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Adomian Decomposition Method With Inverse Differential Operator and Orthogonal Polynomials for Nonlinear Models

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Abstract. A proficient Adomian decomposition method is proposed amidst the presence of inverse differential operator and orthogonal polynomials for solving nonlinear differential models. The method is indeed a reformation of the standard Adomian method thereby improving the rapidity of the solution's convergence rate. A generalized recurrent scheme for a general nonlinear model was derived and further utilized to solve certain nonlinear test models. Lastly, numerical results are reported in comparative tables, demonstrating absolute error differences between the exact and approximate solutions with regards to various employed orthogonal polynomials.

1. Introduction

George Adomian has in the 1980s introduced an elegant semi-analytical method that is popularly referred to as the Adomian decomposition method [1-2]. The method has in the past and recent times been greatly utilized and at the same time improved to solve a variety of functional equations in science and technological applications. In fact, there have been a large number of different reformations, extensions, and improvements of this method available in the literature [3-6], and the related modification [7-8]. More so, the applicability of this method and its modifications in solving different forms of Initial-Value Problems (IVPs) of both the ordinary and partial differential equation types is notable in different professions.

Besides, as the present study focus on improving the accuracy of the Adomian decomposition method by defining inverse differential operator L^{-1}, and the incorporation of orthogonal polynomials for the source function, let us review related work on the orthogonal polynomials. Orthogonal functions or polynomials [9] are regarded with high esteem in the fields of numerical

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methods, and approximation theories among others. However, in line with their applications, Hosseini [10] demonstrated the relevance of Chebyshev's polynomials in improving the Adomian method. We mention also the excellent work of Liu [11] where Legendre's polynomials were coupled in the standard approach instead of the ordinary Adomian procedure. Additionally, the Adomian approach was equally enhanced using the Gegenbauer's and Jacobi's orthogonal polynomials [12] to solve some important models of mathematical physics; one may in the same fashion read about the relevance of Hermite's, Laguerre's, second kind Chebyshev's, and the Legendre's polynomials in optimizing the standard Adomian procedure in [13-16].

However, the present manuscript proposes a proficient method based on the standard Adomian decomposition method by incorporating inverse differential operator and orthogonal polynomials to solve nonlinear differential models. The method is indeed a reformation of the standard Adomian method thereby improving the rapidity of the solution's convergence rate upon defining inverse differential operator L^{-1}, and at the same time incorporating orthogonal polynomials including the Legendre's, Chebyshev's, Gegenbauer's, and Jacobi's polynomials; in addition to Taylor's series. A generalized recurrent scheme for a general nonlinear model will be derived and further utilized to solve certain nonlinear test models. The study will also report the numerical results via comparative tables to demonstrate the absolute error differences between the exact and approximate solutions. Lastly, we organize the paper in the following manner: Section 2 gives the standard Adomian procedure; while its modifications based on the inverse differential operator and orthogonal polynomials are given in Section 3. Section 4 gives certain illustrative test examples; while Section 5 gives the concluding remarks.

2. Standard Adomian decomposition method

The present section gives a generalized derivation procedure for tackling nonlinear Initial-Value Problems (IVPs) based on the ADM. To do so, let us consider the following differential equation

$$G(u(x)) = g(x), \tag{2.1}$$

with G representing a generalized ordinary (or partial) differential operator, and $g(x)$ as a source term. This operator being general, it can equally be expressed to involve both linear and nonlinear operators. Thus, we decompose the operator further, and rewrite the above equation as follows

$$Lu + Ru + Nu = g, \tag{2.2}$$

where L is the highest linear operator that is invertible, with $R < L$; while N is specifically the nonlinear operator. More so, we rewrite the latter equation as follows

$$Lu = g - Ru - Nu,$$

such that applying the inverse linear operator L^{-1} to both sides of the above equation yields

$$u = \phi(x) + L^{-1}g(x) - L^{-1}Ru - L^{-1}Nu. \tag{2.3}$$

where $\phi(x)$ is the function emanating from the prescribed initial data.
Further, the iterative procedure by the name ADM decomposes the solution $u(x)$ using an infinite series of the following form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \tag{2.4}$$

while the nonlinear component Nu is equally decomposed using the following infinite series

$$N(u) = \sum_{n=0}^{\infty} A_n(u_0, u_1, ...), \tag{2.5}$$

where A_n's are polynomials devised by Adomian, and recursively determined using the following scheme

$$A_n(u_0, u_1, ...) = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{j=0}^{n} \lambda^j u_j\right)\right]_{\lambda=0}, \qquad n = 0, 1, 2, ... \tag{2.6}$$

Therefore, upon substituting Eqs. (2.4) and (2.5) into Eq. (2.3), one gets

$$\sum_{n=0}^{\infty} u_n(x) = \phi(x) + L^{-1}g(x) - L^{-1}R\sum_{n=0}^{\infty} u_n(x) - L^{-1}\sum_{n=0}^{\infty} A_n(u_0, u_1, ...),$$

Furthermore, the ADM procedure swiftly reveals the generalized recursive solution for the problem from the above equation as follows

$$\begin{cases} u_0 = \phi(x) + L^{-1}g(x), \\ \\ u_{n+1} = -L^{-1}Ru_n - L^{-1}A_n(u_0, u_1, ...), \quad n \geq 0, \end{cases} \tag{2.7}$$

where A_n's are the Adomian polynomials computed from Eq. (2.6). Expressing few of these terms, we get

$$\begin{aligned} A_0(u_0) &= N(u_0), \\ A_1(u_0, u_1) &= \frac{dN(u_0)}{du_0}u_1, \\ A_2(u_0, u_1, u_2) &= \frac{dN(u_0)}{du_0}u_2 + \frac{1}{2}\frac{d^2N(u_0)}{du_0^2}u_1^2, \\ A_3(u_0, u_1, u_2, u_3) &= \frac{dN(u_0)}{du_0}u_3 + \frac{d^2N(u_0)}{du_0^2}u_1u_2 + \frac{1}{3!}\frac{d^3N(u_0)}{du_0^3}u_1^3, \\ &\vdots \end{aligned}$$

Remarkable, it is obvious that the Adomian polynomials A_n's depend on the solution components u_n. For instance, A_0 relies merely on u_0; A_1 relies merely on u_0 and u_1; A_2 relies merely on u_0, u_1 and u_2 , and so on.

Finally, a realistic solution is obtained by considering the following m-term approximations as

$$\Psi_n = \sum_{j=0}^{n-1} u_j, \tag{2.8}$$

where

$$u(x) = \lim_{n\to\infty} \Psi_n(x) = \sum_{j=0}^{\infty} u_j(x). \tag{2.9}$$

3. Modified Adomian decomposition method

The present section gives a generalized derivation procedure for tackling second-order nonlinear Initial-Value Problems (IVPs) based on the standard Adomian procedure amidst the presence of an inverse differential operator, and certain known orthogonal polynomials. To do so, let us consider the following differential equation

$$\begin{gathered} u'' + p(x)u' + G(x,u) = g(x), \\ u(0) = A, \quad u'(0) = B, \end{gathered} \tag{3.1}$$

with $G(x,u)$ representing a generalized nonlinear operator, and $g(x)$ as a source term; while A and B are prescribed real constants

Furthermore, if we go against the standard ADM procedure, a linear differential operator L that was proposed in [17] will be utilized in the present modification method. The linear differential operator takes the following form

$$L = e^{-\int p(x)dx} \frac{d}{dx}\left(e^{\int p(x)dx} \frac{d}{dx}\right). \tag{3.2}$$

where the inverse operator L^{-1} is thus taken as a 2-fold integral representation as follows

$$L^{-1}(.) = \int_0^x e^{-\int p(x)dx} \int_0^x e^{\int p(x)dx}(.)dxdx. \tag{3.3}$$

More so, the source function $g(x)$ in the governing model is further expressed through the following series expansions

(i). **Taylor's expansion**

Taylor's series expansion is used here for an arbitrary positive integer, say m, to expand $g(x)$ as follows

$$g(x) = \sum_{n=0}^{m} \frac{g^n(0)}{n!} x^n. \tag{3.4}$$

(ii). **Legendre's expansion**

Legendre's polynomials is used to expand the source term $g(x)$ into a series of Legendre's polynomial as follows [11]

$$g(x) = \sum_{n=0}^{m} c_n P_n(x), \tag{3.5}$$

where $P_n(x)$ are the orthogonal Legendre's polynomials, and the coefficients of Legendre's expansion c_i are determined through

$$c_i = \frac{2i+1}{2}\int_{-1}^{1} g(x)P_i(x)dx, \quad i = 0, 1, \cdots$$

(iii). **Chebyshev's expansion**

Chebyshev's polynomials of the first kind [10] will be used to expand $g(x)$ as follows

$$g(x) = \sum_{n=0}^{m} c_n T_n(x), \tag{3.6}$$

where $T_n(x)$ are the orthogonal Chebyshev's polynomial of the first kind; while the coefficient of Chebyshev's expansion c_i are expressed as follows

$$c_0 = \frac{1}{\pi}\int_{-1}^{1} \frac{g(x)T_0(x)}{\sqrt{1-x^2}}dx, \quad c_i = \frac{2}{\pi}\int_{-1}^{1} \frac{g(x)T_i(x)}{\sqrt{1-x^2}}dx, i = 1, 2, \cdots \tag{3.7}$$

(iv). **Gegenbauer's expansion**

Gegenbauer's polynomials will be utilized to express the source term $g(x)$ via the Gegenbauer's series [12] as follows

$$g(x) = \sum_{n=0}^{m} c_n C_n^{\alpha}(x), \tag{3.8}$$

where $C_n^{\alpha}(x)$ are the orthogonal Gegenbauer's polynomials, and the coefficients of Gegenbauer's expansion c_i are defined as follows

$$c_i = \frac{\int_{-1}^{1} g(x)C_i^{\alpha}(x)(1-x^2)^{\alpha-1/2}dx}{\int_{-1}^{1}[C_i^{\alpha}(x)]^2(1-x^2)^{\alpha-1/2}dx}, \quad i = 0, 1, 2, \cdots \tag{3.9}$$

where the normalization of the functions are done using as $(1-x^2)^{\alpha-1/2}$ as the weight function.

(v). **Jacobi's expansion**

Considering Jacobi's orthogonal polynomials over $[\check{}1, 1]$, we in the same way decompose the source term $g(x)$ as follows [12]

$$g(x) = \sum_{n=0}^{m} c_n P_n^{(\alpha,\beta)}(x), \tag{3.10}$$

where $\alpha, \beta > -1$ and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi's polynomials that are orthogonal, and the coefficients c_i of Jacobi's expansion are defined as follows

$$c_i = \frac{\int_{-1}^{1} g(x)P_i^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta}dx}{\int_{-1}^{1}[P_i^{(\alpha,\beta)}(x)]^2(1-x)^{\alpha}(1+x)^{\beta}dx}, \qquad i = 0, 1, 2, \cdots \tag{3.11}$$

where the weight function $(1-x)^{\alpha}(1+x)^{\beta}$ is used for the normalization in this equation.

Therefore, with the aforementioned expansions for $g(x)$, we proceed further to express the governing model in Eq. (3.1) as follows

$$Lu = g(x) - G(x, u), \tag{3.12}$$

Therefore, operating L^{-1} on Eq. (3.12), one gets

$$u(x) = \phi(x) + L^{-1}g(x) - L^{-1}G(x, u),$$

with

$$L\phi(x) = 0.$$

Finally, the procedure continues as in the standard Adomian's method with the following recurrent relation

$$\begin{cases} u_0 = \phi(x) + L^{-1}g(x), \\ u_{n+1} = -L^{-1}A_n, \quad n \geq 0, \end{cases} \tag{3.13}$$

where practical solutions via these polynomials coupled with the inverse differential operator are obtained recurrently by considering m-term approximations using $u(x) = \sum_{n=0}^{m} u_n$.

4. Numerical examples

This section demonstrates the application of the proposed modified Adomian decomposition via orthogonal polynomials and the inverse differential operator. Three Initial-Value Problems (IVPs) of ordinary differential equations will be considered as test examples. Additionally, we shall utilize seven-term approximations via the Maple 18 package programmer for the computational simulation.

Example 4.1. *Consider the nonlinear IVP [17]*

$$\begin{gathered} u'' + xu' + x^2u^3 = (2 + 6x^2)e^{x^2} + x^2e^{3x^2}, \\ u(0) = 1, \quad u'(0) = 0, \end{gathered} \tag{4.1}$$

that admits the exact solution $u(x) = e^{x^2}$.

Firstly, we consider the following direct and its corresponding inverse operators

$$L = e^{-\frac{x^2}{2}} \frac{d}{dx} e^{\frac{x^2}{2}} \frac{d}{dx}, \qquad L^{-1}(.) = \int_0^x e^{-\frac{x^2}{2}} \int_0^x e^{\frac{x^2}{2}} (.)dxdx.$$

So, we express Eq. (4.1) in an operator notation as follows

$$Lu = g(x) - x^2u^3, \tag{4.2}$$

where $g(x) = (2 + 6x^2)e^{x^2} + x^2e^{3x^2}$.

More so, applying the inverse operator L^{-1} to both sides of Eq. (4.2), one gets

$$u = u(0) + xu'(0) + L^{-1}(g(x)) - x^2 * L^{-1}(A_n),$$

where A_n's denote the Adomian polynomials associated with the nonlinear term u^3.

$$\begin{cases} A_0 = u_0^3, \\ A_1 = 3u_0^2 u_1, \\ A_2 = 3u_0^2 u_2 + 3u_0 u_1^2, \\ A_3 = 3u_0^2 u_3 + 6u_0 u_1 u_2 + u_1^3, \\ \vdots \end{cases} \tag{4.3}$$

So, proceeding as presented in the methodology, the recurrent scheme is thus obtained as follows

$$\begin{cases} u_0 = u(0) + xu'(0) + L^{-1}(g(x)), \\ u_{n+1} = -x^2 L^{-1}(A_n), \quad n \geq 0. \end{cases}$$

So, in what follows, we give different approximate solutions based on the decomposition of the function $g(x)$. These solutions are in line with the proposed modification methods through the application of the respective orthogonal functions. Additionally, Taylor's series expansion solution will first be provided to give more scope for comparison.

Taylor's expansion

Now, expanding the function $g(x)$ using Taylor's series expansion for $m = 6$, one gets

$$g(x) = 2 + 9x^2 + 10x^4 + \frac{47}{6}x^6.$$

Therefore, with the presence of $g(x)$ in form of Taylor's series, we equally expand $e^{-\frac{x^2}{2}}$ and $e^{\frac{x^2}{2}}$ via the same Taylor's expansion for $m = 6$ and thereafter make use of the proposed scheme to obtain the following solution components

$$\begin{aligned} u_0 &= 1 + x^2 + \tfrac{7}{12}x^4 + \tfrac{23}{90}x^6 + \cdots \\ u_1 &= -\tfrac{1}{12}x^4 - \tfrac{4}{45}x^6 + \cdots \\ &\vdots \end{aligned}$$

that sums to the following series solution

$$u_t(x) = \sum_{n=0}^{6} u_n(x) = 1 + x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \cdots \tag{4.4}$$

Legendre's expansion

Using the Legendre's expansion on the function $g(x)$, we use the following formula for $m = 6$

$$g(x) = \sum_{n=0}^{6} c_n P_n(2x-1), \quad 0 \leq x \leq 1,$$

where

$$c_i = \frac{2i+1}{2}\int_{-1}^{1} g(0.5x + 0.5)P_i(x)dx, \quad i = 0, 1, \cdots$$

Then, the approximate function is thus obtained as follows

$$g(x) \approx 2.071456258 - 3.74043326x + 55.32635998x^2 + \cdots$$

Additionally, with the presence of $g(x)$ in form of Legendre's polynomials, we expand $e^{-\frac{x^2}{2}}$ and $e^{\frac{x^2}{2}}$ via the Taylor's expansion for $m = 6$ and thereafter make use of the proposed scheme to obtain the following solution components

$$\begin{aligned}
u_0 &= 1 + 1.035728129x^2 - 0.6234055433x^3 + 4.437908643x^4 + \cdots \\
u_1 &= -0.08333333332x^4 - 0.09246170179x^6 + \cdots \\
&\vdots
\end{aligned}$$

that leads to the following series solution

$$u_P(x) = \sum_{n=0}^{6} u_n(x) = 1 + 1.035728129x^2 - 0.6234055433x^3 + 4.354575310x^4 + \cdots \tag{4.5}$$

Chebyshev's expansion

Using the Chebyshev's expansion on the function $g(x)$, we use the following formula for $m = 6$

$$g(x) = \sum_{n=0}^{6} c_n T_n(2x-1), \quad 0 \le x \le 1,$$

where

$$c_0 = \tfrac{1}{\pi}\int_{-1}^{1} \frac{g(0.5x+0.5)T_0(x)}{\sqrt{1-x^2}}dx \quad c_i = \tfrac{2}{\pi}\int_{-1}^{1} \frac{g(0.5x+0.5)T_i(x)}{\sqrt{1-x^2}}dx, \quad i = 1, 2, \cdots$$

Thus, $g(x)$ takes the following approximate form

$$g(x) \approx 2.031640879 - 2.89636358x + 51.4781262x^2 + \cdots$$

What's more, we get the following solution components

$$\begin{aligned}
u_0 &= 1 + 1.015820440x^2 - 0.4827272633x^3 + 4.120540450x^4 + \cdots \\
u_1 &= -0.08333333332x^4 - 0.09047093289x^6 + \cdots \\
&\vdots
\end{aligned}$$

that gives the series solution

$$u_T(x) = \sum_{n=0}^{6} u_n(x) = 1 + 1.015820440x^2 - 0.4827272633x^3 + 4.03720711x^4 + \cdots \tag{4.6}$$

Gegenbauer's expansion

Through the Gegenbauer's expansion, we expand $g(x)$ for $m = 6$ as follows

$$g(x) = \sum_{n=0}^{6} c_n C_n^{\alpha}(2x-1), \quad 0 \le x \le 1,$$

where

$$c_i = \frac{\int_{-1}^{1} g(0.5x+0.5)C_i^{\alpha}(x)(1-x^2)^{\alpha-1/2}dx}{\int_{-1}^{1} [C_i^{\alpha}(x)]^2(1-x^2)^{\alpha-1/2}dx}, \quad i = 0, 1, 2, \cdots$$

and get explicitly for $\alpha = 1$ the following expression

$$g(x) \approx 2.114011841 - 4.47896860x + 58.34871046x^2 + \cdots$$

The solution components are thus determined as

$$\begin{aligned} u_0 &= 1 + 1.057005920x^2 - 0.7464947667x^3 + 4.686224885x^4 + \cdots \\ u_1 &= -0.08333333332x^4 - 0.09458948089x^6 + \cdots \\ &\vdots \end{aligned}$$

that sums to the following series solution

$$u_g(x) = \sum_{n=0}^{6} u_n(x) = 1 + 1.057005920x^2 - 0.7464947667x^3 + 4.602891552x^4 + \cdots \tag{4.7}$$

Jacobi's expansion

Moreover, through the Jacobi's expansion, we expand $g(x)$ for $m = 6$ as follows

$$g(x) = \sum_{n=0}^{6} c_n P_n^{(\alpha,\beta)}(2x-1), \quad 0 \leq x \leq 1,$$

where

$$c_i = \frac{\int_{-1}^{1} g(0.5x+0.5) P_i^{(\alpha,\beta)}(x)(1-x)^{\alpha}(1+x)^{\beta} dx}{\int_{-1}^{1} [P_i^{(\alpha,\beta)}(x)]^2 (1-x)^{\alpha}(1+x)^{\beta} dx}, \quad i = 0, 1, 2, \cdots$$

such that for $\alpha, \beta = 1$, the following explicit expression is obtained

$$g(x) \approx 2.1562947 - 5.119789x + 60.745654x^2 + \cdots$$

Therefore, the resulting solution components take the form

$$\begin{aligned} u_0 &= 1 + 1.078147350x^2 - 0.8532981667x^3 + 4.882446608x^4 + \cdots \\ u_1 &= -0.08333333332x^4 - 0.09670228889x^6 + \cdots \\ &\vdots \end{aligned}$$

that leads to the series solution

$$u_j(x) = \sum_{n=0}^{6} u_n(x) = 1 + 1.078147350x^2 - 0.8532981667x^3 + 4.799113275x^4 + \cdots \tag{4.8}$$

Hence, we report the absolute error differences between the exact solution $u(x)$ and the respective approximate solutions in Table 1. In this table, $u_t(x)$ stands for the modification via the Taylor's expansion; $u_P(x)$ via the Legendre's expansion; $u_T(x)$ via the Chebyshev's expansion; $u_g(x)$ via the Gegenbauer's expansion $(\alpha = 1)$; and $u_j(x)$ via the Jacobi's expansion $(\alpha = 1, \beta = 1)$.

TABLE 1. Absolute error comparisons via the proposed modifications for Example 1

x	$\lvert u(x)-u_t(x)\rvert$	$\lvert u(x)-u_P(x)\rvert$	$\lvert u(x)-u_T(x)\rvert$	$\lvert u(x)-u_g(x)\rvert$	$\lvert u(x)-u_j(x)\rvert$
0	0	0	0	0	0
0.25	6.1×10^{-8}	6.3648×10^{-5}	1.59577×10^{-4}	1.88664×10^{-4}	5.27907×10^{-4}
0.50	6.7513×10^{-5}	2.0067×10^{-5}	1.4384×10^{-4}	4.65886×10^{-4}	1.09333×10^{-3}
0.75	4.4539×10^{-3}	1.0382×10^{-5}	2.66662×10^{-4}	5.87464×10^{-4}	1.42133×10^{-3}
1	9.5528×10^{-2}	2.011028×10^{-3}	1.8978×10^{-3}	1.69942×10^{-3}	8.23638×10^{-4}

Example 4.2. *Consider the nonlinear IVP [18]*

$$\begin{aligned} u'' + \tfrac{\sin x}{x}u' + u^2 &= (x^2+\tfrac{1}{x})\sin^2 x + \tfrac{1}{2}\sin(2x) - x\sin x + 2\cos x, \\ u(0) = 0, \quad u'(0) &= 0, \end{aligned} \tag{4.9}$$

that admits the exact solution $u(x) = x\sin x$.

We start off by expressing the linear operator L as follows

$$L^{-1}(.) = \int_0^x e^{-\int p(x)dx} \int_0^x e^{\int p(x)dx}(.)dxdx, \tag{4.10}$$

while $e^{-\int p(x)dx}$ and $e^{\int p(x)dx}$ through Eq. (4.9) as follows

$$\begin{aligned} e^{-\int p(x)dx} &= e^{-\int \frac{\sin x}{x}dx} \approx e^{-x+\frac{1}{18}x^3-\frac{1}{600}x^5+\frac{1}{35280}x^7}. \\ e^{\int p(x)dx} &= e^{\int \frac{\sin x}{x}dx} \approx e^{x-\frac{1}{18}x^3+\frac{1}{600}x^5-\frac{1}{35280}x^7}, \end{aligned} \tag{4.11}$$

In addition, substitution of the Taylor's series expansion of Eq. (4.11) (of order 6) into Eq. (4.10) yields

$$L^{-1}(.) = \int_0^x X(x) \int_0^x Y(x)(.)dxdx$$

where,

$$X(x) = 1 - x + \frac{1}{2}x^2 - \frac{1}{9}x^3 - \frac{1}{72}x^4 + \frac{4}{225}x^5 - \frac{151}{32400}x^6,$$

$$Y(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{9}x^3 - \frac{1}{72}x^4 - \frac{4}{225}x^5 - \frac{151}{32400}x^6.$$

Then, we write the model given in Eq. (4.9) in operator notation as follows

$$Lu = g(x) - u^2, \tag{4.12}$$

where $g(x) = (x^2+\frac{1}{x})\sin^2 x + \frac{1}{2}\sin(2x) - x\sin x + 2\cos x$.

More so, upon applying the inverse operator L^{-1} into both sides of Eq. (4.12), one gets

$$u = u(0) + xu'(0) + L^{-1}(g(x)) - L^{-1}(A_n)$$

where A_n's represent the Adomian polynomials associated with the nonlinear term u^2.

Therefore, the method yield the following overall recurrent scheme

$$\begin{cases} u_0 = u(0) + xu'(0) + L^{-1}(g(x)), \\ u_{n+1} = -L^{-1}(A_n), \quad n \geq 0. \end{cases}$$

So, in what follows, we give different approximate solutions based on the decomposition of the function $g(x)$. These solutions are in line with the proposed modification methods through the application of the respective orthogonal functions. Additionally, the Taylor's series expansion solution will first be provided to give more scope for comparison.

Accordingly, we report only the obtained series expansion for $g(x)$ and the approximate series solution in each case to suppress the working as follows:

Taylor's expansion

$$g(x) \approx 2 + 2x - 2x^2 - x^3 + \frac{5}{4}x^4 + \frac{8}{45}x^5 - \frac{31}{90}x^6$$

$$u_t(x) = \sum_{n=0}^{6} u_n(x) = x^2 - \frac{1}{6}x^4 + \frac{1}{120}x^6 + \cdots \tag{4.13}$$

Legendre's expansion

$$g(x) \approx 2.000033382 + 1.998157591x - 1.975644053x^2 + \cdots$$

$$u_P(x) = \sum_{n=0}^{6} u_n(x) = 1.000016691x^2 - 0.0003126320000x^3 - 0.1645588463x^4 + \cdots \tag{4.14}$$

Chebyshev's expansion

$$g(x) \approx 2.000013982 + 1.998649195x - 1.978831555x^2 + \cdots$$

$$u_T(x) = \sum_{n=0}^{6} u_n(x) = 1.000006991x^2 - 0.0002274646667x^3 - 0.1648457633x^4 - 0.006513456376x^5 + 0.02050074912x^6 + \cdots \tag{4.15}$$

Gegenbauer's expansion

$$g(x) \approx 2.000055948 + 1.997684458x - 1.972795597x^2 + \cdots$$

$$u_g(x) = \sum_{n=0}^{6} u_n(x) = 1.000027974x^2 - 0.0003952483333x^3 - 0.1643008210x^4 - 0.007368497593x^5 + 0.02114802609x^6 + \cdots \tag{4.16}$$

Jacobi's expansion

$$g(x) \approx 2.000080135 + 1.997236537x - 1.970243595x^2 + \cdots$$

$$u_j(x) = \sum_{n=0}^{6} u_n(x) = 1.000040068x^2 - 0.0004739330000x^3 - 0.1640684830x^4 + \cdots$$

Hence, we report the absolute error differences between the exact solution $u(x)$ and the respective approximate solutions in Table 2. In this table, $u_t(x)$ stands for the modification via the Taylor's expansion; $u_P(x)$ via the Legendre's expansion; $u_T(x)$ via the Chebyshev's expansion; $u_g(x)$ via the Gegenbauer's expansion ($\alpha = 1$); and $u_j(x)$ via the Jacobi's expansion ($\alpha = 1, \beta = 1$).

Table 2. Absolute error comparisons via the proposed modifications for Example 2

x	$\lvert u(x) - u_t(x) \rvert$	$\lvert u(x) - u_P(x) \rvert$	$\lvert u(x) - u_T(x) \rvert$	$\lvert u(x) - u_g(x) \rvert$	$\lvert u(x) - u_j(x) \rvert$
0	0	0	0	0	0
0.25	4.5×10^{-10}	2.0930000×10^{-8}	4.952×10^{-8}	8.052×10^{-8}	$2.32560000 \times 10^{-7}$
0.50	4.84×10^{-8}	1.3000000×10^{-9}	6.18×10^{-8}	1.774×10^{-7}	$4.43600000 \times 10^{-7}$
0.75	1.23218×10^{-5}	1.9093000×10^{-6}	1.9921×10^{-6}	1.6831×10^{-6}	$1.33740000 \times 10^{-6}$
1	2.91925×10^{-4}	5.1750100×10^{-5}	5.1843×10^{-5}	5.1488×10^{-5}	$5.10952000 \times 10^{-5}$

Example 4.3. *Consider the following nonlinear IVP [19]*

$$\begin{aligned} u'' + u' + u + u^3 &= \cos^3 x - \sin x, \\ u(0) = 1, \quad u'(0) &= 0, \end{aligned} \tag{4.17}$$

that admits the exact solution $u(x) = \cos x$.

As precede, we make use of the following direct and its corresponding inverse operators as follows

$$L = e^{-x}\frac{d}{dx}e^{x}\frac{d}{dx}, \qquad L^{-1}(.) = \int_0^x e^{-x} \int_0^x e^{x}(.)dxdx,$$

in such a way that Eq. (4.17) is expressed via this operator as follows

$$Lu = g(x) - u - u^3, \qquad g(x) = \cos^3 x - \sin x. \tag{4.18}$$

More so, we apply the inverse operator L^{-1} to both sides of Eq. (4.18) to obatin

$$u = u(0) + xu'(0) + L^{-1}(g(x)) - L^{-1}(u_n) - L^{-1}(A_n),$$

where A_n's are the polynomials by Adomian for the nonlinear expression u^3.

Finally, the resulting recurrent scheme is thus obtained without further delay as follows

$$\begin{cases} u_0 = u(0) + xu'(0) + L^{-1}(g(x)), \\ u_{n+1} = -L^{-1}(u_n) - L^{-1}(A_n), \quad n \geq 0. \end{cases} \tag{4.19}$$

Proceeding as in the previous examples, we report the approximate series expansion of $g(x)$ and the approximate series solution in each modification as follows:

Taylor's expansion

$$g(x) \approx 1 - x - \frac{3}{2}x^2 + \frac{1}{6}x^3 + \frac{7}{8}x^4 - \frac{1}{120}x^5 - \frac{61}{240}x^6,$$
$$u_t(x) = \sum_{n=0}^{6} u_n(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \cdots \tag{4.20}$$

Legendre's expansion

$$g(x) \approx 1.000029103 - 1.001617626x - 1.478418664x^2 + \cdots$$
$$u_P(x) = \sum_{n=0}^{6} u_n(x) = 1 - 0.4999854485x^2 - 0.0002744548x^3 + 0.0435288744x^4 + \cdots \tag{4.21}$$

Chebyshev's expansion

$$g(x) \approx 1.000012159 - 1.001183038x - 1.481288191x^2 + \cdots$$
$$u_T(x) = \sum_{n=0}^{6} u_n(x) = 1 + 0.49999392x^2 - 0.0001991994x^3 + 0.043273757x^4 + \cdots \tag{4.22}$$

Gegenbauer's expansion

$$g(x) \approx 1.000048886 - 1.002037497x - 1.475843723x^2 + \cdots$$
$$u_g(x) = \sum_{n=0}^{6} u_n(x) = 1 - 0.4999755570x^2 - 0.0003477304x^3 + 0.043758474x^4 + \cdots \tag{4.23}$$

Jacobi's expansion

$$g(x) \approx 1.000070161 - 1.002436342x - 1.473528280x^2 + \cdots$$
$$u_j(x) = \sum_{n=0}^{6} u_n(x) = 1 - 0.4999649195x^2 - 0.0004177504x^3 + 0.0439653874x^4 + \cdots \tag{4.24}$$

In the same manner, we report the absolute error differences between the exact solution $u(x)$ and the respective approximate solutions in Table 3. In this table, $u_t(x)$ stands for the modification via the Taylor's expansion series; $u_P(x)$ via the Legendre's expansion; $u_T(x)$ via the Chebyshev's expansion; $u_g(x)$ via the Gegenbauer's expansion ($\alpha = 1$); and $u_j(x)$ via the Jacobi's expansion ($\alpha = 1, \beta = 1$).

Table 3. Absolute error comparisons via the proposed modifications for Example 3

x	$\lvert u(x) - u_t(x)\rvert$	$\lvert u(x) - u_P(x)\rvert$	$\lvert u(x) - u_T(x)\rvert$	$\lvert u(x) - u_g(x)\rvert$	$\lvert u(x) - u_j(x)\rvert$
0	0	0	0	0	0
0.25	5.0×10^{-10}	1.7300000×10^{-8}	3.97×10^{-8}	6.65×10^{-8}	$1.93600000 \times 10^{-7}$
0.50	2.819×10^{-7}	1.3300000×10^{-7}	8.63×10^{-8}	2.63×10^{-7}	4.6200000×10^{-7}
0.75	2.03161×10^{-5}	4.3186900×10^{-5}	4.31366×10^{-5}	4.33261×10^{-5}	4.3543800×10^{-5}
1	1.64997×10^{-3}	$2.03334810 \times 10^{-3}$	2.0333×10^{-3}	2.03348×10^{-3}	2.0336930×10^{-3}

5. Conclusion

In conclusion, the present manuscript proposed a proficient Adomian decomposition method by incorporating inverse differential operator, and orthogonal polynomials to solve nonlinear differential models. The method was indeed a reformation of the standard Adomian method thereby improving the rapidity of the solution's convergence rate upon defining inverse differential operator L^{-1}, and the incorporation of orthogonal polynomials including the Legendre's, Chebyshev's, Gegenbauer's, and Jacobi's polynomials; in addition to the Taylor's series. A generalized recurrent scheme for a general nonlinear model was derived and further utilized to solve certain nonlinear test models. The reported numerical results via comparative tables demonstrated the absolute error differences between the exact and approximate solutions. Lastly, it is remarkable that the present study gives more accurate options to obtaining approximate exact solutions within the framework of the original Adomian approach.

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