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Estimate on Logarithmic Co-Efficients of Sokol-Stankiewicz Type Star-Like Function Associated with Caratheodory Functions

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Abstract. The fundamental focus of researching coefficient problems for various families of univalent functions involves characterizing the coefficients of functions within a particular family based on the coefficients of Caratheodory functions. Consequently, by employing known inequalities for the class of Caratheodory functions, coefficient functionals can be scrutinized. This study will tackle several coefficient problems by applying the methodology to the aforementioned family of functions. Our investigation centers on the family of Sokol-Stankiewicz star-like functions which is defined in the open unit disk $\mathbb{D}$. We explore the bounds of certain initial coefficients, including the Fekete-Szego inequality and other results concerning logarithmic coefficients for functions within this class.

1. Introduction

Logarithmic functions find application in various branches of mathematics and other scientific disciplines. In order to provide a comprehensive grasp of the principal outcomes detailed in this paper, we elucidate the fundamental terminology employed throughout our key findings, accompanied by preliminary definitions and relevant results. We denote by $\mathcal{A}$ the class of analytic, holomorphic normalized functions $f : \mathbb{D} \longrightarrow \mathbb{C}$ defined in open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$, which satisfy the following normalization conditions

$$f(0) = 0 = f'(0) - 1.$$

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Thus, each $f \in \mathcal{A}$ has the following series of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{D}). \tag{1.1}$$

Moreover, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ of functions which are univalent in $\mathbb{D}$. For two functions $g_1, g_2 \in \mathcal{A}$, we say that the function g_1 is subordinate to the function g_2 (written as $g_1 \prec g_2$) if there exists an analytic function w with the property

$$|w(z)| \leq 1 \text{ and } w(0) = 0$$

such that

$$g_1(z) = g_2(w(z)) \qquad (z \in \mathbb{D}).$$

In particular, if g_2 is univalent in $\mathcal{S}$, then we have the following equivalence

$$g_1(z) \prec g_2(z) \iff g_1(0) = g_2(0) \text{ and } g_1(|z| < 1) \subset g_2(|z| < 1). \tag{1.2}$$

In 1992, Ma and Minda [1] introduce the $\mathcal{S}^*(\delta)$ as follows

$$\mathcal{S}^*(\delta) = \left\{ f \in \mathcal{A} : \left(\frac{zf'(z)}{f(z)} \right) \prec \delta, \quad (z \in \mathbb{D}) \right\}. \tag{1.3}$$

The function δ is expected to be analytic within a region $\mathbb{D}$, where its real part is positive. In simpler terms, $\mathcal{S}^*(\delta)$ is imagined to have a symmetric shape like a star, but it's confined within a certain area that is $\delta(0) = 1$ and $\delta'(0) > 0$. Additionally, they explored several beneficial geometric characteristics like expansion, deformation and coverage outcomes. This was achieved by taking

$$\delta(z) = (1 + z)(1 - z)^{-1}.$$

Specifically, we observe that the function class $\mathcal{S}^*(\delta)$ resembles the well-established class of starlike functions. Depending on the particular function δ chosen, we encounter the following distinct function classes.

1. If we let

   $$\delta(z) = 1 + \sin(z),$$

   then we obtain the class

   $$\mathcal{S}^*_{sin} = \mathcal{S}^*\left(1 + \sin(z)\right),$$

   the class of starlike functions which maps to an eight-shaped figure within the open unit disk $\mathbb{D}$. This distinctive shape emerges when considering their image under the unit disk [2].
2. If we put the class of functions

   $$\delta(z) = 1 + z - \frac{1}{3}z^3,$$

   then we get the class

   $$\mathcal{S}^*_{nep} = \mathcal{S}^*\left(1 + z - \frac{1}{3}z^3\right),$$

the class of starlike functions which exhibits a unique feature when visualized within the open unit disk $\mathbb{D}$. It forms a nephroid-shaped region. This distinct shape becomes apparent when examining their representation under the unit disk [3].

3. If we opt the class of functions

$$\delta(z) = \sqrt{1+z},$$

then we acquire the class of functions

$$\mathcal{S}^*_{\mathcal{L}} = \mathcal{S}^*\left(\sqrt{1+z}\right),$$

the function $\delta(z) = \sqrt{1+z}$ transforms the domain $\mathbb{D}$ onto the image domain bounded by the right half of the Bernoulli lemniscate represented by $|w^2-1|$ [4].

4. If we opt the class of functions

$$\delta(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2,$$

then we get the class

$$\mathcal{S}^*_{Card} = \mathcal{S}^*\left(1 + \frac{4}{3}z + \frac{2}{3}z^2\right),$$

which is the class of starlike functions whose image under open unit is cardioid shaped given by $(9x^2+9y^2-18x+5)^2 - 16(9x^2+9y^2-6x+1) = 0$ and this was introduced by Sharma et al [5].

5. If we let

$$\delta(z) = e^z,$$

then we derive the class of functions

$$\mathcal{S}^*_{Exp} = \mathcal{S}^*\left(e^z\right).$$

This is the class of starlike functions associated with exponential function and this was introduced and studied by Mendiratta et al [6].

6. If we opt the class of functions

$$\delta(z) = \left(\sqrt{1+z}\right) + z,$$

then we procure the class

$$\mathcal{S}^*_{Cre} = \mathcal{S}^*\left(\sqrt{1+z} + z\right),$$

which is the class of starlike functions associated with the crescent-shaped region as discussed in [7].

7. If we let

$$\delta(z) = 1 + \frac{4}{5}z + \frac{1}{5}z^4,$$

then we obtain the class

$$\mathcal{S}^*_{Three\ leaf} = \mathcal{S}^*\left(1 + \frac{4}{5}z + \frac{1}{5}z^4\right),$$

which is the class of starlike functions linked to the defined geometric area called three leaf shaped domain and studied in [44].

8. If we assign the class of functions

$$\delta(z) = 1 + \frac{5}{6}z + \frac{1}{6}z^5,$$

then we obtain the class

$$\mathcal{S}^*_{Four\ leaf} = \mathcal{S}^*\left(1 + \frac{5}{6}z + \frac{1}{6}z^5\right),$$

which in the class of starlike functions linked with outlined four - shaped region which was introduced and studied is [51].

9. If we assign the class of functions

$$\delta(z) = 1 + \sinh^{-1}(z),$$

then we get the class

$$\mathcal{S}^*_{Petal} = \mathcal{S}^*\left(1 + \sinh^{-1}(z)\right),$$

which is the class of starlike functions associated with the petal - shaped region as discussed in [45].

10. Moreover, if we take

$$\delta(z) = \cosh(z),$$

then we derive the class

$$\mathcal{S}^*_{cosh} = \mathcal{S}^*\left(\cosh(z)\right),$$

whose image is bounded by the cosine of the functions which were contributed by A. Alotaibi, M. Arif, M. A. Alghamdi, and S. Hussain [49] .

11. Furthermore if we pick

$$\delta(z) = 1 + \tanh(z),$$

then we get the class of functions

$$\mathcal{S}^*_{tanh} = \mathcal{S}^*\left(1 + \tanh(z)\right),$$

this can be studied in [48] and starlike function associated with tan hyperbolic function.

12. If we put the class of functions

$$\delta(z) = \cos(z),$$

then we obtain the class

$$\mathcal{S}^*_{cos} = \mathcal{S}^*\left(\cos(z)\right).$$

This is the class of starlike functions associated with the cosine function as discussed in [47].

13. If we assign the class of functions

$$\delta(z) = \frac{1 + (1-2\varphi)z}{1-z}$$

with $0 \leq \varphi < 1$, we get the class

$$\mathcal{S}^* = \mathcal{S}^*\left(\frac{1 + (1-2\varphi)z}{1-z}\right)$$

of starlike functions of order φ [14].

The class $\mathcal{S}^*$ has been extensively explored by various researchers. They have substituted f with different sequences such as Fibonacci numbers, Bell numbers, shell-like curves, conic domains and a modified sigmoid function [8–11]. Furthermore, these researchers have established additional subclasses within the broader class of starlike functions. The two most important and extensively studied families of univalent functions are the class $\mathcal{S}^*(\delta(z))$ which represents starlike functions with respect to symmetric points of order $\delta(z)$ $(0 \leq \delta < 1)$ analytically defined by

$$\mathcal{S}^*(\delta(z)) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \delta, \quad (z \in \mathbb{D}) \right\}. \tag{1.4}$$

The class $\mathcal{K}(\delta(z)) \subset \mathcal{S}^*$ of convex functions of order δ, $(0 \leq \delta < 1)$ is defined by

$$\mathcal{K}(\delta(z)) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \delta, \quad (z \in \mathbb{D}) \right\}. \tag{1.5}$$

The class $\mathcal{V}(\delta(z)) \subset \mathcal{S}^*$ of closed - to - convex functions of order δ, $(0 \leq \delta < 1)$ is defined by

$$\mathcal{V}(\delta(z)) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{g(z)}\right) > \delta, \quad (z \in \mathbb{D}) \right\} \tag{1.6}$$

where $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ belongs to starlike functions and so on. Thus we get, subclasses of starlike varieties, convex types, close-to - convex forms, functions with bounded turning points and more.

Consider an analytic and univalent function $\mathcal{P}$ within domain $\mathbb{D}$, with a positive real component, satisfying $p(0) = 0$, $p'(0) = 1$, $Re(p(z)) > 0$. $\mathcal{P}$ transforms the unit disk $\mathbb{D}$ into a space of star-like functions, respecting symmetric points along the real axis. Now, we delve into the Taylor series expansion

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n, \qquad |p_n| \leq 2. \tag{1.7}$$

In this paper, we assume that the function p satisfies the aforementioned conditions unless stated otherwise. We denote the classes of functions by $\mathcal{S}^*(p)$ and $\mathcal{K}(p)$ where all coefficients are real and $p > 0$.

By we represent the following classes of functions

$$\mathcal{S}^*(p) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) \prec p(z), \quad (z \in \mathbb{D}) \right\} \tag{1.8}$$

and

$$\mathcal{K}(p)=\left\{f\in\mathcal{A}:\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)\prec p(z),\quad (z\in\mathbb{D})\right\}. \tag{1.9}$$

The categories $\mathcal{S}^*(p),\mathcal{K}(p)$ expand upon the traditional collections of star shaped and convex functions, as detailed in the work of Ma and Minda [1]. These functions act as the fundamental basis from which subsequent subclasses inherit their characteristics, all originating from the Caratheodory function category $\mathcal{P}$. Sokól and Stankiewicz [12] introduced a group designated as $\mathcal{SL}^*$, encompassing normalized analytic functions f in $\mathbb{D}$ that adhere to specific criteria.

$$\left|\left[\frac{zf'(z)}{f(z)}\right]^2-1\right|<1.$$

The category is commonly known as Sokól-Stankiewicz starlike functions. Moreover, Raza and Malik [13] have established the upper limit of the third Hankel determinant $H_3(1)$ for the class $\mathcal{SL}^*$. Furthermore, Sahoo and Patel [14] obtained some upper bound to the second Hankel determinant for the class

$$\tilde{\mathcal{R}}=\left\{f\in\mathcal{A}:|f'(z)^2-1|<1,\quad (z\in\mathbb{D})\right\}. \tag{1.10}$$

Inspired by the aforementioned research findings, as presented by previous scholars, Trailokya Panigrahi and Janusz Sokól [18] introduced the following subclass of analytic functions

Definition 1.1. *A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{AR}^*_\delta$, $0\le\delta\le1$, if it satisfies the condition*

$$\left|\left[\frac{zf'(z)}{(1-\delta)f(z)+\delta z}\right]^2-1\right|<1,\quad (z\in\mathbb{D}). \tag{1.11}$$

The family $\mathcal{A}(\delta)$ of new subclasses in analytical functions of type δ; $0\le\delta\le1$ provides a transition from the class of starlike functions to the class of functions of bounded boundary rotation. To see this, we note that for $\delta=0$, we have $\mathcal{A}(\delta)\equiv\mathcal{S}^*(0)\equiv\mathcal{S}^*$ the class of starlike functions $f>\mathcal{A}$, so that $\Re\left(\frac{zf'}{f}\right)>0$ in $\mathbb{D}$. For $\delta=1$, we get the family of functions $\tilde{\mathcal{R}}$ of functions $f>\mathcal{A}$, of bounded boundary rotation so that $\Re(f')>0$ in $\mathbb{D}$. (For further details. [19]) Note that for $\delta=0$, the class $\mathcal{AR}^*_0$, reduces to the class $\mathcal{SL}^*$, studied by Raza and Malik [13] and while $\delta=1$, the class $\mathcal{AR}^*_1$, reduces to $\tilde{\mathcal{R}}$ studied by Sahoo and Patel [14]. In terms of subordination, relation (1.11) can be written as

$$\mathcal{A}(\delta)=\frac{zf'(z)}{(1-\delta)f(z)+\delta z}\prec p(z),\quad (z\in\mathbb{D}). \tag{1.12}$$

In this research paper, we investigate the bounds of some initial coefficients and estimates of Fekete-Szego functionals which are extensively studied structured matrices with applications in various fields such as statistics, image processing, quantum mechanics and more.

For each function $f(z)\in\mathcal{S}$, we can define logarithmic function $\mathcal{A}_a(z)$ as follows

$$\mathcal{A}_a(z)=\log\left(\frac{f(z)}{z}\right)=2\sum_{n=1}^{\infty}s_nz^n,\quad (z\in\mathbb{D}). \tag{1.13}$$

The logarithmic coefficients s_n play a central role in the theory of univalent functions [42, 45–47]. A very few exact upper bounds for s_n seem to have been established. The significance of this problem in the context of Bieberbach conjecture was pointed by Milin [23] in his conjecture. Milin [23] conjectured that for $f \in \mathcal{S}$ and $n \geq 2$.

$$\sum_{m=1}^{n}\sum_{k=1}^{m}\left(k|s_k|^2 - \frac{1}{k}\right) \leq 0 \quad , \tag{1.14}$$

which led De Branges, by proving this conjecture, to the proof of Bieberbach conjecture [24]. For the Koebe function $k(z) = \dfrac{z}{(1-z)^2}$, the logarithmic coefficients are $s_n = \frac{1}{n}$. Since the Koebe function k plays the role of extremal function for most of the extremal problems in the class $\mathcal{S}$, it is expected that $s_n \leq \frac{1}{n}$ holds for functions in $\mathcal{S}$. But this is not true in general, even in order of magnitude. Indeed, there exists a bounded function f in the class $\mathcal{S}$ with logarithmic coefficients $s_n \neq O(n^{-0.83})$. By differentiating (1.13) and on equating coefficients we obtain

$$s_1 = \frac{1}{2}a_2 , \tag{1.15}$$

$$s_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right) , \tag{1.16}$$

$$s_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right) , \tag{1.17}$$

$$s_4 = \frac{1}{2}\left(a_5 - a_2a_4 + a_2^2a_3 - \frac{1}{2}a_3^2 - \frac{1}{4}a_2^4\right) , \tag{1.18}$$

$$s_5 = \frac{1}{2}\left(a_6 - a_2a_5 - a_3a_4 + a_2a_3^2 + a_2^2a_4 - a_2^3a_3 + \frac{1}{5}a_2^5\right) . \tag{1.19}$$

If $f \in \mathcal{S}$, it is easy to see that $|s_1| \leq 1$, because $|a_2| \leq 2$. Using the Fekete-Szego inequality for functions in $\mathcal{S}$ in (1.16), we obtain the sharp estimate

$$|s_2| \leq \frac{1}{2}(1 + 2e^{-2}) = 0.635\cdots .$$

For $n \geq 3$, the problem seems much harder, and no significant bound for $|s_n|$ when $f \in \mathcal{S}$ appear to be known. In 2017, Ali and Allu [25] obtained the bounds for the initial three logarithmic coefficients for a subclass of $f \in \mathcal{S}$. The problem of computing the bounds for the logarithmic coefficients is also considered in [6, 18, 21] for several subclasses of close to convex functions. In 2021, Zaprawa [26] obtained the sharp bounds of the initial logarithmic coefficients s_n for functions in the classes $\mathcal{S}^*$ and $\mathcal{K}$.

We recall the definition of the Hankel determinant with k as a parameter and $n \in \mathbb{N} := \{1, 2, 3, ...\}$.

$$H_{k,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+k-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+k-1} & a_{n+k} & \cdots & a_{n+2k-2} \end{vmatrix} \quad (n, k \in \mathbb{N} = 1, 2, 3, ...). \tag{1.20}$$

For example,

$$H_{2,1}(f)=\begin{vmatrix}a_1 & a_2\\ a_2 & a_3\end{vmatrix},\quad H_{2,2}(f)=\begin{vmatrix}a_2 & a_3\\ a_3 & a_4\end{vmatrix},\quad H_{3,1}(f)=\begin{vmatrix}a_1 & a_2 & a_3\\ a_2 & a_3 & a_4\\ a_3 & a_4 & a_5\end{vmatrix}. \tag{1.21}$$

The evaluation of the upper bound of $H_{k,n}(f)$ across different subfamilies of $\mathbb{A}$ is an interesting area of research within Geometric Function Theory in Complex Analysis. Noonan and Thomas [27], as well as Noor [28], examined the growth rate of $H_{k,n}(f)$ as $n \longrightarrow \infty$ for fixed k and n, focusing on various subfamilies of the class of univalent function $\mathcal{S}^*$. The Hankel determinant $H_{2,1}(f) = a_3 - a_2^2$ and $H_{2,2}(f) = a_2a_4 - a_3^2$ are known as the Fekete-Szego functional and second Hankel determinant respectively. The functional $H_{2,1}(f)$ is further generalized as $a_3 - \mu a_2^2$ for some real or complex parameter μ. Various researchers have obtained upper bounds of $H_{2,1}(f)$ for different subfamilies of class $\mathcal{S}^*$ (refer to [29–31]). Recently, Srivastava et al. [29] derived bounds for the second Hankel determinant for q-starlike and q-convex functions. Additionally, several studies have focused on obtaining bounds for initial coefficients, exploring the Fekete-Szegö inequality, and estimating Hankel determinants of different orders for various subclasses of univalent and bi-univalent functions [30–33].

Building upon the previous concepts, we suggest investigating the Hankel determinant, where its elements comprise of logarithmic coefficients of $\mathcal{S}$. This exploration could unveil fascinating insights into the interplay between logarithmic coefficients and Hankel determinants given by

$$H_{k,n}(f)=\begin{vmatrix} s_n & s_{n+1} & \cdots & s_{n+k-1}\\ s_{n+1} & s_{n+2} & \cdots & s_{n+k}\\ \vdots & \vdots & \cdots & \vdots\\ s_{n+k-1} & s_{n+k} & \cdots & s_{n+2k-2}\end{vmatrix}\quad (n,k\in\mathbb{N}=1,2,3,...). \tag{1.22}$$

The principal goal of this paper is to ascertain upper bounds for $H_{2,1}(f) = a_3 - a_2^2$ over the category of Sokol-Stankiewicz starlike functions linked with Caratheodory functions. This pursuit focuses on delineating boundaries for the behavior and characteristics of Sokol-Stankiewicz starlike functions within the framework of caratheodory functions.

2. Definitions and Preliminaries

Lemma 2.1. *[40] Let $p \in \mathcal{P}$ be given by* (1.7)*, then*

$$\left|p_k\right| \leq 2 \; for \; k \geq 1\,, \tag{2.1}$$

$$\left|p_{n+k} - \mu p_n p_k\right| < 2 \; for \; 0 \leq \mu \leq 1\,, \tag{2.2}$$

$$\left|p_n p_k - p_n p_l\right| \leq 4 \; for \; m+k = k+l, \tag{2.3}$$

$$\left|p_2 - \frac{1}{2}p_1^2\right| \leq 2 - \frac{1}{2}\left|p_1\right|^2 . \tag{2.4}$$

Lemma 2.2. *[19, 39] Let $p \in \mathcal{P}$ be given by (1.7), for complex number μ, we have*

$$\left|c_{n+k} - \mu c_n c_k\right| \leq 2\ max\left\{1, \left|2\mu - 1\right|\right\} = \begin{cases} 2, & \text{for } 0 \leq \mu \leq 1; \\ 2\left|2\mu - 1\right|, & \text{otherwise.} \end{cases} \tag{2.5}$$

Also, if $k_1 \in [0, 1]$ and $k_1(2k_1 - 1) \leq k_2 \leq k_1$, we get

$$\left|p_3 - 2k_1 p_1 p_2 + k_2 p_1^3\right| \leq 2. \tag{2.6}$$

Lemma 2.3. *[40] Let $p \in \mathcal{P}$ be given by (1.7), then*

$$\left|\xi p_1^3 - \varpi p_1 p_2 + \sigma p_3\right| \leq 2\left(|\xi| + |\varpi - 2\xi| + |\xi - \varpi + \sigma|\right) \tag{2.7}$$

Lemma 2.4. *[41] Let α, β, r and b satisfy the inequalities $0 < \alpha < 1$, $0 < b < 1$ and*

$$8b(1-b)\left[(\alpha\beta - 2r)^2 + (\alpha(b+\alpha) - \beta)^2\right] + \alpha(1-\alpha)(\beta - 2b\alpha)^2 \leq 4\alpha^2(1-\alpha)^2 b(1-b). \tag{2.8}$$

If $p \in \mathcal{P}$, be given by (1.7), then

$$\left|rp_1^4 + bp_2^2 + 2\alpha p_1 p_3 - \frac{3}{2}\beta p_1^2 p_2 - p_4\right| \leq 2. \tag{2.9}$$

Lemma 2.5. *[42] If $p \in \mathcal{P}$ be given by (1.7), then*

$$\left|p_1^5 + 3p_1 p_2^2 + 3p_1^2 p_3 - 4p_1^3 p_2 - 2p_1 p_4 - 2p_2 p_3 + p_5\right| \leq 2. \tag{2.10}$$

$$\left|p_1^6 + 6p_1^2 p_2^2 + 4p_1^3 p_3 + 2p_1 p_5 + 2p_2 p_4 + p_3^2 - p_2^3 - 5p_1^4 p_2 - 3p_1^2 p_4 - 6p_1 p_2 p_3 - p_6\right| \leq 2. \tag{2.11}$$

Lemma 2.6. *[43] Let $p \in \mathcal{P}$ be given by (1.7), then for some complex valued x with $|x| \leq 1$, some complex valued ϱ with $|\varrho| \leq 1$ and some complex valued ψ with $|\psi| \leq 1$, we have*

$$2p_2 = p_1^2 + x\left(4 - p_1^2\right), \tag{2.12}$$

$$4p_3 = p_1^3 + 2\left(4 - p_1^2\right)p_1 x - p_1\left(4 - p_1^2\right)x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)\varrho, \tag{2.13}$$

$$\begin{aligned} 8p_4 &= p_1^4 + \left(4 - p_1^2\right)x\left[p_1^2\left(x^2 - 3x + 3\right) + 4x\right] \\ &- 4\left(4 - p_1^2\right)\left(1 - |x|^2\right)\left[p\left(x - 1\right)\varrho + \bar{x}\varrho^2 - \left(1 - |\varrho|^2\psi\right)\right]. \end{aligned} \tag{2.14}$$

In this section, we start with finding the bounds of the first few initial logarithmic coefficients for the category of Sokol-Stankiewicz starlike functions linked with caratheodory functions.

3. Coefficient Estimates for Logarithmic $\mathcal{A}(\delta)$

Theorem 3.1. *If $f \in \mathcal{A}(\delta)$; $(0 \leq \delta \leq 1)$, then we have the sharp bounds*

$$
\begin{aligned}
|s_1| &\leq \frac{1}{(1+\delta)}, \\
|s_2| &\leq \frac{1}{(2+\delta)}, \\
|s_3| &\leq \frac{1}{(3+\delta)}, \\
|s_4| &\leq \frac{1}{(4+\delta)}, \\
|s_5| &\leq \frac{1}{(5+\delta)}.
\end{aligned}
$$

Proof. First note that by equating the corresponding coefficients in the equation

$$\frac{zf'(z)}{(1-\delta)f(z)+\delta z} = p(z)\,, \tag{3.1}$$

we get

$$a_2 = \frac{p_1}{\delta+1}\,, \tag{3.2}$$

$$a_3 = \frac{p_1^2(1-\delta)}{(1+\delta)(\delta+2)} + \frac{p_2}{\delta+2}\,, \tag{3.3}$$

$$a_4 = \frac{p_1^3(1-\delta)^2}{(\delta+1)(\delta+2)(\delta+3)} + \frac{p_1p_2(1-\delta)(3+2\delta)}{(\delta+1)(\delta+2)(\delta+3)} + \frac{p_3}{\delta+3}\,, \tag{3.4}$$

$$
\begin{aligned}
a_5 &= \frac{p_1^4(1-\delta)^3}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)} + \frac{p_1^2p_2(1-\delta)^2(6+3\delta)}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)} \\
&+ \frac{p_1p_3(1-\delta)}{(\delta+3)(\delta+4)} + \frac{p_2^2(1-\delta)}{(\delta+2)(\delta+4)} + \frac{p_1p_3(1-\delta)}{(\delta+1)(\delta+4)} + \frac{p_4}{\delta+4},
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
a_6 &= \frac{p_1^5(1-\delta)^4}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)(\delta+5)} + \frac{p_1^3p_2(1-\delta)^3(10+4\delta)}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)(\delta+5)} \\
&+ \frac{p_1^2p_3(1-\delta)^2(3\delta^2+15\delta+20)}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)(\delta+5)} + \frac{p_1p_2^2(1-\delta)^2(3\delta^2+15\delta+15)}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)(\delta+5)} \\
&+ \frac{p_1p_4(1-\delta)(5+2\delta)}{(\delta+1)(\delta+4)(\delta+5)} + \frac{p_2p_3(1-\delta)(5+2\delta)}{(\delta+2)(\delta+3)(\delta+5)} + \frac{p_5}{(\delta+5)}.
\end{aligned} \tag{3.6}
$$

By making use of (3.2)-(3.6) in (1.16)-(1.19), we get

$$s_1 = \frac{p_1}{2(\delta+1)}\,, \tag{3.7}$$

$$s_2 = \frac{1}{2(\delta+2)}\left[p_2 - \left[\frac{2\delta^2-\delta}{2(1+\delta)^2}\right]p_1^2\right], \tag{3.8}$$

$$s_3 = \frac{1}{2(\delta+3)}\left[\left[\frac{3\delta^4+3\delta^3+4\delta^2+2\delta}{3(\delta+1)^3(\delta+2)}\right]p_1^3 - \left[\frac{2\delta^2+2\delta}{(1+\delta)(\delta+2)}\right]p_1p_2 + p_3\right], \tag{3.9}$$

$$\begin{aligned} s_4 &= \left[\frac{-4\delta^7-14\delta^6-30\delta^5-57\delta^4-47\delta^3-22\delta^2-6\delta}{8(1+\delta)^4(2+\delta)^2(3+\delta)(\delta+4)}\right]p_1^4-\left[\frac{3\delta+2\delta^2}{4(2+\delta)^2(4+\delta)}\right]p_2^2 \\ &- \left[\frac{2\delta^2+3\delta}{2(\delta+1)(3+\delta)(4+\delta)}\right]p_1p_3+\left[\frac{3\delta^5+12\delta^4+18\delta^3+18\delta^2+9\delta}{2(1+\delta)^2(2+\delta)^2(3+\delta)(4+\delta)}\right]p_1^2p_2 \\ &+ \left[\frac{1}{2(\delta+4)}\right]p_4\,, \end{aligned} \tag{3.10}$$

$$\begin{aligned} s_5 &= \left[\frac{5\delta^9+20\delta^8+70\delta^7+200\delta^6+266\delta^5+276\delta^4+179\delta^3+56\delta^2+8\delta}{10(1+\delta)^5(2+\delta)^2(3+\delta)(4+\delta)(5+\delta)}\right]p_1^5 \\ &- \left[\frac{4\delta^7+20\delta^6+50\delta^5+104\delta^4+110\delta^3+56\delta^2+16\delta}{2(1+\delta)^3(2+\delta)^2(3+\delta)(4+\delta)(5+\delta)}\right]p_1^3p_2 \\ &+ \left[\frac{3\delta^5+15\delta^4+28\delta^3+28\delta^2+16\delta}{2(1+\delta)^2(2+\delta)(3+\delta)(4+\delta)(5+\delta)}\right]p_1^2p_3 \\ &+ \left[\frac{3\delta^5+18\delta^4+35\delta^3+40\delta^2+24\delta}{2(1+\delta)(2+\delta)^2(3+\delta)(4+\delta)(5+\delta)}\right]p_1p_2^2 \\ &- \left[\frac{2\delta^2+4\delta}{2(1+\delta)(4+\delta)(5+\delta)}\right]p_1p_4-\left[\frac{2\delta^2+2\delta}{2(2+\delta)(3+\delta)(5+\delta)}\right]p_3p_2+\left[\frac{1}{2(5+\delta)}\right]p_5. \end{aligned} \tag{3.11}$$

For s_1, using lemma (2.1) in (3.7), we get

$$|s_1| \leq \frac{1}{1+\delta}. \tag{3.12}$$

Since $0 \leq \left[\frac{2\delta^2-\delta}{2(1+\delta)^2}\right] \leq 1$, by application of lemma (2.3) and lemma (2.1) in (3.8), we obtain

$$|s_2| \leq \frac{1}{(2+\delta)} \tag{3.13}$$

For s_3, relation (3.9) can be written as

$$s_3 = \frac{1}{2(\delta+3)}\left[p_3-2\left[\frac{\delta}{(\delta+2)}\right]p_1p_2+\left[\frac{3\delta^4+3\delta^3+4\delta^2+2\delta}{3(\delta+1)^2(\delta+2)}\right]p_1^3\right]. \tag{3.14}$$

Comparing the equation (3.14) with lemma (2.3), we obtain

$$k_1 = \frac{\delta}{(\delta+2)} \tag{3.15}$$

and

$$k_2 = \frac{3\delta^4+3\delta^3+4\delta^2+2\delta}{3(\delta+1)^2(\delta+2)}\quad. \tag{3.16}$$

Clearly $0 \leq k_1 \leq 1$ and $k_1 \geq k_2$. Moreover,

$$k_1\,(2k_1-1) = \frac{\delta(\delta-2)}{(\delta+2)^2} \leq k_2\ . \tag{3.17}$$

Given that all conditions outlined in lemma (2.2) are met, applying (2.6) yields the following conclusion

$$|s_3| = \frac{1}{(3+\delta)}. \tag{3.18}$$

To obtain the bound of s_4, we use lemma (2.4) in (3.10), so that

$$s_4 = \frac{1}{2(\delta+4)}\left[\lambda_1 p_1^4 + \lambda_2 p_2^2 + 2\lambda_3 p_1 p_3 - \frac{3}{2}\lambda_4 p_1 p_2 + p_4\right] \tag{3.19}$$

where,

$$\begin{aligned}
\lambda_1 &= \left[\frac{-2\delta^8 - 14\delta^7 - 6\delta^6 + 38\delta^5 - 67\delta^4 - 153\delta^3 - 18\delta^2 + 42\delta}{4(1+\delta)^4(2+\delta)^2(3+\delta)}\right],\\
\lambda_2 &= \left[\frac{-3\delta - 2\delta^2}{2(2+\delta)^2}\right],\\
\lambda_3 &= \left[\frac{-2\delta^2 - 3\delta}{2(\delta+1)(3+\delta)}\right],\\
\lambda_4 &= \left[\frac{2\left(-\delta^5 - 4\delta^4 - 6\delta^3 - 6\delta^2 - 3\delta\right)}{(1+\delta)^2(2+\delta)^2(3+\delta)}\right].
\end{aligned}$$

Consequently, all the conditions outlined in lemma (2.2) and (2.4) are fulfilled and utilizing equation (2.9), we get

$$|s_4| \leq \frac{1}{4+\delta}. \tag{3.20}$$

To calculate the bound of s_5, compare the relation (3.11) with (2.10)

$$\frac{1}{2(\delta+5)}\left|\xi_1 p_1^5 + 3\xi_2 p_1 p_2^2 + 3\xi_3 p_1^2 p_3 - 4\xi_4 p_1^3 p_2 - 2\xi_5 p_1 p_4 - 2\xi_6 p_2 p_3 + p_5\right| \leq 2. \tag{3.21}$$

where,

$$\begin{aligned}
\xi_1 &= \frac{5\delta^9 + 20\delta^8 + 70\delta^7 + 200\delta^6 + 266\delta^5 + 276\delta^4 + 179\delta^3 + 56\delta^2 + 8\delta}{5(1+\delta)^5(2+\delta)^2(3+\delta)(4+\delta)},\\
\xi_2 &= \frac{3\delta^5 + 18\delta^4 + 35\delta^3 + 40\delta^2 + 24\delta}{3(1+\delta)(2+\delta)^2(3+\delta)(4+\delta)},\\
\xi_3 &= \frac{3\delta^5 + 15\delta^4 + 28\delta^3 + 28\delta^2 + 16\delta}{(1+\delta)^2(2+\delta)(3+\delta)(4+\delta)},\\
\xi_4 &= \frac{-\left(4\delta^7 + 20\delta^6 + 50\delta^5 + 104\delta^4 + 110\delta^3 + 56\delta^2 + 16\delta\right)}{4(1+\delta)^3(2+\delta)^2(3+\delta)(4+\delta)},\\
\xi_5 &= \frac{\delta(\delta+2)}{(1+\delta)(4+\delta)},\\
\xi_6 &= \frac{\delta(\delta+1)}{(2+\delta)(3+\delta)}.
\end{aligned}$$

Thus, all the conditions of lemma (2.5) are satisfied and applications of (2.10) gives

$$|s_5| \leq \frac{1}{(5+\delta)} \tag{3.22}$$

which completes the proof of theorem (3.1). □

Remark 3.1. *Taking* $\delta = 0$ *in theorem* (3.1)*, we get the following results*

$$\begin{aligned} |s_1| &\leq 1, \\ |s_2| &\leq \frac{1}{2}, \\ |s_3| &\leq \frac{1}{3}, \\ |s_4| &\leq \frac{1}{4}, \\ |s_5| &\leq \frac{1}{5}. \end{aligned}$$

Theorem 3.2. *For any* $\alpha \in \mathbb{C}$*, where* $\mathbb{C}$ *is the set of complex numbers, if we consider the function* f *defined by* (1.4) *to belong to the class* $\mathcal{A}(\delta)$*;* $(0 \leq \delta \leq 1)$*, then*

$$\left|s_1 - \alpha s_2^2\right| \leq \frac{1}{2+\delta} \max\left\{1, \left|\frac{\delta^2 - \delta + \alpha(2+\delta) - 1}{(1+\delta)^2}\right|\right\}. \tag{3.23}$$

Proof. Making use of (3.7) and (3.8), we get

$$\begin{aligned} \left|s_1 - \alpha s_2^2\right| &= \left|\left[\frac{p_1^2(1-\delta)}{(\delta+1)(\delta+2)} + \frac{p_2}{2(\delta)} - \frac{p_1^2}{4(\delta+1)^2}\right] - \alpha\left[\frac{p_1^2}{4(1+\delta)^2}\right]\right| \\ &= \frac{1}{2(\delta+2)}\left|p_2 - \left[\frac{2+\delta}{2(\delta+1)^2} + \frac{\alpha(2+\delta)}{2(\delta+1)^2} - \frac{1-\delta}{2(\delta+1)(\delta+2)}\right]p_1^2\right| \\ &= \frac{1}{2(\delta+2)}\left|p_2 - \left[\frac{2\delta^2 + \delta + \alpha(\delta+2)}{2(\delta+1)^2}\right]p_1^2\right| \\ &= \frac{1}{2(\delta+2)}\left|p_2 - \kappa p_1^2\right| \end{aligned}$$

where,

$$\kappa = \frac{2\delta^2 + \delta + \alpha(\delta+2)}{2(\delta+1)^2}.$$

An application of lemma (2.2), we get

$$\left|s_1 - \alpha s_2^2\right| \leq \frac{1}{2+\delta} \max\left\{1, \left|\frac{\delta^2 - \delta + \alpha(2+\delta) - 1}{(1+\delta)^2}\right|\right\}$$

which completes the proof of theorem (3.2). □

Putting $\delta = 0$ in theorem (3.2), we get the following result.

Corollary 3.1. *Let* f *given by* (1.1)*, be in the class* $\mathcal{A}(\delta)$*;* $(0 \leq \delta \leq 1)$*. Then for any* $\alpha \in \mathbb{C}$*, we have*

$$\left|s_1 - \alpha s_2^2\right| \leq \frac{1}{2} \max\{1, |2\alpha - 1|\}.$$

$\alpha = 1$, we get the following result in form of corollary

Corollary 3.2. *Let* f *given by* (1.1)*, be in the class* $\mathcal{A}(\delta)$*;* $(0 \leq \delta \leq 1)$*. Then for any* $\alpha \in \mathbb{C}$*, we have*

$$\left|s_1 - s_2^2\right| \leq \frac{1}{2+\delta} \max\left\{1, \left|\frac{\delta^2 + 1}{(\delta+1)^2}\right|\right\}.$$

Conclusion

In this paper, the authors present a new category of Sokol-Stankiewicz starlike functions, specifically tailored for the open unit disk $\mathbb{D}$ and subordinate to a family, based on the coefficients of caratheodory functions within this class, we examine the limits of certain initial coefficients and provide an estimation for the Fekete-Szego functional. Utilizing quantum or q-calculus, researchers can define this class and subsequently derive the corresponding results.

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