_2)\widetilde{\cap}(L, \rho_2) = \Phi_{\rho_2}$. Since h is a soft continuous, then there exists a non-null soft open set $h^{-1}((H, \rho_2))$ in X_{ρ_1} such that $h^{-1}((H, \rho_2))\widetilde{\subseteq}(K, \rho_1)$ and $h^{-1}((H, \rho_2))\widetilde{\cap}h^{-1}((L, \rho_2)) = \Phi_{\rho_1}$. Therefore, $h^{-1}((L, \rho_2))$ is a strongly soft nowhere dense set in X_{ρ_1}.

(3) Let (K, ρ_1) be a strongly soft second category set in X_{ρ_1}. Assume that $h((K, \rho_1))$ is a strongly soft first category set in Y_{ρ_2}. Then, $h((K, \rho_1)) = \widetilde{\bigcup}_{n\in\mathbb{N}}(H_n, \rho_2)$, where each (H_n, ρ_2) is a strongly soft nowhere dense set in Y_{ρ_2} for all $n \in \mathbb{N}$. By part (2), $h^{-1}((H_n, \rho_2))$ is a strongly soft nowhere dense set in X_{ρ_1} for all $n \in \mathbb{N}$. Now, $h^{-1}(h((K, \rho_1))) = h^{-1}(\widetilde{\bigcup}_{n\in\mathbb{N}}(H_n, \rho_2)) = \widetilde{\bigcup}_{n\in\mathbb{N}}h^{-1}((H_n, \rho_2))$ which implies that $h^{-1}(h((K, \rho_1)))$ is a strongly soft first category in X_{ρ_1}; hence (K, ρ_1) is a strongly soft first category in X_{ρ_1} since the subset of a strongly soft nowhere dense set is a strongly soft nowhere dense which contradicts the fact that (K, ρ_1) is a strongly soft second category set in X_{ρ_1}. Therefore, $h((K, \rho_1))$ is a strongly soft second category set in Y_{ρ_2}.

(4) Let (L, ρ_2) be a strongly soft second category set in Y_{ρ_2} and let h be a bijective function. Assume that $h^{-1}((L, \rho_2))$ is a strongly soft first category in X_{ρ_1}. Then, $h^{-1}((L, \rho_2)) = \widetilde{\bigcup}_{n\in\mathbb{N}}(H_n, \rho_1)$, where (H_n, ρ_1) is a strongly soft nowhere dense set in X_{ρ_1} for all $n \in \mathbb{N}$. By part (1), $h((H_n, \rho_1))$ is s strongly soft nowhere dense set in Y_{ρ_2} for each $n \in \mathbb{N}$. Now, $h(h^{-1}((L, \rho_2))) = h(\widetilde{\bigcup}_{n\in\mathbb{N}}(H_n, \rho_1))$.

Since h is a bijective soft function, then $(L, \rho_2) = \widetilde{\bigcup}_{n\in\mathbb{N}} h((H_n, \rho_1))$ which implies that (L, ρ_2) is a strongly soft first category set in Y_{ρ_2}, which contradicts the fact that (L, ρ_2) is a strongly soft second category set in Y_{ρ_2}. Therefore, $h^{-1}((L, \rho_2))$ is a strongly soft second category set in X_{ρ_1}. □

The following theorem shows that the image and the inverse-image of a strongly soft nodec space is strongly soft nodec under a bijective soft quasi homeomorphism between soft generalized topological spaces.

Theorem 5.1. *Let (X, μ_1, ρ_1), (Y, μ_2, ρ_2) be soft generalized topological spaces and $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ be a bijective soft quasi-homeomorphism. Then, X_{ρ_1} is a strongly soft nodec space if and only if Y is a strongly soft nodec space.*

Proof. Suppose that X_{ρ_1} is a strongly soft nodec space. Let (L, ρ_2) be a non-null strongly soft nowhere dense set in Y_{ρ_2}. By hypothesis and Lemma 5.1, $h^{-1}((L, \rho_2))$ is a strongly soft nowhere dense set in X_{ρ_1}. Then, $h^{-1}((L, \rho_2))$ is a soft closed set in X_{ρ_1}. Since h is a soft quasi-homeomorphism, there exists a unique soft closed set (H, ρ_2) in Y_{ρ_2} such that $h^{-1}((L, \rho_2)) = h^{-1}((H, \rho_2))$. Since h is a surjective soft function, then (L, ρ_2) is a soft closed set in Y_{ρ_2}. Therefore, Y_{ρ_2} is a strongly soft nodec space.

Conversely, assume that Y is a strongly soft nodec space. Let (K, ρ_1) be a strongly soft nowhere dense set in X_{ρ_1}. By hypothesis and Lemma 5.1, $h((K, \rho_1))$ is a strongly soft nowhere dense set in Y_{ρ_2}. Then, $h((K, \rho_1))$ is a soft closed set in Y_{ρ_2}. Since h is a bijective soft continuous function, (K, ρ_1) is a soft closed set in X_{ρ_1}. Therefore, X_{ρ_1} is a soft strongly nodec space. □

Theorem 5.2. *Let (X, μ, ρ) be a strong soft generalized topology space and $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ be a surjective soft quotient function. If X_ρ is a strongly soft nodec space, then X_ρ is a T_0-strongly soft nodec space.*

Proof. Let X be a strongly soft nodec space. Let (L, ρ) be a strongly soft nowhere dense set in $T_0(X_\rho)$. By Lemma 5.1, $h^{-1}((L, \rho))$ is a strongly soft nowhere dense set in X_ρ; hence $h^{-1}((L, \rho))$ is a soft closed set in X_ρ. By Lemma 4.1, $h(h^{-1}((L, \rho)))$ is a soft closed set in $T_0(X_\rho)$. Then, (L, ρ) a soft closed set in $T_0(X_\rho)$ since h is a surjective function. Thus, $T_0(X_\rho)$ is a strongly soft nodec space. Therefore, X_ρ is a T_0-strongly soft nodec space. □

Theorem 5.3. *Let (X, μ, ρ) be a strong soft generalized topology space and $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ be a surjective soft quotient function. If π is an injective function and X_ρ is a T_0-strongly soft nodec space, then X_ρ is a strongly soft nodec space.*

Proof. Let (K, ρ) be a non-null strongly soft nowhere dense set in X_ρ. By Lemma 5.1, $\pi((K, \rho))$ is a non-null strongly soft nowhere dense set in $T_0(X_\rho)$; hence by hypothesis, $\pi((K, \rho))$ is a soft closed set in $T_0(X_\rho)$. By Lemma 4.1, π is a soft continuous function which implies that $\pi^{-1}(\pi((K, \rho)))$ is a soft closed set in X_ρ. Since π is an injective function, then (K, ρ) is a soft closed set in X_ρ. Therefore, X_ρ is a strongly soft nodec space. □

Definition 5.1. *Let (X, μ, ρ) be a soft generalized topological space. A space X_ρ is said to be a soft weak Baire space if every non-null soft open set in X_ρ is a strongly soft second category in X_ρ.*

The next theorems indicates basic relationships between the concepts introduced above.

Theorem 5.4. *Let (X, μ, ρ) be a soft weak Baire soft generalized topological space, then X_ρ is a strongly soft second category.*

Proof. Let (X, μ, ρ) be a soft weak Baire soft generalized topological space. Suppose that X_ρ is a strongly soft first category. Then, $X_\rho = \widetilde{\bigcup}_{n\in\mathbb{N}}(H_n, \rho)$ such that (H_n, ρ) is a strongly soft nowhere dense set in X_ρ for all $n \in \mathbb{N}$. Clearly, (H_n, ρ) is a soft nowhere dense set in X_ρ for all $n \in \mathbb{N}$. Hence, $cl_{g\mu}((H_n, \rho))$ has no soft interior soft points so any non-null soft open set in X_ρ must intersect $(K_n, \rho) = X_\rho \setminus cl_{g\mu}((H_n, \rho))$ for all $n \in \mathbb{N}$. If $\{(K_n, \rho)\}_{n\in\mathbb{N}}$ is a family of non-null soft open soft dense sets in X_ρ, then $cl_{g\mu}((K_n, \rho)) = X$ for all $n \in \mathbb{N}$ which implies that $cl_{g\mu}((K_n, \rho))$ is a strongly soft first category set in X_ρ. By Lemma 3.1 and $(K_n, \rho)\widetilde{\subseteq} cl_{g\mu}((K_n, \rho))$, (K_n, ρ) is a strongly soft first category set in X_ρ for all $n \in \mathbb{N}$. Thus, a non-null soft open set (K_n, ρ) is not strongly soft second category in X_ρ for all $n \in \mathbb{N}$ which is a contradiction to that X_ρ is a soft weak Baire space. Therefore, X_ρ is a strongly soft second category. □

Theorem 5.5. *Let (X, μ_1, ρ_1), (Y, μ_2, ρ_2) be soft generalized topological spaces and $h : (X, \mu_1, \rho_1) \to (Y, \mu_2, \rho_2)$ be a surjective soft quasi-homeomorphism. Then the following statements hold:*

1. *If X_ρ is a soft weak Baire space, then Y is a strongly soft second category.*
2. *If h is an injective function and Y is a soft weak Baire space, then X_ρ is a strongly soft second category.*

Proof. (1) Let (X, μ_1, ρ_1) be a soft weak Baire space. Let (L, ρ_2) be a non-null soft open set in Y_{ρ_2}. Since h is a soft continuous soft function, then $h^{-1}((L, \rho_2))$ is a non-null soft open set in X_ρ. By hypothesis, $h^{-1}((L, \rho_2))$ is a strongly soft second category in X_ρ. By Lemma 5.1, $h(h^{-1}((L, \rho_2)))$ is a strongly soft second category set in Y_{ρ_2}. Since h is a surjective soft function, then $h(h^{-1}((L, \rho_2))) = (L, \rho_2)$ is a strongly soft second category set in Y_{ρ_2}. Hence, Y is a soft weak Baire space. By Theorem 5.4, Y is a strongly soft second category.

(2) Let (Y, μ_2, ρ_2) be a soft weak Baire space. Let (K, ρ) be a non-null soft open set in X_ρ. Since h is a soft quasi-homeomorphism soft function, there exists a non-null soft open set (G, ρ_2) in Y_{ρ_2} such that $(K, \rho) = h^{-1}((G, \rho_2))$. Since h is a surjective function, then $h((K, \rho)) = (G, \rho_2)$. By hypothesis, (G, ρ_2) is a strongly soft second category. Hence, $h((K, \rho))$ is strongly soft second category in Y_{ρ_2}. By Lemma 5.1, $h^{-1}(h((K, \rho)))$ is a strongly soft second category in X_ρ. Since h is an injective function, then $h^{-1}(h((K, \rho))) = (K, \rho)$ is strongly soft second category in X_ρ. Hence, X_ρ is a soft weak Baire space. By Theorem 5.4, X_ρ is strongly soft second category. □

Theorem 5.6. *Let (X, μ, ρ) be a soft generalized topological space and $\pi : (X, \mu, \rho) \to (T_0(X), \mu_\pi, \rho)$ be a surjective soft quotient function. Then, the following statements hold:*

1. *If X_ρ is a soft weak Baire space, then $T_0(X_\rho)$ is a soft weak Baire space.*

(2) *If $T_0(X_\rho)$ is a soft weak Baire space and π is an injective function, then X_ρ is a soft weak Baire space.*

Proof. (1) Let (X, μ_1, ρ_1) be a soft weak Baire space. Let (L, ρ_2) be a non-null soft open set in $T_0(X_\rho)$. Since h is a soft continuous function, then $h^{-1}((L, \rho_2))$ is a non-null soft open set in X_ρ. By hypothesis, $h^{-1}((L, \rho_2))$ is a strongly soft second category in X_ρ. By Lemma 5.1, $h(h^{-1}((L, \rho_2)))$ is a strongly soft second category set in $T_0(X_\rho)$. Since h is a surjective soft function, then $h(h^{-1}((L, \rho_2))) = (L, \rho_2)$ is a strongly soft second category set in $T_0(X_\rho)$. Hence, $T_0(X_\rho)$ is a soft weak Baire space.

(2) Let $(T_0(X), \mu_2, \rho_2)$ be a soft weak Baire space. Let (K, ρ) be a non-null soft open set in X_ρ. By Lemma 4.1, h is a soft open soft quotient function, then $h((G, \rho_2))$ is a non-null soft open set in $T_0(X_\rho)$. By hypothesis, $h((G, \rho_2))$ is a strongly soft second category in $T_0(X_\rho)$. By Lemma 5.1, $h^{-1}(h((G, \rho_2)))$ is a strongly soft second category in X_ρ. Since h is an injective function, then $h^{-1}(h((G, \rho_2))) = (G, \rho_2)$ is a strongly soft second category in X_ρ. Hence, X_ρ is a soft weak Baire space. □

6. Conclusion

In recent years, several types of soft generalized topological properties have been presented, such as soft basis, subspace soft generalized topology, soft interior, soft closure, soft neighborhood, soft limit point, soft boundary, soft exterior, and soft continuity of soft functions. We have continued working in the same direction by presenting definitions of a strongly soft nowhere dense, a strongly soft dense, a strongly soft codense, a strongly soft first category, and a strongly soft second category set in a soft generalized topological space. By following the idea of the definition of a strongly soft nowhere dense set, we define a new space named strongly soft nodec generalized topological space along with their basic properties and show that if the product of two soft generalized topological spaces is a strongly soft nodec space, then each one is a strongly soft nodec space. Then, we extend these notions to T_0-strongly soft nodec generalized topological space by using the soft quotient function and discussing its main properties.

To ensure our results, some counterexamples have been studied. Moreover, we use soft quasi-homeomorphism and soft quotient functions to make comparisons and connections between these spaces with the support of appropriate counterexamples. Then, we show that an image and inverse-image of a strongly soft nodec space is strongly soft nodec under a bijective soft quasi homeomorphism in soft generalized topological space. We finalize this work by successfully determining the conditions under which the soft generalized topological space is a soft weak Baire space; hence, a strongly soft second category.

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