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Paley Wiener Theorem on a Reductive Lie Group

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Abstract. Let G be a locally compact group, K a maximal compact subgroup of G and δ on arbitrary class of irreducible unitary representations of K. The spherical Grassmannian $\mathcal{G}_{p,\delta}$ is an equivalence class of spherical functions of type $\delta-$positive of height p. In this work, we give an extension of orbital integral with respect to δ, when G is reductive Lie group. Moreover, if the discret serie is not empty, we give an extension of Paley-Wiener theorem using a compact Cartan subgroup of G.

1. Introduction

Let G be a reductive Lie group with non empty discret serie, and K be a compact subgroup of G. We denote by $\widehat{K}$ the unitary dual of K and by $\mathcal{K}(G)$ the subspace of continuous complex functions on G with compact support.

For all class δ of $\widehat{K}$, we denote by $I_{c,\delta}(G)$, the subspace of $\mathcal{K}(G)$ containing K-$\delta-$ invariant central functions on G.

Let E be a space of representation of δ.

Let set $\chi_\delta = d(\delta)\zeta_\delta where\, d(\delta)$ is the degree of δ and ζ_δ the character of δ .

A spherical function of type δ is a quasi-bounded continuous and central function ϕ on G with values in $End_{\mathbb{C}}(E)$ such that:

$$\chi_\delta \star \phi = \phi = \phi \star \chi_\delta \text{ and the map the}$$

$$u_\phi : f \mapsto u_\phi(f) = \int_G f(x)\phi(x^{-1})dx$$

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is an irreducible representation of the algebra $I_{c,\delta}(G)$.
The spherical Grassmannian $\mathcal{G}_{p,\delta}(G)$ of order p according to δ is the set of classes of spherical functions on G of type δ of height p.

2. The δ-Orbital Integral

Let G be a reductive Lie group with non-empty discret serie, $\mathcal{G}$ the Lie algebra of G ($\mathcal{G} = \mathrm{Lie}(G)$). Let H be a Cartan compact subgroup of G and $\mathcal{H}$ the associated Cartan subalgebra, $\mathcal{U} \subset G$ a completely invariant open set, γ a regular element of $\mathcal{U}$ ($\gamma \in \mathcal{U}_{reg}$).
One denotes by H^0 the identity component of H and $K = \gamma H^0$. $\mathcal{K} = Lie(K)$, K is a compact and normal Cartan subgroup of G and the Weyl group $W(G,K)$ acts on $G/K \times K_{reg}$, $d\overline{g}$ a measure of G which is invariant on G/K
If $\check{\delta}$ is the class of contragredient representation of δ in $\widehat{K}$, we have $\overline{\chi_\delta} = \chi_{\check{\delta}}$ and we can verify easily, thanks to Schur orthogonality relationships, that $\chi_{\check{\delta}} \star \chi_{\check{\delta}} = \chi_{\check{\delta}}$.
Let $\mathcal{K}(\mathcal{U})$ be the subspace of continuous complex functions on $\mathcal{U}$ with compact support and $\mathcal{D}(\mathcal{U})$ the subspace of indefinitely differentiable functions on $\mathcal{U}$.
Identifyting χ_δ to a bounded measure on G, we put, for any function $f \in \mathcal{K}(\mathcal{U})$,

$$_\delta f(x) = \overline{\chi_\delta} \star f(x) = \int_K \chi_{\check{\delta}}(k) f(kx) dk$$

$$f_\delta(x) = f \star \chi_\delta(x) = \int_K \chi_\delta(k^{-1}) f(xk) dk$$

$$\text{and } I_\delta(\mathcal{U}) = \{f \in \mathcal{K}(\mathcal{U}) : f =_\delta f = f_{\check{\delta}}\}.$$

Let $\mathcal{J}_c(\mathcal{U})$ be the set of the $K-$central functions of $\mathcal{K}(\mathcal{U})$ (ie $f(kx) = f(xk), \ \forall x \in \mathcal{U}, k \in K$).
$I^\infty_{c,\delta}(\mathcal{U}) = I_{c,\delta}(\mathcal{U}) \cap \mathcal{D}(\mathcal{U})$ where $I_{c,\delta}(\mathcal{U}) = I_\delta(\mathcal{U}) \cap \mathcal{J}_c(\mathcal{U})$.

Remark 2.1. Denote by $u_{\check{\delta}}$ an irreducible unitary representation of K in the dual class $\check{\delta}$ on a space $E_{\check{\delta}}$; for an arbitrary endomorphism T of $E_{\check{\delta}}$ defined by the number $\sigma(T) = d(\check{\delta}) tr(T)$ then, thanks to Schur Orthogonality Relations, for any

$$T \in F_{\check{\delta}} = Hom_{\mathbb{C}}(E_{\check{\delta}}, E_{\check{\delta}}) \text{ one has}$$

$$T = \int_K u_{\check{\delta}}(k^{-1}) \sigma(u_{\check{\delta}}(k) T) dk$$

In the sequel, we assume that $\mathcal{U}_{reg} = \mathcal{U}$.
Let $f \in \mathcal{D}(\mathcal{U}_{reg})$, the function $\gamma \mapsto J_G(f)(\gamma)$ defined by .

$$J_G(f)(\gamma) = \mid det(1 - Ad(\gamma^{-1}))_{\mathcal{G}/\mathcal{K}} \mid^{1/2} \int_{G/K} f(g.\gamma) d\overline{g}$$

is the classical orbital integral of f. The function $\gamma \mapsto J_G(f)(\gamma)$ is G- invariant distribution on $C^\infty(\mathcal{U}_{reg})$.
The map $J_G : \mathcal{D}(\mathcal{U}) \to I(\mathcal{U})$=$J_G(\mathcal{U})$ is linear, continuous and its transpose ${}^tJ_G$ is a bijection of

$I(\mathcal{U})'$ on $\mathcal{D}(\mathcal{U})'$ the space of the G- invariant distribution on $\mathcal{U}$.

$$\begin{aligned} {}^tJ_G : & I(\mathcal{U})' \to \mathcal{D}(\mathcal{U})' \\ & \theta \mapsto {}^t J_G(\theta) \end{aligned}$$

$\forall \theta \in I(\mathcal{U})'$, ${}^tJ_G(\theta)$ is a $G-$ invariant distribution on $\mathcal{U}$.
If $f \in I^{\infty}_{c,\delta}(\mathcal{U}_{reg})$ and thanks to the isomorphism $f \mapsto \Psi^{\delta}_f$ of $I^{\infty}_{c,\delta}(\mathcal{U})$ onto $U^{\infty}_{c,\delta}(\mathcal{U})$,
we can put: $J^{\delta}_G(f)(\gamma) =| det(1 - Ad(\gamma^{-1}))_{\mathcal{G}/\mathcal{K}} |^{1/2} \int_{G/K} \Psi^{\delta}_f({}^g\gamma) d\overline{g}$
where $\Psi^{\delta}_f({}^g\gamma) = \int_K u_{\check{\delta}}(k^{-1}) f(k^g\gamma) dk$. The $\delta-$ orbital integral $J^{\delta}_G(f)$ of f is defined by
$J^{\delta}_G(f)(\gamma) =| det(1 - Ad(\gamma^{-1}))_{\mathcal{G}/\mathcal{K}} |^{1/2} \int_K \int_{G/K} u_{\check{\delta}}(k^{-1}) f(k^g\gamma) d\overline{g} dk$

Remark 2.2. If δ is trivial and one dimension, we obtain the classical orbital integral.

Theorem 2.3. *Let $f \in I^{\infty}_{c,\delta}(\mathcal{U})$. The map:*

$$\begin{aligned} J^{\delta}_G(f) : & \mathcal{U} \to F_{\delta} \\ & \gamma \mapsto J^{\delta}_G(f)(\gamma) \end{aligned}$$

is $K - \delta -$invariant.
proof.
Let's put $M(\gamma) = |det(1 - Ad(\gamma^{-1}))_{\mathcal{G}/\mathcal{K}}|^{1/2}$.
we have:

$$\begin{aligned} M(k\gamma k^{-1}) &= \left| det(1 - Ad(k\gamma^{-1}k^{-1}))_{\mathcal{G}/\mathcal{K}} \right|^{1/2} \\ &= \left| det\left(Ad(k)Ad(k^{-1})Ad(k)Ad(k^{-1}) - Ad(k\gamma^{-1}k^{-1}) \right)_{\mathcal{G}/\mathcal{K}} \right|^{1/2} \\ &= \left| det\left(Ad(k)Ad(k^{-1})Ad(k)Ad(k^{-1}) - Ad(k)Ad(\gamma^{-1})Ad(k^{-1}) \right)_{\mathcal{G}/\mathcal{K}} \right|^{1/2} \\ &= \left| det\left(Ad(k)\left[Ad(k^{-1})Ad(k) - Ad(\gamma^{-1}) \right] Ad(k^{-1}) \right)_{\mathcal{G}/\mathcal{K}} \right|^{1/2} \\ M(k\gamma k^{-1}) &= \left| det(1 - Ad(\gamma^{-1}))_{\mathcal{G}/\mathcal{K}} \right|^{1/2} \\ so\ M(k\gamma k^{-1}) &= M(\gamma) \end{aligned}$$

Let's show that $J^{\delta}_G(f)$ is K-central

$$\begin{aligned} J^{\delta}_G(f)(k\gamma k^{-1}) &= M(\gamma) \int_{G/K} \Psi^{\delta}_f({}^g k\gamma k^{-1}) d\overline{g} \\ &= M(\gamma) \int_{G/K} \int_K u_{\check{\delta}}(\tilde{k}^{-1}) f(\tilde{k} g k\gamma k^{-1} g^{-1}) d\tilde{k} d\overline{g} \\ &= M(\gamma) \int_K \int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1}) f(\tilde{k}(gk).\gamma) d\tilde{k} d\overline{g} \end{aligned}$$

$$\begin{aligned}
J_G^\delta(f)(k\gamma k^{-1}) &= M(\gamma)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})f(\tilde{k}g.\gamma)d\overline{g}d\tilde{k}, \quad d\overline{g} \textit{ is a } G-\textit{invariant mesure on } G/K\\
&= M(\gamma)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})f(\tilde{k}g.\gamma)d\overline{g}d\tilde{k}\\
&= M(\gamma)\int_{G/K}\left(\int_K u_{\check{\delta}}(\tilde{k}^{-1})f(\tilde{k}g.\gamma)d\tilde{k}\right)d\overline{g}\\
&= M(\gamma)\int_{G/K}\Psi_f^\delta({}^g\gamma)d\overline{g} = J_G^\delta(f)(\gamma)\\
\textit{so } J_G^\delta(f)(k\gamma k^{-1}) &= J_G^\delta(f)(\gamma)
\end{aligned}$$

Let's show that $\chi_\delta \star J_G^\delta(f)(\gamma) = J_G^\delta(f)(\gamma)$.

$$\begin{aligned}
\chi_\delta \star J_G^\delta(f)(\gamma) &= \int_K \chi_\delta(k)J_G^\delta(f)(k^{-1}\gamma)dk\\
&= \int_K \chi_\delta(k)\left(M(k^{-1}.\gamma)\int_{G/K}\Psi_f^\delta({}^g k^{-1}.\gamma)d\overline{g}\right)dk\\
&= \int_K \chi_\delta(k)\left(M(k^{-1}\gamma k)\int_{G/K}\Psi_f^\delta({}^g k^{-1}.\gamma)d\overline{g}\right)dk\\
&= \int_K \chi_\delta(k)\left(M(\gamma)\int_{G/K}\Psi_f^\delta({}^g k^{-1}.\gamma)d\overline{g}\right)dk\\
&= M(\gamma)\int_K\chi_\delta(k)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})f(\tilde{k}(gk^{-1}).\gamma)d\overline{g}d\tilde{k}dk\\
&= M(\gamma)\int_K\chi_\delta(k)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})f(k^{-1}.(\tilde{k}(g.\gamma)))d\overline{g}d\tilde{k}dk\\
&= M(\gamma)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})\left(\int_K\chi_\delta(k)f(k^{-1}.(\tilde{k}(g.\gamma)))\right)dkd\overline{g}d\tilde{k}\\
&= M(\gamma)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})(\chi_\delta\star f)(\tilde{k}(g.\gamma))d\overline{g}d\tilde{k}
\end{aligned}$$

$$\begin{aligned}
\chi_\delta \star J_G^\delta(f)(\gamma) &= M(\gamma)\int_K\int_{G/K} u_{\check{\delta}}(\tilde{k}^{-1})f(\tilde{k}g.\gamma)d\overline{g}d\tilde{k}.\\
&= M(\gamma)\int_{G/K}\Psi_f^\delta(g.\gamma)d\overline{g}\\
&= J_G^\delta(f)(\gamma)
\end{aligned}$$

then $\chi_\delta \star J_G^\delta(f) = J_G^\delta(f)$.
Therefore $J_G^\delta(f)$ *is* $K-\delta$*-invariant.*

In addition we have $\forall f \in \mathcal{D}(\mathcal{U})$, $J_G^\delta(f) \in I_{c,\delta}^\infty(\mathcal{U}, F_\delta)$.

Consider the space $\Gamma_\delta(\mathcal{U}, F_\delta) = I_{c,\delta}^\infty(\mathcal{U}, F_\delta) \cap I(\mathcal{U}, F_\delta)$ *with* $I(\mathcal{U}, F_\delta) = J_G^\delta(\mathcal{D}(\mathcal{U}, F_\delta))$ $\square$

Theorem 2.4. *The map* $J_G^\delta : I_{c,\delta}^\infty(\mathcal{U}) \longrightarrow \Gamma_\delta(\mathcal{U}, F_\delta)$
$f \longmapsto J_G^\delta(f)$
is linear and surjective.

proof.

J_G^δ *is linear because of the linearity of integral* $M(k.\gamma) = M(\gamma)$
Let's show that J_G^δ *is surjective.*
If $\varphi \in \Gamma_\delta(\mathcal{U}, F_\delta)$ *then* $\varphi \in I(\mathcal{U}, F_\delta)$ *and there exists* $f \in \mathcal{D}(\mathcal{U}, F_\delta)$ *such that* $J_G^\delta(f) = \varphi$, f_K *is* $K-$*central and* $\chi_\delta \star f_K \in I_{c,\delta}^\infty(\mathcal{U})$
$\chi_\delta \star f_K \in I_{c,\delta}^\infty(\mathcal{U}) \Rightarrow \chi_\delta \star f_K \in \mathcal{D}(\mathcal{U})$.

$$\begin{aligned}
J_G^\delta(\chi_\delta \star f_K)(\gamma) &= M(\gamma)\int_{G/K}(\chi_\delta \star f_K)({}^g\gamma)d\overline{g}\\
&= M(\gamma)\int_{G/K}(\chi_\delta \star f_K)(g.\gamma)d\overline{g}\\
&= M(\gamma)\int_{G/K}\left(\int_K \chi_\delta(k) f_K(k^{-1}.(g.\gamma))dk\right)d\overline{g}\\
&= M(\gamma)\int_{G/K}\int_K \chi_\delta(k) f_K((k^{-1}g).(\gamma))dkd\overline{g}\\
&= M(\gamma)\int_{G/K}\int_K \chi_\delta(k)\left(\int_K f\left(k_1((k^{-1}g).\gamma)k^{-1}\right)dk_1\right)dkd\overline{g}\\
&= M(\gamma)\int_K \chi_\delta(k)\int_{G/K}\int_K f(k_1((gk^{-1}).\gamma)k_1^{-1})dk_1 d\overline{g}dk\\
&= M(\gamma)\int_K \chi_\delta(k)\int_K\int_{G/K} f((k_1 g).(k_1^{-1}.\gamma))d\overline{g}dk_1 dk\\
&= \int_K \chi_\delta(k)M(\gamma)\int_{G/K}\int_K f(k_1(g.(k^{-1}.\gamma)))dk_1 d\overline{g}dk\\
&= \int_K \chi_\delta(k)M(\gamma)\int_{G/K} f(g.(k^{-1}.\gamma))d\overline{g}dk\\
&= \int_K \chi_\delta(k)\left(J_G^\delta(f)(k^{-1}.\gamma)\right)dk\\
&= \int_K \chi_\delta(k)J_G^\delta(f)(k^{-1}.\gamma)dk\\
&= \int_K \chi_\delta(k)\varphi(k^{-1}.\gamma)dk\\
&= \chi_\delta \star \varphi(\gamma).
\end{aligned}$$

$J_G^\delta(\chi_\delta \star f_K)(\gamma) = \varphi(\gamma)$, *because* $\varphi \in \Gamma_\delta(\mathcal{U}, F_\delta)$ *then* $\delta-$*invariant.*
Thus, $J_G^\delta(\chi_\delta \star f_K) = \varphi$, *then* J_G^δ *is surjective.*

We have ${}^tJ_G^\delta : I_{c,\delta}^\infty(\mathcal{U}, F_\delta)' \to I_{c,\delta}^\infty(\mathcal{U})'$ its transpose
Clearly if $T \in I_{c,\delta}(\mathcal{U}, F_\delta)'$ then ${}^tJ_G^\delta(T)$ is a G- invariant distribution of type δ.

If $T \in I^{\infty}_{c,\delta}(\mathcal{U})'$, then T is $K-\delta-$invariant.
Let us put $J^{\delta}_{G}(I^{\infty}_{c,\delta}(\mathcal{U})) = I_{\delta}(\mathcal{U}, F_{\delta})$.

Theorem 2.5. *Let H be a Cartan subgroup of G and A its vector subgroup such that $H = KA$. Let $\mathcal{A}$ be the Lie algebra of A.*
Let $\delta \in \widehat{K}$ such that $\int_K u_{\check{\delta}}(k)dk = Id_{E_{\check{\delta}}}$, then the mapping $(u_{\check{\delta}}, \nu) \in \widehat{K} \times \mathcal{A}^*_{\mathbb{C}}$ defined by $(u_{\check{\delta}}, \nu)(kh) = \int_K u_{\check{\delta}}(k_1 k)(k_1.h)^{i\nu} dk_1$ is $K-\delta-$invariant.

Proof. Let's show that $(u_{\check{\delta}}, \nu)$ is K-central.

$$\begin{aligned}
(u_{\check{\delta}}, \nu)_K(\tilde{k}\tilde{h}) &= \int_K (u_{\check{\delta}}, \nu)(k\tilde{k}\tilde{h}k^{-1})dk \\
&= \int_K (u_{\check{\delta}}, \nu)(k\tilde{k}k^{-1}k\tilde{h}k^{-1})dk \\
&= \int_K (u_{\check{\delta}}, \nu)(k\tilde{k}k^{-1}(k.\tilde{h}))dk \\
&= \int_K \int_K u_{\check{\delta}}(k_1 k\tilde{k}k^{-1})(k_1.(k.\tilde{h}))^{i\nu} dk_1 dk \\
&= \int_K \int_K u_{\check{\delta}}(k_1 k) u_{\delta}(\tilde{k}k^{-1})(k_1.k.\tilde{h})^{i\nu} dk_1 dk
\end{aligned}$$

$$u_{\check{\delta}}(\tilde{k}k^{-1}) \in F_{\check{\delta}} \Leftrightarrow u_{\delta}(\tilde{k}k^{-1}) = \int_K u_{\check{\delta}}(\tilde{k}_3^{-1})\sigma(u_{\check{\delta}}(\tilde{k}_3)u_{\check{k}}(\tilde{k}k^{-1}))d\tilde{k}_3$$

$$\begin{aligned}
(u_{\check{\delta}}, \nu)_K(\tilde{k}\tilde{h}) &= \int_K \int_K u_{\check{\delta}}(k_1 k)\left(\int_K u_{\delta}(\tilde{k}_3^{-1})\sigma(u_{\delta}(\tilde{k}_3)u_{\check{k}}(\tilde{k}k^{-1}))d\tilde{k}_3\right) \times (k_1 k.\tilde{h})^{i\nu} dk_1 dk \\
&= \int_K \int_K u_{\check{\delta}}(k_1 k) u_{\check{\delta}}(\tilde{k}_3^{-1})\sigma(u_{\delta}(\tilde{k}_3)u_{\check{k}}(\tilde{k}k^{-1}))(k_1 k.\tilde{h})^{i\nu} d\tilde{k}_3 dk_1 dk \\
&= \int_K \int_K u_{\check{\delta}}(k_1 k) u_{\check{\delta}}(\tilde{k}_3^{-1})\sigma(u_{\delta}(k^{-1}\tilde{k}_3)u_{\check{k}}(\tilde{k}))(k_1 k.\tilde{h})^{i\nu} d\tilde{k}_3 dk_1 dk
\end{aligned}$$

$$\begin{aligned}
(u_{\check{\delta}}, \nu)_K(\tilde{k}\tilde{h}) &= \int_K \int_K \int_K u_{\check{\delta}}(t) u_{\check{\delta}}(\tilde{k}_3^{-1})\sigma(u_{\delta}(k^{-1}\tilde{k}_3)u_{\check{k}}(\tilde{k}))(t.\tilde{h})^{i\nu} dt dk d\tilde{k}_3 \\
&= \int_K \int_K \int_K u_{\check{\delta}}(t) u_{\check{\delta}}(\tilde{k}_3^{-1}k)\sigma(u_{\delta}(k^{-1}\tilde{k}_3)u_{\check{k}}(\tilde{k}))(t.\tilde{h})^{i\nu} dt dk d\tilde{k}_3 \\
&= \int_K \int_K \int_K u_{\check{\delta}}(t) u_{\check{\delta}}((k^{-1}\tilde{k}_3)^{-1})\sigma(u_{\delta}(k^{-1}\tilde{k}_3)u_{\check{k}}(\tilde{k}))(t.\tilde{h})^{i\nu} dt dk d\tilde{k}_3 \\
&= \int_K \left(\int_K \int_K u_{\delta}(t) u_{\delta}((k^{-1}\tilde{k}_3)^{-1})\sigma(u_{\delta}(k^{-1}\tilde{k}_3)u_{\check{k}}(\tilde{k}))dk d\tilde{k}_3\right)(t.\tilde{h})^{i\nu} dt \\
&= \int_K u_{\check{\delta}}(t) u_{\delta}(\tilde{k})(t.\tilde{h})^{i\nu} dt
\end{aligned}$$

$$= \int_K u_{\check{\delta}}(t\tilde{k})(t.\tilde{h})^{iv}dt$$

$$(u_{\check{\delta}},\nu)_K(\tilde{k}\tilde{h}) = (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h}).$$

Then $(u_{\check{\delta}},\nu)$ is K-central.

Let's show that $\overline{\chi}_\delta \star (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h}) = (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h})$

$$\begin{aligned}
\overline{\chi}_\delta \star (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h}) &= \int_K \overline{\chi}_\delta(k)(u_{\check{\delta}},\nu)(k^{-1}\tilde{k}\tilde{h})dk \\
&= \int_K \chi_{\check{\delta}}(k)(u_{\check{\delta}},\nu)(k^{-1}\tilde{k}\tilde{h})dk \\
&= \int_K \chi_{\check{\delta}}(k)\left(\int_K u_{\check{\delta}}(k_1k^{-1}\tilde{k})(k_1.\tilde{h})^{iv}dk_1\right)dk \\
&= \int_K d(\check{\delta})tr(u_{\check{\delta}}(k))\left(\int_K u_{\check{\delta}}(k_1k^{-1}\tilde{k})(k_1.\tilde{h})^{iv}dk_1\right)dk \\
&= \int_K\int_K \sigma(u_{\check{\delta}}(k))u_{\check{\delta}}(k_1k^{-1}\tilde{k})(k_1.\tilde{h})^{iv}dk_1dk \\
&= \int_K\int_K u_{\check{\delta}}(k_1)u_{\check{\delta}}(k^{-1}\tilde{k})\sigma(u_{\check{\delta}}(k))(k_1.\tilde{h})^{iv}dk_1dk.
\end{aligned}$$

$$\begin{aligned}
&= \int_K\int_K u_{\check{\delta}}(k_1)u_{\check{\delta}}(t)\sigma(u_{\check{\delta}}(\tilde{k}t^{-1}))(k_1.\tilde{h})^{iv}dk_1dt \\
&= \int_K\int_K u_{\check{\delta}}(k_1)u_{\check{\delta}}(t)\sigma(u_{\check{\delta}}(t^{-1})u_{\check{\delta}}(\tilde{k}))(k_1.\tilde{h})^{iv}dk_1dt \\
&= \int_K u_{\check{\delta}}(k_1)\left(\int_K u_{\check{\delta}}(t^{-1})\sigma(u_{\check{\delta}}(t)u_{\check{\delta}}(\tilde{k}))dt\right)(k_1.\tilde{h})^{iv}dk_1 \\
&= \int_K u_{\check{\delta}}(k_1)u_{\check{\delta}}(\tilde{k})(k_1.\tilde{h})^{iv}dk_1 \\
&= \int_K u_{\check{\delta}}(k_1\tilde{k})(k_1.\tilde{h})^{iv}dk_1 \\
&= (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h}).
\end{aligned}$$

Then $\overline{\chi}_\delta \star (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h}) = (u_{\check{\delta}},\nu)(\tilde{k}\tilde{h})$.

Therefore $(u_{\check{\delta}},\nu) \in I_{c,\delta}(H)$. □

If $(u_{\check{\delta}},\nu) \in \mathcal{G}_{p,\delta}(H)$ then the spherical Fourier transform of type δ is defined by this relation:

$$\mathcal{F}(f)(u_{\check{\delta}},\nu) = \int_H f(kh)(u_{\check{\delta}},\nu)((kh)^{-1})dkdh, \quad \text{where } f \in I_{c,\delta}(H)$$

Let G be a reductive Lie group, H a Cartan subgroup of G, $\mathfrak{H} = Lie(H)$ and $M = Z(G,\mathfrak{H}_{\mathbb{R}})$. the centralizer of $\mathfrak{H}_{\mathbb{R}} \in G$

Let $P = MN$ a parabolic subgroup of G containing M, N nilradical and K a compact maximal subgroup of G and $\eta = Lie(N)$.

Let $X(M)$ be the set of characters of M.
Put

$$^0M = \bigcap_{\zeta \in X(M)} ker(|\zeta|)$$

where $|\zeta|(x) = |\zeta(x)| \quad (x \in M)$, $M = ^0MA$ is a Langlands decomposition of M.

Let $M = ^0MA$ be the Langland decomposition of M and Δ a positive root system of the pair $(\mathcal{G}_{\mathbb{C}}, \mathcal{K}_{\mathbb{C}})$. $\Theta^{^0M}_{(u_{\check{\delta}}, \Delta)}$ the invariant distribution associated to $(u_{\check{\delta}}, \Delta)$
let put

$$\Theta^{M}_{((u_{\check{\delta}}, \nu), \Delta)} = I^G_M \left(\Theta^{^0M}_{(u_{\delta}, \Delta)} \otimes \nu \right)$$

3. Paley-Wiener Theorem of Type δ

We denote by ρ_Δ one half the sum of positive roots associated to lie algebra of G. ε the sigature of Weyl group, and Δ a positive root system.

For all $r > 0$, and we denote by $PW_\delta(H)_r$ the space of function $F : \mathcal{G}_{p,\delta}(H) \longrightarrow End(E)$ such that:

(1) $\nu \mapsto F(u_{\check{\delta}}, \nu)$ is holomorphic on $\mathcal{A}^\star_c$
(2) $\forall N \in \mathbb{N}; \exists C_N \geq 0$: such that $|\, F(u_{\check{\delta}}, \nu) \,| \leq C_N \left(1 + \|(u_{\delta}, \nu)\|\right)^{-N} e^{r\|I_m \nu\|}$.
(3) $F\left(\omega.(u_{\check{\delta}}, \nu)\zeta_{\omega.\rho_\Delta - \rho_\Delta}\right) = \varepsilon(\omega) F(u_{\delta}, \nu), \quad \forall \omega \in W$.

Let put: $PW_\delta(H) = \bigcup_{r>0} PW_\delta(H)_r$. We endow $PW_\delta(H)_r$ with the topology defined by these semi-norms :

$$S_k(F) = \sup_{(u_{\check{\delta}}, \nu)} \left(1 + \|(u_{\delta}, \nu)\|\right)^k |\, F(u_{\check{\delta}}, \nu) \,|$$

with inductive limit topology.

$$PW_\delta(H)^\Delta = \left\{F \in PW_\delta(H) / F\left((u_{\check{\delta}}, \nu)(\omega^{-1} h \omega)\zeta_{\omega.\rho_\Delta - \rho_\Delta}(h)\right) = \varepsilon(\omega) F(u_{\check{\delta}}, \nu)\right\}$$
$$I^\infty_{c,\delta}(H)^{-\Delta} = \left\{f \in I^\infty_{c,\delta}(H) / f(g^{-1} h g) = \varepsilon(g) \zeta_{\rho_\Delta - g.\rho_\Delta} f(h); \quad h \in H_{reg}\right\}$$

Theorem 3.1.

$$\mathcal{F}\left(I^\infty_{c,\delta}(H)^{-\Delta}\right) = PW_\delta(H)^\Delta$$

where the map $f \mapsto \mathcal{F} f$ is the spherical Fourier transform of type δ.

Proof. $PW_\delta(H)$ is a closed subspace of $PW(H, End_C(E))$, then $PW_\delta(H)$ is a Frechet space.
Let put $Z_{\delta,P}(H)$ the subspace of functions φ of $\mathcal{G}_{p,\delta}(H)$.
There exists $z \in (End(E))^\star$ such that

$$\forall c_1, c_2, \cdots, c_n \in \mathbb{C}; x_i \in G \quad \text{and} \quad i \in \overline{1, n}; \langle \sum c_i \overline{c_j} \varphi(x_i; x_j^{-1}), z\rangle \geq 0$$

Thanks to the Bochner theorem, there exists a measure $\widehat{\mu}$ such that:

$$f(kh) = \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_\delta,\nu)(kh)d\widehat{\mu}(u_{\check{\delta}},\nu)$$

$$\forall F \in PW_\delta(H);\ \text{we have}\ \mathcal{F}f = F$$

$$\mathcal{F}f(u_{\check{\delta}},\nu) = \int_{Z_{\delta,P}(H)} f(kh)(u_{\check{\delta}},\nu)^{-1}(kh)dkdh.$$

$\phi \in \mathcal{G}_{P,\delta}(H) \Leftrightarrow (u_{\check{\delta}},\nu) \in \mathcal{G}_{P,\delta}(H)$
$f_K(x) = \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)_K(x)d\widehat{\mu}(u_{\check{\delta}},\nu).$
As $(u_{\check{\delta}},\nu)_K(x) = (u_{\check{\delta}},\nu)(x)$
$f_K(x) = \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)(x)d\widehat{\mu}(u_{\check{\delta}},\nu).$

$$\begin{aligned}
\text{and}\quad f\star\overline{\chi_\delta}(x) &= \int_K \chi_\delta(k^{-1})f(xk)dk\\
f(kh) &= \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)(kh)d\widehat{\mu}(u_{\check{\delta}},\nu)\\
f &= \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)d\widehat{\mu}(u_{\check{\delta}},\nu)\\
f\star\overline{\chi_\delta}(x) &= \left(\int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)d\widehat{\mu}(u_{\check{\delta}},\nu)\right)\star\overline{\chi_\delta}(x)\\
&= \int_{Z_{\delta,P}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)\star\overline{\chi_\delta}(x)d\widehat{\mu}(u_{\check{\delta}},\nu).
\end{aligned}$$

Moreover, $(u_{\check{\delta}},\nu) \in \mathcal{G}_{P,\delta}(H)$ then $(u_{\check{\delta}},\nu)\star\overline{\chi_\delta}(x) = (u_{\check{\delta}},\nu)(x)$ then

$$(f\star\overline{\chi_\delta})(x) = \left(\int_{Z_{\delta,p}(H)} F(u_{\check{\delta}},\nu)(u_{\check{\delta}},\nu)d\widehat{\mu}(u_{\check{\delta}},\nu)\right)(x)$$

then $f\star\overline{\chi_\delta}(x) = f(x)$.
So we have $f \in I^\infty_{c,\delta}(H)$.
If $F \in PW_\delta(H)^\Delta$, $\forall\omega \in W_{\mathbb{C}}$, we have

$$\begin{aligned}
F\Big(g.(u_{\check{\delta}},\nu)\zeta_{g.\rho_\Delta-\rho_\Delta}\Big) &= \varepsilon(g)F(u_{\check{\delta}},\nu), g \in G\\
f(g.h) &= \int_{Z_{\delta,P}(H)} F(u_\delta,\nu)(u_\delta,\nu)(g.h)d\widehat{\mu}(u_\delta,\nu)\\
&= \int_{Z_{\delta,P}(H)} F(g.(u_{\check{\delta}},\nu))(u_\delta,\nu)(h)d\widehat{\mu}(u_{\check{\delta}},\nu)\\
&= \int_{Z_{\delta,P}(H)} \varepsilon(g)\zeta_{\rho_\Delta-g\rho_\Delta}(h)F(u_\delta,\nu)(u_\delta,\nu)(h)d\widehat{\mu}(u_\delta,\nu).\\
&= \varepsilon(g)\zeta_{\rho_\Delta,-g.\rho_\Delta}(h)f(h).
\end{aligned}$$

Thus, $f \in I^\infty_{c,\delta}(H)^{-\Delta}$ and the surjection is proved..
Let fix some conjugacy classes of Cartan subgroup of G, $H_1, ..., H_k \in Car(G)$ and $\Delta_1, ..., \Delta_k$ the correspondant positive root system. Then $\forall f \in I^\infty_{c,\delta}(G)$, and $\mathcal{F}(f)_{H_i} \in PW_\delta(H_i)^{\Delta_i}$. $\square$

Let T be a torus and let Δ be a positive root system in the set of roots of T.
Let H be a compact Cartan subgroup.
For any $\varphi \in I^\infty_{c,\delta}(H)^{-\Delta}$, there exists $f \in I^\infty_{c,\delta}(G)$ such that

$$\left(b_{-\Delta} J^\delta_G(f)\right)(t) = \varphi(t) \quad \forall t \in H_{reg} \quad \text{and} \quad (J^\delta_G(f))_T \equiv 0$$

for any non compact Cartan subgroup.

Theorem 3.2. *The map $\mathcal{F}$ given by*

$$\mathcal{F} : I^\infty_{c,\delta}(G) \longrightarrow \bigoplus_{i=1}^{k} PW_\delta(H_i)^{\Delta_i}$$
$$f \longmapsto \mathcal{F}(f) = \sum_{i=1}^{k} \mathcal{F}(f)_{H_i}$$

is surjective.

Proof. Let put $F = \sum F_i \in \bigoplus_{i=1}^{k} PW_\delta(H_i)^{\Delta_i}$.
Thanks to theorem 3.1, there exists $\phi_i \in I^\infty_{c,\delta}(H_i)^{\Delta_i}$ such that $\mathcal{F}(\phi_i) = F_i$.
Let put $C_{G,H_i}(\gamma) = \mid H_i / Z(G, \gamma H_i) \mid$.
The function $C_{G,H_i}\phi_i \in I^\infty_{c,\delta}(H_i)^{-\Delta_i}$ and there exists $f \in I^\infty_{c,\delta}(G)$ such that

$$b_{-\Delta_i}(J^\delta_G(f))_{H_i} = C_{G,H_i}\phi_i$$

where b_Δ is the projection of $I^\infty_{c,\delta}(G)$ onto $I^\infty_{c,\delta}(G)^\Delta$ defined by
$b_\Delta(\gamma) = \Pi_{\alpha \in \Delta} \dfrac{1 - \zeta_\alpha(\gamma^{-1})}{\mid 1 - \zeta_\alpha(\gamma^{-1}) \mid}$. We have

$$C^{-1}_{G,H_i}\left(b_{-\Delta_i} J^\delta_G(f)_{H_i}\right) = \phi_i$$
$$\mathcal{F}\left(C^{-1}_{G,H_i}\left(b_{-\Delta_i} J^\delta_G(f)_{H_i}\right)\right) = \mathcal{F}(\phi_i)$$
$$\mathcal{F}\left(C^{-1}_{G,H_i}\left(b_{-\Delta_i} J^\delta_G(f)_{H_i}\right)\right) = \mathcal{F}(f)_{H_i}.$$

Then $\mathcal{F} : I^\infty_{c,\delta}(G) \to \bigoplus_{i=1}^{k} PW_\delta(H_i)^{\Delta_i}$ is surjective $\square$

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