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Some Novel Coincidence and Fixed Point Results Based on $\mathcal{Z}$ Family of Functions in Fuzzy Metrics

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Abstract. This paper suggests some coincidence and fixed point theorems based on the $\mathcal{Z}$ family functions for set valued mappings. After that, we provide the concept of $\mathcal{Z}^*$ family of functions, and prove some coincidence and fixed points results for strongly demicompact mappings in fuzzy metric space. We also suggest some examples to support our results. Eventually, we give the existence and uniqueness of a solution for a functional equation involved in a dynamic programming. Our results are novel and suggest a new direction to researchers who working in the theory of coincidence and fixed point.

1. Introduction and preliminaries

Fixed point theory is a branch of analysis that offers effective and productive methods for determining the existence and approximation of solutions in various types of distance spaces (see, e.g., [1–3] and others). In [4], Banach suggested his famous theorem for contraction mappings in a complete metric space setting and applied for finding sought solutions of integral equations. Since then, many extensions of Banach's theorem have been given by various researchers in the the setting of different spaces for single valued mappings and multi-valued mappings. In 1969, Nadler's

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generalization of the Banach contraction for set valued mappings was the motivation behind multivalued results in various spaces. As in probabilistic metric space, using weakly demicompact mappings and strongly demicompact the Nadler's theorem for multivalued mappings is proved (See, for instance [13,18,20]). Later on, Došenović et al. [21] also proved multivalued fixed point theorems using altering distance function and demicompact mappings in fuzzy metric settings. Fixed point result of the set and single valued mappings are applicable to several other fields for attaining the existence and uniqueness of the solutions of integral inclusion, partial and ordinary differential, functional and fractional equations.

George and Veeramani [6] modified the notion of fuzzy metric space (FMS), which was initially introduced by Kramosil and Michalek [5], to study a Hausdorff topology on the said space. FMS is defined by means of a fuzzy set [8] and a continuous t-norm [7].

Definition 1.1. *[7] A given function, namely, $* : [0,1] \times [0,1] \to [0,1]$ is known as a continuous triangular norm (for short, t-norm) whenever for any choice of $a, b, c \in [0,1]$, one have the following properties:*

- (t_1) *The function $*$ is commutative and associated, that is, $a * b = b * a$ and $a * (b * c) = (a * b) * a$.*
- (t_2) *The function $*$ is essentially continuous.*
- (t_3) *The condition $1 * a = a$ is also hold.*
- (t_4) *If $a \leq c$ and $b \leq d$, then one has, $a * b \leq c * d$.*

Now we provide the definition of a FMS as follows.

Definition 1.2. *[6] Assume that we have a possibly a nonempty set $\mathcal{X}$. Then a FMS is denoted by $(\mathcal{X}, \mathfrak{M}, *)$, where $\mathfrak{M}$ is any given fuzzy set on $\mathcal{X}^2 \times (0,\infty)$ and $*$ is any given continuous t-norm such that for any choice of $\upsilon, \varrho, z \in \mathcal{X}$ and $t, s > 0$, one have*

- (f_1) $\mathfrak{M}(\upsilon, \varrho, t) > 0$.
- (f_2) $\upsilon = \varrho \iff \mathfrak{M}(\upsilon, \varrho, t) = 1$.
- (f_3) $\mathfrak{M}(\upsilon, \varrho, t) = \mathfrak{M}(\varrho, \upsilon, t)$.
- (f_4) $\mathfrak{M}(\upsilon, z, t + s) \geq \mathfrak{M}(\upsilon, \varrho, t) * \mathfrak{M}(\varrho, z, s)$.
- (f_5) $\mathfrak{M}(\upsilon, \varrho, .) : (0,\infty) \longrightarrow (0,1]$ *is essentially continuous.*

If $(\mathcal{X}, \mathfrak{M}, *)$ is a FMS, then the fuzzy metric $\mathfrak{M}$ on $\mathcal{X}$ generates the Hausdorff topology, with open set $\{\mathcal{B}(\upsilon, r, t) : \upsilon \in \mathcal{X}, r \in (0,1), t > 0\}$ as its base, where $\mathcal{B}(\upsilon, r, t) = \{\varrho \in \mathcal{X} : \mathfrak{M}(\upsilon, \varrho, t) > 1 - r\}$ is a open ball with centre $\upsilon \in \mathcal{X}$, radius $r \in (0,1)$ and $t > 0$. The topology induced is not only Hausdorff, actually it is metrizable (see, [6]). In FMS, $\mathfrak{M}$ is a continuous function (see, [9]).

Definition 1.3. *[6] Suppose we have a given a FMS $\mathcal{X} = (\mathcal{X}, \mathfrak{M}, *)$. Then*

- (i) *A given sequence, namely, $\upsilon_n \in \mathcal{X}$ is known as a convergent sequence with limit $\upsilon \in \mathcal{X}$, if one has $\lim_{n\to\infty} \mathfrak{M}(\upsilon_n, \upsilon, t) = 1$, for any choice of $t > 0$, that is, if we have given any $r \in (0,1)$ and $t > 0$, one is able to find $n_0 \in \mathbb{N}$ such that*

$$\mathfrak{M}(\upsilon_n, \upsilon, t) > 1 - r, \text{ for every } n \geq n_0.$$

(ii) *A given sequence, namely, $\upsilon_n \in \mathcal{X}$ is said to be a Cauchy sequence, if one has for any choice of $\epsilon > 0$ and $t > 0$, one is able to find $n_0 \in \mathbb{N}$ such that*

$$\mathfrak{M}(\upsilon_n, \upsilon_m, t) > 1 - \epsilon \text{ for every } n, m > n_0.$$

(iii) *The space $\mathcal{X}$ is known as a complete FMS, if any Cauchy sequence in $\mathcal{X}$ is essentially convergent to an element of $\mathcal{X}$.*

Lemma 1.1. *[12] Let $(\mathcal{X}, \mathfrak{M}, *)$ be a FMS. $\mathfrak{M}(\upsilon, \varrho, .)$ is non-decreasing with respect to t, for any $\upsilon, \varrho \in \mathcal{X}$.*

Throughout, in this paper $\mathcal{K}(\mathcal{X})$ is all nonempty compact subsets of $\mathcal{X}$. In 2004, Hausdorff fuzzy metric induced by $\mathfrak{M}$ was introduced by Rodríguez-López and Romaguera [9] as:
$\mathcal{H}_{\mathfrak{M}}(\mathcal{A}, \mathcal{B}, t) = min\left\{\inf_{a \in \mathcal{A}} \mathfrak{M}(a, \mathcal{B}, t), \inf_{b \in \mathcal{B}} \mathfrak{M}(\mathcal{A}, b, t)\right\}$, for $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\mathcal{X})$ and $t > 0$, where $\mathfrak{M}(a, \mathcal{A}, t) = \sup_{b \in \mathcal{A}} \mathfrak{M}(a, b, t)$. Then $(\mathcal{K}(\mathcal{X}), H_{\mathfrak{M}}, *)$ is called the Hausdorff fuzzy metric space (HFMS), where $\mathcal{H}_{\mathfrak{M}}$ is the Hausdorff fuzzy metric induced by $\mathfrak{M}$.

Consistent with [9], we recall the following Lemma which we shall use in our sequel.

Lemma 1.2. *[9] Let $(\mathcal{X}, \mathfrak{M}, *)$ be a FMS and $\mathcal{A}, \mathcal{B} \in \mathcal{K}(\mathcal{X})$. For any $\upsilon \in \mathcal{A}$ and $t > 0$, there exists $\varrho \in \mathcal{B}$ such that $\sup_{\varrho' \in \mathcal{B}} \mathfrak{M}(\upsilon, \varrho', t) = \mathfrak{M}(\upsilon, \varrho, t)$.*

Recently, Shukla et al. [10] introduced a family of functions as:

Definition 1.4. *[10] $\mathcal{Z}$ denote the family of all functions $\zeta : (0, 1] \times (0, 1] \to \mathbb{R}$ satisfying:*

$$\zeta(\iota, q) > q, \text{ for each } \iota, q \in (0, 1).$$

With the help of these functions authors unified classes of different fuzzy contractive mappings presented in [[14], [15], [16] and [17]] and introduced fuzzy $\mathcal{Z}$-contractive mappings in FMS.

Definition 1.5. *([13], [18]) Let $\mathcal{A}$ be a non-empty subset of a FMS $(\mathcal{X}, \mathfrak{M}, *)$. A mapping $\mathcal{S} : \mathcal{A} \to 2^{\mathcal{X}}/\{0\}$ ($2^{\mathcal{X}}$ is the family of all non-empty subsets of the set $\mathcal{X}$) is weakly demicompact, if for each sequence $\{\upsilon_n\}_{n \in \mathbb{N}}$ from $\mathcal{A}$, with $\upsilon_{n+1} \subseteq \mathcal{S}\upsilon_n$ and $\lim_{n \to \infty} \mathfrak{M}(\upsilon_{n+1}, \upsilon_n, t) = 1$ for $t > 0$, there exists any subsequence, namely, $\{\upsilon_{n_j}\}_{j \in \mathbb{N}}$ of $\{\upsilon_n\}$, that is essentially convergent.*

Example 1.1. *Let $\mathcal{X} = [0, 1]$, $\mathfrak{M}(\upsilon, \varrho, t) = \frac{t}{t + d(x, y)}$, d is Euclidean metric on $\mathcal{X}$, then $(\mathcal{X}, \mathfrak{M}, *)$ is a FMS, with t-norm $a * b = \min\{a, b\}$. Let $\mathcal{S}(\upsilon) = \{0, 1\}, \upsilon \in \mathcal{X}$. Then it can be easily verified that mapping $\mathcal{S}$ is is weakly demicompact. Indeed, if for any sequence $\{\upsilon_n\}_{n \in \mathbb{N}} \in \mathcal{X}$, with $\upsilon_{n+1} \in \mathcal{S}\upsilon_n$ and $\lim_{n \to \infty} \mathfrak{M}(\upsilon_{n+1}, \upsilon_n, t) = 1$ for $t > 0$, subsequence is either $\{0\}$ or $\{1\}$.*

Definition 1.6. *[19] Let $\mathcal{A}$ be a non-empty subset of a FMS $(\mathcal{X}, \mathfrak{M}, *)$ and $g : \mathcal{A} \to \mathcal{A}$ be a mapping. $\mathcal{S} : \mathcal{A} \to \mathcal{K}(\mathcal{A})$ is g-strongly demicompact if for every sequence $\{\upsilon_n\}_{n \in \mathbb{N}} \in \mathcal{A}$, with $\lim_{n \to \infty} \mathfrak{M}(g\upsilon_n, \varrho_n, t) = 1$ for some sequence $\{\varrho_n\}_{n \in \mathbb{N}} \subset \mathcal{S}\upsilon_n$ and $t > 0$, then there exists a subsequence $\{g\upsilon_{n_j}\}_{j \in \mathbb{N}}$, which is convergent.*

Example 1.2. *Let $\mathcal{X} = \{1, 2, 3\}$ and define $\mathfrak{M}(\upsilon, \varrho, t) = \dfrac{\min\{\upsilon, \varrho\}}{\max\{\upsilon, \varrho\}}$, for all $\upsilon, \varrho \in \mathcal{X}$ and $t > 0$. Then $(\mathcal{X}, \mathfrak{M}, *)$ is a complete, where $*$ is any continuous t-norm.*

Define $\mathcal{S} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ *as*

$$\mathcal{S}(\upsilon) = \{1, 3\}, \text{ for all } \upsilon \in \mathcal{X}$$

and $g : \mathcal{X} \to \mathcal{X}$ *as*

$$g(\upsilon) = \begin{cases} 1 \text{ if } \upsilon = 2, \\ 3 \text{ if } \upsilon \neq 2. \end{cases}$$

Now, if for every sequence $\{\upsilon_n\}_{n\in\mathbb{N}} \in \mathcal{X}$*, with* $\lim_{n\to\infty} \mathfrak{M}(g\upsilon_n, \varrho_n, t) = 1$ *for some sequence* $\{\varrho_n\}_{n\in\mathbb{N}} \subset \mathcal{S}\upsilon_n \in \{1, 3\}$ *and* $g\upsilon_n \in \{1, 3\}$*. The possibility of subsequence of* $\{g\upsilon_n\}$ *is either singleton* $\{1\}$ *or* $\{3\}$*. Hence,* $\mathcal{S}$ *is g-strongly demicompact.*

Definition 1.7. *([20], [18]) Let* $g : \mathcal{X} \to \mathcal{X}$ *be a mapping.* $\mathcal{S} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ *is a weakly commuting mapping with g, if* $g(\mathcal{S}\upsilon) \subseteq \mathcal{S}(g\upsilon)$ *for every* $\upsilon \in \mathcal{X}$*.*

Example 1.3. *Let* $\mathcal{X} = \{4, 6, 8\}$*. Define* $\mathcal{S} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ *as*

$$\mathcal{S}(\upsilon) = \{4, 6\}, \text{ for all } \upsilon \in \mathcal{X}$$

and $g : \mathcal{X} \to \mathcal{X}$ *as*

$$g(\upsilon) = \begin{cases} 4 \text{ if } \upsilon = 8, \\ 6 \text{ if } \upsilon \neq 8. \end{cases}$$

Now, $\mathcal{S}(g\upsilon) = \{4, 6\}$ *and* $g(\mathcal{S}\upsilon) = \{6\}$*, for all* $\upsilon \in \mathcal{X}$*. Hence,* $\mathcal{S}$ *is weakly commuting mapping with g.*

Lemma 1.3. *[11] Let* $(\mathcal{X}, \mathfrak{M}, *)$ *be a FMS with an additional condition* $\lim_{t\to\infty} \mathfrak{M}(\upsilon, \varrho, t) = 1$*, where* $\upsilon, \varrho \in \mathcal{X}$ *and* $t > 0$*. If* $\{\upsilon_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ *and there is some* $k \in (0, 1)$ *such that*

$$\mathfrak{M}(\upsilon_n, \upsilon_{n+1}, t) \geq \mathfrak{M}(\upsilon_{n-1}, \upsilon_n, \frac{t}{k}),\ t > 0$$

and

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathfrak{M}(\upsilon_0, \upsilon_1, \frac{1}{\alpha^i}) = 1,$$

for $\alpha \in (0, 1)$*, then* $\{\upsilon_n\}$ *is a Cauchy sequence.*

In this paper, our aim is to present multivalued generalization of existing coincidence and fixed point results using fuzzy $\mathcal{Z}-$contractive mappings in a HFMS. As fuzzy $\mathcal{Z}-$contractive mappings unifies and generalized different fuzzy contractive mappings, which motivates us to prove some multivalued coincidence and fixed points results so that results are automatically weaker and extensions of several results in the existing literature. We also provide examples to support our claims. We give existence and uniqueness result for solution to a functional equation as an application.

2. Main results

We begin with the our first multivalued coincidence point theorem, as following.

Theorem 2.1. *Assume that $(\mathcal{X}, \mathfrak{M}, *)$ denotes a FMS and $\lim_{t\to\infty} \mathfrak{M}(\upsilon, \varrho, t) = 1$, for any choice of $\upsilon, \varrho \in \mathcal{X}$ and $t > 0$. Suppose that $\mathcal{S}, \mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ is any set valued mappings and $g : \mathcal{X} \to \mathcal{X}$ is essentially non-constant continuous with the following properties:*

(i) for some $\zeta \in \mathcal{Z}$ and $k \in (0,1)$,

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon, \mathcal{Q}\varrho, t) \geq \zeta\Big(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon, \mathcal{Q}\varrho, t), \mathfrak{M}(g\upsilon.g\varrho, \frac{t}{k})\Big),\ \upsilon, \varrho \in \mathcal{X},\ g\upsilon \neq g\varrho \text{ and } t > 0, \tag{2.1}$$

(ii) there exist $\upsilon_0, \upsilon_1 \in \mathcal{X}$ such that $g\upsilon_1 \in \mathcal{S}\upsilon_0$ and

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathfrak{M}(g\upsilon_0, g\upsilon_1, \frac{1}{\alpha^i}) = 1,\ \alpha \in (0,1),$$

(iii) $\mathcal{S}$ and $\mathcal{Q}$ are weakly commuting with f.

Then, there exists some $\upsilon \in \mathcal{X}$ for which $g\upsilon \in \mathcal{S}\upsilon \cap \mathcal{Q}\upsilon$.

Proof. Let $\upsilon_0 \in \mathcal{X}$. Since $\mathcal{S}\upsilon_0 \in \mathcal{K}(\mathcal{X})$, there exist $\upsilon_1 \in \mathcal{X}$ such that $g\upsilon_1 \in \mathcal{S}\upsilon_0$. Also, $\mathcal{Q}\upsilon_1 \in \mathcal{K}(\mathcal{X})$. As $g\upsilon_1 \in \mathcal{S}\upsilon_0$ and $\mathcal{Q}\upsilon_1 \in \mathcal{K}(\mathcal{X})$ then by Lemma 1.2, condition (i) and Definition 1.4, there exists $\upsilon_2 \in \mathcal{X}$ such that $g\upsilon_2 \in \mathcal{Q}\upsilon_1$ satisfying

$$\begin{aligned}
\mathfrak{M}(g\upsilon_1, g\upsilon_2, t) &= \sup_{\upsilon_2' \in \mathcal{Q}\upsilon_1} \mathfrak{M}(g\upsilon_1, \upsilon_2', t) \\
&\geq \mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon_0, \mathcal{Q}\upsilon_1, t) \\
&\geq \zeta\Big(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon_0, \mathcal{Q}\upsilon_1, t), \mathfrak{M}(g\upsilon_0, g\upsilon_1, \frac{t}{k})\Big) \\
&> \mathfrak{M}(g\upsilon_0, g\upsilon_1, \frac{t}{k}),\ t > 0,\ \text{for some } k \in (0,1).
\end{aligned}$$

As $g\upsilon_2 \in \mathcal{Q}\upsilon_1$ and $\mathcal{S}\upsilon_2 \in \mathcal{K}(\mathcal{X})$ then by Lemma 1.2, condition (i) and Definition 1.4, there exists $\upsilon_3 \in \mathcal{X}$ such that $g\upsilon_3 \in \mathcal{S}\upsilon_2$ satisfying

$$\begin{aligned}
\mathfrak{M}(g\upsilon_2, g\upsilon_3, t) &= \sup_{\upsilon_3' \in \mathcal{S}\upsilon_2} \mathfrak{M}(\upsilon_3', g\upsilon_2, t) \\
&\geq \mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon_2, \mathcal{Q}\upsilon_1, t) \\
&\geq \zeta\Big(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon_2, \mathcal{Q}\upsilon_1, t), \mathfrak{M}(g\upsilon_2, g\upsilon_1, \frac{t}{k})\Big) \\
&> \mathfrak{M}(g\upsilon_2, g\upsilon_1, \frac{t}{k}) \\
&= \mathfrak{M}(g\upsilon_1, g\upsilon_2, \frac{t}{k}),\ t > 0,\ \text{for some } k \in (0,1).
\end{aligned}$$

Repeating the above procedure, we obtain a sequence $\{\upsilon_n\}_{n\in\mathbb{N}} \in \mathcal{X}$ such that

(i) $g\upsilon_{2n+1} \in \mathcal{S}\upsilon_{2n}$, $g\upsilon_{2n+2} \in \mathcal{Q}\upsilon_{2n+1}$, $n \in \mathbb{N}$,

(ii) $\mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t) > \mathfrak{M}(g\upsilon_{n-1}, g\upsilon_n, \frac{t}{k})$, $t > 0$, $n \in \mathbb{N}$ and for $k \in (0,1)$.

Since $v_0, v_1 \in \mathcal{X}$ and $gv_1 \in \mathcal{S}v_0$, by using condition (ii) and Lemma 1.3, we get $\{gv_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. As $(\mathcal{X}, \mathfrak{M}, *)$ is complete, there exists $v \in \mathcal{X}$ with

$$\lim_{n\to\infty} gv_n = v. \tag{2.2}$$

Now, we have to prove that $gv \in \mathcal{S}v \cap \mathcal{Q}v$. Since $\mathcal{S}v \cap \mathcal{Q}v = \overline{\mathcal{S}v} \cap \overline{\mathcal{Q}v}$, we have to show that, there exists $\rho_1 = \rho_1(t,\gamma) \in \mathcal{S}v$ and $\rho_2 = \rho_2(t,\gamma) \in \mathcal{Q}v$ such that $\rho_1, \rho_2 \in \mathcal{B}(gv, t, \gamma)$, for $\gamma \in (0,1)$ and $t > 0$, i.e., $\mathfrak{M}(gv, \rho_1, t) > 1-\gamma$ and $\mathfrak{M}(gv, \rho_1, t) > 1-\gamma$.

By Definition 1.1, we have $\epsilon = \epsilon(\gamma) \in (0,1)$ for which

$$*(1-\epsilon, *(1-\epsilon, 1-\epsilon)) > 1-\gamma. \tag{2.3}$$

Let $t_1 > 0$ and $k \in (0,1)$, using continuity of g, (2.2) and Lemma 1.1, there exist $n_0, n_1 \in \mathbb{N}$ such that

$$\mathfrak{M}(ggv_{2n}, ggv_{2n+1}, \frac{t_1}{3k}) \geq \mathfrak{M}(ggv_{2n}, ggv_{2n+1}, \frac{t_1}{3}) > 1-\epsilon,\ n \geq n_0, \tag{2.4}$$

and

$$\mathfrak{M}(gv, ggv_{2n}, \frac{t_1}{3k}) \geq \mathfrak{M}(gv, ggv_{2n}, \frac{t_1}{3}) > 1-\epsilon,\ n \geq n_1. \tag{2.5}$$

By condition (iii), we have

$$ggv_{2n+1} \in g(\mathcal{S}v_{2n}) \subseteq \mathcal{S}(gv_{2n}),$$

and $\mathcal{Q}v \in \mathcal{K}(\mathcal{X})$ then by Lemma 1.2, condition (i), Definition 1.4 and equation (2.5), there exists $\rho_2 \in \mathcal{Q}v$ satisfying

$$\begin{aligned}
\mathfrak{M}(\rho_2, ggv_{2n_2+1}, \frac{t_1}{3}) &= \sup_{\rho' \in \mathcal{Q}v} \mathfrak{M}(ggv_{2n_2+1}, \rho', \frac{t_1}{3}) \\
&\geq \mathcal{H}_{\mathfrak{M}}(\mathcal{S}gv_{2n_2}, \mathcal{Q}v, \frac{t_1}{3}) \\
&\geq \zeta\Big(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}gv_{2n_2}, \mathcal{Q}v, \frac{t_1}{3}), \mathfrak{M}(ggv_{2n_2}, gv, \frac{t_1}{3})\Big) \\
&> \mathfrak{M}(ggv_{2n_2}, gv, \frac{t_1}{3k}) \\
&> 1-\epsilon,
\end{aligned}$$

for arbitrary $n_2 \geq max\{n_0, n_1\}$.

Now, by using (2.3), we get

$$\begin{aligned}
\mathfrak{M}(gv, \rho_2, t_1) &\geq *\Big(\mathfrak{M}(ggv_{2n_2}, gv, \frac{t_1}{3}), *\Big(\mathfrak{M}(ggv_{2n_2}, ggv_{2n_2+1}, \frac{t_1}{3}), \mathfrak{M}(\rho_2, ggv_{2n_2+1}, \frac{t_1}{3})\Big)\Big) \\
&\geq *(1-\epsilon, *(1-\epsilon, 1-\epsilon)) \\
&> 1-\gamma,
\end{aligned}$$

for arbitrary $t_1 > 0$ and $\gamma \in (0,1)$, which implies that $gv \in \mathcal{S}v$. So $\rho_2 \in \mathcal{B}(gv, t, \gamma)$. Similarly, it can be prove that $\rho_1 \in \mathcal{B}(gv, t, \gamma)$, $t > 0$, $\gamma \in (0,1)$, i.e., $gv \in \mathcal{Q}v$, too. □

Example 2.1. *Let $\mathcal{X} = \mathbb{N} \cup \{0\}$. Define $\mathfrak{M} = \dfrac{t}{t+|\upsilon-\varrho|}$ for all $\upsilon, \varrho \in \mathcal{X}$ and $t > 0$. Then FMS $(\mathcal{X}, \mathfrak{M}, *)$ is a complete, where $*$ is a continuous t-norm. Define set valued mappings $\mathcal{S}, \mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ such that $\mathcal{S}(\upsilon) = \{0, 1\}$ and $\mathcal{Q}(\upsilon) = \{0, 2\}$, for every $\upsilon \in \mathcal{X}$, and a continuous mapping $g : \mathcal{X} \to \mathcal{X}$ such that*

$$g(\upsilon) = \begin{cases} 0, \text{ if } \upsilon = 0, 1, 2, \\ 16, \text{ otherwise.} \end{cases}$$

Let $\zeta : (0, 1] \times (0, 1] \to \mathbb{R}$ be given by

$$\zeta(\iota, q) = \begin{cases} \iota, \text{ if } \iota > q, \\ \sqrt{q}, \text{ if } \iota \leq q, \end{cases} \quad \text{for all } \iota, q \in (0, 1].$$

Now, for $k = \frac{1}{2}$, it can be easily verified that all the conditions of Theorem 2.1 hold. Hence, there exists some $\upsilon \in \mathcal{X}$ for which $g\upsilon \in \mathcal{S}\upsilon \cap \mathcal{Q}\upsilon$. Indeed, $\upsilon = 0$.

Corollary 2.1. *Let $(\mathcal{X}, \mathfrak{M}, *)$ be a FMS with an additional condition $\lim_{t\to\infty} \mathfrak{M}(\upsilon, \varrho, t) = 1$, $\upsilon, \varrho \in \mathcal{X}$, $t > 0$ and $\mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ a set valued mapping satisfying:*

(i) for some $\zeta \in \mathcal{Z}$ and $k \in (0, 1)$,

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{Q}\upsilon, \mathcal{Q}\varrho, t) \geq \zeta\Big(\mathcal{H}_{\mathfrak{M}}(\mathcal{Q}\upsilon, \mathcal{Q}\varrho, t), \mathfrak{M}(\upsilon, \varrho, \frac{t}{k})\Big), \ \upsilon, \varrho \in \mathcal{X}, \ \upsilon \neq \varrho \text{ and } t > 0, \tag{2.6}$$

(ii) there exist $\upsilon_0, \upsilon_1 \in \mathcal{X}$ such that $\upsilon_1 \in \mathcal{Q}\upsilon_0$ and

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathfrak{M}(\upsilon_0, \upsilon_1, \frac{1}{\alpha^i}) = 1, \ \alpha \in (0, 1).$$

Then, $\mathcal{Q}$ has a fixed point.

Corollary 2.2. *Let $(\mathcal{X}, \mathfrak{M}, *)$ be a FMS with an additional condition $\lim_{t\to\infty} \mathfrak{M}(\upsilon, \varrho, t) = 1$, $\upsilon, \varrho \in \mathcal{X}$ and $t > 0$ and a mapping $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$ satisfies:*

(i) for some $\zeta \in \mathcal{Z}$ and $k \in (0, 1)$,

$$\mathfrak{M}(\mathcal{Q}\upsilon, \mathcal{Q}\varrho, t) \geq \zeta\Big(\mathfrak{M}(\mathcal{Q}\upsilon, \mathcal{Q}\varrho, t), \mathfrak{M}(\upsilon, \varrho, \frac{t}{k})\Big), \ \upsilon, \varrho \in \mathcal{X}, \ \upsilon \neq \varrho \text{ and } t > 0, \tag{2.7}$$

(ii) there exist $\upsilon_0, \upsilon_1 \in \mathcal{X}$ such that $\upsilon_1 = \mathcal{Q}\upsilon_0$ and

$$\lim_{n\to\infty} *_{i=n}^{\infty} \mathfrak{M}(\upsilon_0, \upsilon_1, \frac{1}{\alpha^i}) = 1, \ \alpha \in (0, 1),$$

Then, $\mathcal{Q}$ has a unique fixed point.

Proof. Suppose mapping $\mathcal{Q}$ is single valued in Corollary 2.1, then $\mathcal{Q}$ has a fixed point. Now, for uniqueness, suppose $\upsilon, \varrho \in \mathcal{X}$ be two distinct fixed points of $\mathcal{Q}$, then from condition (i), we obtain

$$\begin{aligned} \mathfrak{M}(\upsilon, \varrho, t) &= \mathfrak{M}(\mathcal{Q}(\upsilon), \mathcal{Q}(\varrho), t) \\ &\geq \zeta(\mathfrak{M}(\mathcal{Q}(\upsilon), \mathcal{Q}(\varrho), t), \mathfrak{M}(\upsilon, \varrho, \frac{t}{k})) \\ &> \mathfrak{M}(\upsilon, \varrho, \frac{t}{k}), \end{aligned}$$

for some $k \in (0, 1)$ and for every $t > 0$, i.e.,

$$\mathfrak{M}(\upsilon, \varrho, kt) > \mathfrak{M}(\upsilon, \varrho, t). \tag{2.8}$$

As $\mathfrak{M}(\upsilon, \varrho, t)$ is nondecreasing w.r.t. t and $kt < t$, for every $t > 0$,

$$\mathfrak{M}(\upsilon, \varrho, kt) \leq \mathfrak{M}(\upsilon, \varrho, t), \tag{2.9}$$

for all $t > 0$. Hence from (2.8) and (2.9), we get $\upsilon = \varrho$. □

Definition 2.1. *Let $\mathcal{Z}^*$ denote set of all functions $\zeta^* : (0,1] \times (0,1] \to \mathbb{R}$, satisfying:*

- *$\zeta^*(q, \iota) > \iota$, $\forall\, \iota, q \in (0,1)$.*
- *Let $\{\iota_n\}$ and $\{q_n\}$ are two sequences in $(0,1)$ such that $\iota_n \leq q_n$, for every $n \in \mathbb{N}$ and $\lim_{n\to\infty} \iota_n \in (0,1)$, then $\limsup_{n\to\infty} \zeta^*(q_n, \iota_n) = 1$.*

Theorem 2.2. *Let $(\mathcal{X}, \mathfrak{M}, *)$ be a FMS. Let $\mathcal{S}, \mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ be the set valued mappings and $g : \mathcal{X} \to \mathcal{X}$ is a non-constant continuous mapping satisfying:*

(i) for some $\zeta^* \in \mathcal{Z}^*$,

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon, \mathcal{Q}\varrho, t) \geq \zeta^*\left(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon, \mathcal{Q}\varrho, t), \mathfrak{M}(g\upsilon . g\varrho, t)\right),\ \upsilon, \varrho \in \mathcal{X},\ g\upsilon \neq g\varrho \text{ and } t > 0, \tag{2.10}$$

(ii) either $\mathcal{S}$ or $\mathcal{Q}$ is g-strongly demicompact,

(iii) $\mathcal{S}$ and $\mathcal{Q}$ are weakly commuting with f.

Then, there exists some $\upsilon \in \mathcal{X}$ for which $g\upsilon \in \mathcal{S}\upsilon \cap \mathcal{Q}\upsilon$.

Proof. Using Lemma 1.2, condition (i) and Definition 1.4, we define a sequence $\{\upsilon_n\} \in \mathcal{X}$ for which $g\upsilon_{2n+1} \in \mathcal{S}\upsilon_{2n}$, $g\upsilon_{2n+2} \in \mathcal{Q}\upsilon_{2n+1}$, $n \in \mathbb{N}$ and

$$\mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t) > \mathfrak{M}(g\upsilon_{n-1}, g\upsilon_n, t),\ t > 0,\ n \in \mathbb{N}. \tag{2.11}$$

The proof of the above is on the lines of proof of Theorem 2.1.

Therefore, $\{\mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t)\}_{n\in\mathbb{N}}$ is a bounded and increasing sequence for $t > 0$. So $\alpha : (0, \infty) \to [0,1]$ exists in such a way that

$$\lim_{n\to\infty} \mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t) = \alpha(t),\ t > 0.$$

Our claim is $\alpha(t) = 1$ for every $t > 0$. If for some $t_1 > 0$, $\alpha(t) < 1$. Denote $q_n = \mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t_1)$ and $p_n = \mathfrak{M}(g\upsilon_{n-1}, g\upsilon_n, t_1)$, for every $n \in \mathbb{N}$, then in the light of (2.11), we have $p_n \leq q_n$, for every $n \in \mathbb{N}$. Now, using Definition 2.1, we get

$$1 = \limsup_{n\to\infty} \zeta^*(\mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t_1), \mathfrak{M}(g\upsilon_{n-1}, g\upsilon_n, t_1)) > \limsup_{n\to\infty} \mathfrak{M}(g\upsilon_{n-1}, g\upsilon_n, t_1) < 1,\ \text{for } t_1 > 0,$$

which is a contradiction. Hence,

$$\lim_{n\to\infty} \mathfrak{M}(g\upsilon_n, g\upsilon_{n+1}, t) = \alpha(t) = 1,\ t > 0. \tag{2.12}$$

Since $g\upsilon_{2n+1} \in \mathcal{S}\upsilon_{2n}$ or $g\upsilon_{2n+2} \in \mathcal{Q}\upsilon_{2n+1}$, $n \in \mathbb{N}$ and form (2.12) we have $\lim_{n\to\infty} \mathfrak{M}(g\upsilon_{2n}, g\upsilon_{2n+1}, t) = 1$, $t > 0$. Using condition (ii), there exists a subsequence either $\{g\upsilon_{2n_j}\}_{j\in\mathbb{N}}$ or $\{g\upsilon_{2n_j+1}\}_{j\in\mathbb{N}}$, which is convergent.

In the last part of proof, in place of $\{g\upsilon_n\}_{n\in\mathbb{N}}$, we deal with $\{g\upsilon_{2n_j}\}_{j\in\mathbb{N}}$ or $\{g\upsilon_{2n_j+1}\}_{j\in\mathbb{N}}$ and following the lines of proof of Theorem 2.1, we get our result. □

Example 2.2. *Let $\mathcal{X} = \{1,2,3\}$ and define $\mathfrak{M}(\upsilon,\varrho,t) = \dfrac{\min\{\upsilon,\varrho\}}{\max\{\upsilon,\varrho\}}$, for all $\upsilon,\varrho \in \mathcal{X}$ and $t > 0$. Then $(\mathcal{X},\mathfrak{M},*)$ is a complete, where $*$ is a continuous t-norm.*

Define $\mathcal{S},\mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ as

$$\mathcal{S}(\upsilon) = \{1,3\}, \text{ for all } \upsilon \in \mathcal{X} \text{ and } \mathcal{Q}(\upsilon) = \begin{cases} \{1,2\}, \text{ for } \upsilon = 2, \\ \{1,3\}, \text{ for } \upsilon \neq 2, \end{cases}$$

and $g : \mathcal{X} \to \mathcal{X}$ as

$$g(\upsilon) = \begin{cases} 1 \text{ if } \upsilon = 2, \\ 3 \text{ if } \upsilon \neq 2. \end{cases}$$

Now, choose $\zeta^*(\iota,q) = \frac{q}{\iota}$ in $\mathcal{Z}^*$.

Case i: If $\upsilon = 1,2,3$ and $\varrho = 1,3$, then we have

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon,\mathcal{Q}\varrho,t) = 1.$$

Case ii: If $\upsilon = 1,2,3$ and $\varrho = 2$, then we have

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon,\mathcal{Q}\varrho,t) = \frac{2}{3}.$$

Here for every $\upsilon,\varrho \in \mathcal{X}$ and $g\upsilon \neq g\varrho$, we get

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon,\mathcal{Q}\varrho,t) \geq \zeta^*(\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon,\mathcal{Q}\varrho,t),\mathfrak{M}(g\upsilon,g\varrho,t)) = \frac{\mathfrak{M}(g\upsilon,g\varrho,t)}{\mathcal{H}_{\mathfrak{M}}(\mathcal{S}\upsilon,\mathcal{Q}\varrho,t)}.$$

Hence, condition (i) of Theorem 2.2 is satisfied. Other conditions of Theorem 2.2 can be easily verified. Hence, there exists some $\upsilon \in \mathcal{X}$, for which $g\upsilon \in \mathcal{S}\upsilon \cap \mathcal{Q}\upsilon$. Indeed, $\upsilon = 3$.

Corollary 2.3. *Let $(\mathcal{X},\mathfrak{M},*)$ be a FMS. $\mathcal{Q} : \mathcal{X} \to \mathcal{K}(\mathcal{X})$ is a set valued mapping satisfying:*

(i) for some $\zeta^* \in \mathcal{Z}^*$,

$$\mathcal{H}_{\mathfrak{M}}(\mathcal{Q}\upsilon,\mathcal{Q}\varrho,t) \geq \zeta^*\left(\mathcal{H}_{\mathfrak{M}}(\mathcal{Q}\upsilon,\mathcal{Q}\varrho,t),\mathfrak{M}(\upsilon,\varrho,t)\right), \ \upsilon,\varrho \in \mathcal{X}, \ \upsilon \neq \varrho \text{ and } t > 0, \tag{2.13}$$

(ii) $\mathcal{Q}$ is weakly demicompact.

Then, there exists some $\upsilon \in \mathcal{X}$ for which $\upsilon \in \mathcal{Q}\upsilon$.

3. Application

Using our findings, we now demonstrate the solution of a functional equation that arises in dynamic programming will exists and the solution of that equation will be unique.

Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are two Banach spaces. $\mathcal{P} \subseteq \mathcal{X}$ and $\mathcal{Q} \subseteq \mathcal{Y}$ represents state space and decision space, respectively. Consider the following functional equation from dynamic programming.

$$\xi(\iota) = \sup_{q\in\mathcal{Q}}\{g(\iota,q) + \Upsilon(\iota,q,\xi(\tau(\iota,q)))\} \text{ for } \iota \in \mathcal{P}, \tag{3.1}$$

where $\tau : \mathcal{P}\times\mathcal{Q} \to \mathcal{P}$, $g : \mathcal{P}\times\mathcal{Q} \to \mathbb{R}$, $\Upsilon : \mathcal{P}\times\mathcal{Q}\times\mathbb{R} \to \mathbb{R}$.

Let $\mathcal{B}(\mathcal{P})$ be the collection of all bounded real-valued functions defined on $\mathcal{P}$. Define

$$\mathfrak{M}(\xi_1, \xi_2, t) = e^{-\dfrac{d(\xi_1, \xi_2)}{t}}, \text{ for } t > 0,$$

where $d(\xi_1, \xi_2) = \sup\limits_{\iota \in \mathcal{P}} |\xi_1(\iota) - \xi_2(\iota)| = \|\xi_1 - \xi_2\|$ for all $\xi_1, \xi_2 \in \mathcal{B}(\mathcal{P})$. Then $(\mathcal{B}(\mathcal{P}), \mathfrak{M}, *)$ is a complete FMS, where $*$ is given by $a * b = a.b$.

Let $\mathcal{T} : \mathcal{B}(\mathcal{P}) \to \mathcal{B}(\mathcal{P})$ be defined by

$$\mathcal{T}(\xi(\iota)) = \sup_{q \in \mathcal{Q}} \{g(\iota, q) + \Upsilon(\iota, q, \xi(\tau(\iota, q)))\} \tag{3.2}$$

where $(x, \xi) \in \mathcal{P} \times \mathcal{B}(\mathcal{P})$.

Theorem 3.1. *Consider the problem (3.1) with $\tau : \mathcal{P} \times \mathcal{Q} \to \mathcal{P}$, $g : \mathcal{P} \times \mathcal{Q} \to \mathbb{R}$, $\Upsilon : \mathcal{P} \times \mathcal{Q} \times \mathbb{R} \to \mathbb{R}$. Suppose*

(i) g and Υ are bounded,

(ii) for some $k \in (0, 1)$ and $\zeta \in \mathcal{Z}$ such that for every $(\iota, q) \in \mathcal{P} \times \mathcal{Q}$,

$$e^{-\frac{d(\Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) - \Upsilon(\iota, q_2, \xi_2(\tau(\iota, q_2))))}{t}} \geq \zeta\left(e^{-\frac{d(\Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) - \Upsilon(\iota, q_2, \xi_2(\tau(\iota, q_2))))}{t}}, e^{-k\frac{d(\xi_1(\iota) - \xi_2(\iota))}{t}}\right),$$

for $\xi_1, \xi_2 \in \mathcal{B}(\mathcal{P})$, $t > 0$.

(iii) there exists $\xi_0, \xi_1 \in \mathcal{B}(\mathcal{P})$ such that $\xi_1 = \mathcal{T}\xi_0$ and

$$\lim_{n \to \infty} *_{i=n}^{\infty} e^{-\alpha_i d(\xi_1 - \xi_2)} = 1, \ \alpha \in (0, 1).$$

Then (3.1) has a unique solution in $\mathcal{B}(\mathcal{P})$.

Proof. As $(\mathcal{B}(\mathcal{P}), \mathfrak{M}, *)$ is a complete FMS and $\mathcal{T} : \mathcal{B}(\mathcal{P}) \to \mathcal{B}(\mathcal{P})$ defined as equation (3.2). If $\xi_1(\mathcal{P}), \xi_2(\mathcal{P}) \in \mathcal{B}(\mathcal{P})$, then for every $\beta \in \mathbb{R}^+$ and $\iota \in \mathcal{P}$, there exist $q_1, q_2 \in \mathcal{Q}$ such that

$$\mathcal{T}(\xi_1(\iota)) < g(\iota, q_1) + \Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) + \beta \tag{3.3}$$

and

$$\mathcal{T}(\xi_2(\iota)) < g(\iota, q_2) + \Upsilon(\iota, q_2, \xi_1(\tau(\iota, q_2))) + \beta. \tag{3.4}$$

Then we have

$$\mathcal{T}(\xi_1(\iota)) \geq g(\iota, q_1) + \Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) \tag{3.5}$$

and

$$\mathcal{T}(\xi_2(\iota)) \geq g(\iota, q_2) + \Upsilon(\iota, q_2, \xi_1(\tau(\iota, q_2))). \tag{3.6}$$

From (3.3) and (3.6), we get

$$\begin{aligned} \mathcal{T}(\xi_1(\iota)) - \mathcal{T}(\xi_2(\iota)) &< \Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) - \Upsilon(\iota, q_2, \xi_1(\tau(\iota, q_2))) + \beta \\ &\leq |\Upsilon(\iota, q_1, \xi_1(\tau(\iota, q_1))) - \Upsilon(\iota, q_2, \xi_1(\tau(\iota, q_2)))| + \beta. \end{aligned}$$

As β is arbitrary, we get

$$\begin{aligned} d(\mathcal{T}(\xi_1(\iota)),\mathcal{T}(\xi_2(\iota))) &= |\mathcal{T}(\xi_1(\iota))-\mathcal{T}(\xi_2(\iota))| \\ &\leq |\Upsilon(\iota,q_1,\xi_1(\tau(\iota,q_1)))-\Upsilon(\iota,q_2,\xi_1(\tau(\iota,q_2)))|. \end{aligned}$$

From condition (ii), we have

$$\begin{aligned} e^{-\frac{d(\mathcal{T}(\xi_1(\iota)),\mathcal{T}(\xi_2(\iota)))}{t}} &\geq e^{-\frac{|\Upsilon(\iota,q_1,\xi_1(\tau(\iota,q_1)))-\Upsilon(\iota,q_2,\xi_1(\tau(\iota,q_2)))|}{t}} \\ &\geq e^{-\frac{d(\Upsilon(\iota,q_1,\xi_1(\tau(\iota,q_1)))-\Upsilon(\iota,q_2,\xi_2(\tau(\iota,q_2))))}{t}} \\ &\geq \zeta(e^{-\frac{d(\Upsilon(\iota,q_1,\xi_1(\tau(\iota,q_1)))-\Upsilon(\iota,q_2,\xi_2(\tau(\iota,q_2))))}{t}},e^{-k\frac{d(\xi_1(\iota)-\xi_2(\iota))}{t}}). \\ &\geq \zeta(e^{-\frac{d(\mathcal{T}(\xi_1(\iota)),\mathcal{T}(\xi_2(\iota)))}{t}},e^{-k\frac{d(\xi_1(\iota)-\xi_2(\iota))}{t}}). \end{aligned}$$

This implies

$$\mathfrak{M}(\mathcal{T}\xi_1,\mathcal{T}\xi_2,t) \geq \zeta(\mathfrak{M}(\mathcal{T}\xi_1,\mathcal{T}\xi_2,t),\mathfrak{M}(\xi_1,\xi_2,\frac{t}{k})).$$

From condition (ii), it follows that

$$\lim_{n\to\infty} *_{i=n}^{\infty}\mathfrak{M}(\xi_0,\xi_1,\frac{1}{\alpha^i}) = 1,\ \alpha \in (0,1).$$

Hence, all the requirements of Corollary 2.2 are met. Thus, (3.1) has a unique solution.

□

4. Conclusion

We proved multivalued coincidence and fixed point results by using the special class of $\mathcal{Z}$ and $\mathcal{Z}^*$ functions. The multivalued contractive mappings were established in order to extend Nadler's generalization of Banach contraction principle. As Shukla et al. showed that fuzzy $\mathcal{Z}-$contractive mapping is a weaker class obtained by the unification of several classes of contractive mappings. So, our results are extensions of several fixed point and coincidence point results in the existing literature. At last, we proved the existence and uniqueness of solution to a functional equation involved in dynamic programming in order to support our results.

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