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Valuing European Option Under Double 3/2-Volatility Jump-Diffusion Model With Stochastic Interest Rate and Stochastic Intensity Under Approximative Fractional Brownian Motion

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Abstract. In this study, we propose a more comprehensive and realistic option pricing model based on approximative fractional Brownian motion, building upon recent advancements in this area. Specifically, we utilize the double 3/2-volatility Jump-Diffusion model, which incorporates approximative fractional Brownian motion with 3/2-volatility, stochastic interest rate, and stochastic intensity. To account for the stochastic interest rate, we employ a two-factor Vasicek model. Notably, our model accommodates negative interest rates. Consequently, we develop a multi-factor model with a stochastic interest rate structure for pricing European options and derive a closed-form pricing formula with an analytical solution by applying some algebraic calculations and Lie symmetries. In order to demonstrate the superiority of our proposed model over other classical approaches, we present numerical results that showcase the value of a European call option. This comparative analysis underscores the advantages of our model in comparison to traditional models.

1. Introduction

The groundbreaking study by Black & Scholes in 1973 [4] revolutionized option pricing by introducing an efficient model based on Brownian motion. Their closed-form formula for pricing European options, utilizing Brownian motion to explain the complexity of underlying asset prices, became a cornerstone in financial modeling. However, subsequent research by Duan and Wei [7] exposed the limitations of the Black-Scholes model in accurately capturing certain market phenomena, such as the asymmetric phenomenon and the volatility smile observed in real-world markets. To address these

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shortcomings, numerous academic researchers proposed various models that incorporate non-constant volatility alongside the Black-Scholes framework. Notable examples include the Hull and White [14] model, the Scott [22] model, the Stein and Stein [25] model, and the Wiggins [29] model. Despite these advancements, many of these stochastic volatility models remain impractical in real-world applications.

The impact of incorporating stochastic volatility lies in its ability to better reflect the complexities of financial markets. Traditional models that assume constant volatility fail to account for the observed phenomenon of the volatility smile, where options with different strikes exhibit varying implied volatilities. Stochastic volatility models allow for varying volatility over time, resulting in more accurate option pricing, especially for options with different maturities and strikes. Moreover, stochastic volatility models can capture sudden spikes or drops in volatility during periods of market uncertainty, providing a more realistic representation of market dynamics. This feature is particularly important in times of high market volatility, such as during economic crises or major events. Additionally, stochastic volatility models are better equipped to handle the phenomenon of volatility clustering, where periods of high volatility are followed by other periods of high volatility, and low-volatility periods are followed by other low-volatility periods. These characteristics of stochastic volatility models make them highly valuable in option pricing and risk management, as they offer improved hedging strategies and more accurate assessments of option prices in dynamic market conditions.

In 1993, Heston [12] presented a closed-form formula for European options, employing the Cox-Ingersoll-Ross process to model variance, which is the square of volatility. While this model represents an improvement, the limitations of single-factor models in precisely capturing the volatility smile have prompted researchers to explore multi-factor stochastic volatility models. Such models offer more realistic representations of return data, and in this study, we focus on option pricing under a two-factor stochastic volatility model, which better approximates the complexities of real-world financial markets.

Moreover, the financial market exhibits long-range persistence and self-similarity, characteristics fundamental to fractional Brownian motion. However, fractional Brownian motion is neither a semimartingale nor a Markov process, rendering traditional Ito calculus inapplicable. To overcome this limitation, Hu and Oksendal [13] introduced Wick products for analyzing fractional Brownian motion, and Xiao and AI [30] defined a fractional stochastic integral using Wick products. While the initial model lacked economic interpretation, Björk and Hult [3] addressed this concern with the introduction of mixed fractional Brownian motion [8, 20, 26, 31]. Additionally, Approximation Fractional Brownian motion [27] emerged as a viable alternative, and Thao [27] demonstrated that it constitutes a semimartingale. This has led to increased interest in fractional stochastic volatility models among experts and academics [11], with many researchers incorporating Approximation Fractional Brownian motion in constructing stochastic volatility models [6].

In recent years, hybrid models that incorporate stochastic interest rates into stochastic models have gained attention [9,10,15,17,24]. Empirical studies also support the integration of stochastic interest rates into option pricing models, as they contribute to improved model results [21]. The impact of incorporating stochastic interest rates lies in capturing the dynamics of interest rate fluctuations, which can significantly influence option pricing. By considering interest rate uncertainty, the model becomes more reflective of real-world market conditions and provides a more accurate representation of option prices, especially for longer-term options. Additionally, stochastic interest rates enable the model to account for the term structure of interest rates, which is crucial in pricing options with different maturities.

In this paper, we present a comprehensive option pricing model that considers volatility and interest rate fluctuations, along with the occurrence and intensity of jumps over time. We adopt the double 3/2-volatility jump-diffusion (DJD) model with approximative fractional Brownian motion, stochastic intensity, and a two-factor interest rate model in section 2. In section 3, we derive an analytical pricing formula for European call options. We provide numerical illustrations in section 4 to demonstrate the model's effectiveness. Finally, we conclude in section 5. The incorporation of the 3/2-volatility process and the use of approximative fractional Brownian motion, together with the consideration of stochastic interest rates, offer the potential to improve the model's performance in capturing the complexities of financial market dynamics, which may lead to more accurate and robust option pricing results. This research contributes to the advancement of financial modeling and provides valuable insights for market participants and investors in making informed decisions.

2. The Model

We present some basic information on approximative fractional Brownian motion . At the first, we present an analysis of fractional Brownian motion $(W_t^H)_{t\geq 0}$ with the Hurst index $H \in (0,1)$. It is a Gaussian process with zero mean and the following covariance:

$$\mathbb{E}[W_t^H W_s^H] = \frac{1}{2}\Big(|t|^{2H} + |s|^{2H} - |Tt - s|^{2H}\Big). \tag{2.1}$$

The decomposition of a fractional Brownian motion W is as follows:

$$W_t^H = \frac{1}{\Gamma(H+\frac{1}{2})}\Big[Z_t + \int_0^t (t-s)^{H-\frac{1}{2}} dB_s\Big] \tag{2.2}$$

where

$$Z_t = \int_{-\infty}^0 \Big((t-s)^{H-\frac{1}{2}} - (-s)^{H-\frac{1}{2}}\Big) dB_s, \tag{2.3}$$

B_t indicates standard Brownian motion, and Γ indicates the gamma function. It is sufficient to focus exclusively on the term:

$$W_t = \int_0^t (t-s)^{H-\frac{1}{2}} \tag{2.4}$$

that has a long-range memory. Note that The approximation of B_t is $\tilde{B}_t^{\epsilon,H}$ which can be expressed as [26]

$$\tilde{W}_t^{\varepsilon,H} = \int_0^t (t-s+\varepsilon)^{H-\frac{1}{2}} dB_s \tag{2.5}$$

where H is a long-memory parameter, ε is non negative approximation factor. Thao [27] proved that for $\varepsilon \to 0$, $\quad (W_t^{\varepsilon,t})_\varepsilon$ converges uniformly to a non-Markov process. In addition, if $\varepsilon > 0$ then $W_t^{\varepsilon,t}$ is a semi-martingale [27]

$$d\tilde{W}_t^{\varepsilon,H} = (H-\frac{1}{2})\psi_t dt + \varepsilon^{H-\frac{1}{2}} dB_t^v \tag{2.6}$$

ψ_t is a stochastic processes expressed as

$$\psi_t = \int_0^t (t-s+\varepsilon)^{H-\frac{3}{2}} dB_s^\psi, \tag{2.7}$$

where$(B_t^\psi)_{t\in[0,T]}$ and $(B_t^v)_{t\in[0,T]}$, are independent standard Brownian motions.

Let$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{Q})$ be a complete probability space with a filtration and $\mathbb{Q}$ presents a risk-neutral measure.The stock price S_t is expressed by the following dynamic system:

$$\begin{cases} dS_t = (r_1 + r_2 - \lambda_t \mu_J) S_t dt + \sqrt{v_1} S_t dB_t^s + \sqrt{v_2} S_t d\hat{B}_t^s + (J-1) S_t dN_t \\ dv_1 = v_1(\theta_1 - a v_1) dt + \sigma_1 v_1^{\frac{3}{2}} d\tilde{B}_t^{\epsilon,H} \\ dv_2 = v_2(\theta_2 - a_2 v_2) dt + \sigma_2 v_2^{\frac{3}{2}} dB_t^{v_2} \\ d\lambda_t = k_\lambda(\theta - \lambda) dt + \sigma\sqrt{\lambda} dB_t^\lambda \\ dr_1 = \beta_1(\alpha_1 - r_1) dt + \eta_1 dB_t^{r_1} \\ dr_2 = \beta_2(\alpha_2 - r_1) dt + \eta_2 dB_t^{r_2} \end{cases} \tag{2.8}$$

where $B_1^s, \hat{B}_t^s, B_t^v, B_t^{r_1}, B_t^{r_2}$ and B_t^λ are the standard Brownian motions. We assume that B_t^s is correlated with B_t^v, $dB_t^s.dB_t^v = \rho_1 dt$,$\hat{B}_t^s$ correlated with $B_t^{\hat{v}}$, $d\hat{B}_t^s d B_t^{\hat{v}} = \rho_2 dt$ and $B_t^{r_1}$ correlated with $B_t^{r_2}$, $dB_t^{r_1}.dB_t^{r_2} = \rho_r dt$. Any other Brownian motions are pairwise independent.

v_1 *and* v_2 are variances, and λ_t is the jump intensity. k_1, k_2 and k_λ are mean reversion rates, θ_1, θ_2 and θ are mean reversion levels, σ_1, σ_2 and σ are the volatilities of the variances. and the short rate follows the two-factor Vasicek model where the short rate is given as a sum of two factors r_1 and r_2, where α_1, α_2 are their mean-reversion , β_1, β_2 are their mean-reversion speed, η_1, η_2 are their volatilities, N_t represents Poisson process with intensity λ_t and J represents the jump size, and we suppose that lnJ has an asymmetric double exponential distribution with density function $pdh_u(x)$:

$$pdh_u(x) = p\eta_1 e^{\gamma_1 x} 1_{\geq 0} + q\gamma_2 e^{\gamma_2 x} 1_{x<0}, \tag{2.9}$$

where $\gamma_1 > 1, \gamma_2 > 0, p, q > 0$, and $p + q = 1$, where q and p represent the probabilities for positive and negative jumps, respectively. As a result we can obtain that $\mu_J = \mathbb{E}^{\mathbb{Q}}(J-1) = (p\gamma_1/\gamma_1 - 1) + (q\gamma_2/\gamma_2 + 1) - 1$.

We set $\tau = T - t$, $X_t = lnS_t$, $Y = lnJ$, the interest rate r are determined by the sum of the two factors r_1 and r_2 ($r = r_1 + r_2$) and $k = lnK$, where T is the maturity date, and K is the strike price. In the risk-neutral world, the price of a call option $C(S, v1, v2, r, \lambda, t)$ at time $t \in [0, T]$ with strike price K and maturity date T is given by

$$C(S, v, \hat{v}, r_1, r_2, \lambda, t) = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_0^t r_s ds} max(S_T - K, 0)|\mathcal{F}_t\right) \tag{2.10}$$

We convert measure $\mathbb{Q}$ to the measure $\mathbb{Q}^S$ and the T forward measure $\mathbb{Q}^T$. By applying Radon–Nikodym derivatives,

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^S} = \frac{e^X}{e^{-\int_0^T r_s ds + X_T}} \tag{2.11}$$

$$\frac{d\mathbb{Q}}{d\mathbb{Q}^T} = \frac{P(t,T)}{e^{-\int_0^t r_s ds}} \tag{2.12}$$

where

$$S = e^X = \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds + X_T}|\mathcal{F}_t\right), \tag{2.13}$$

$P(t,T) := \mathbb{E}^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds +}|\mathcal{F}_t\right)$, is the price at time t of a zero-coupon bond which matures at time T (see appendix). Then, we can have the following expression:

$$C(S, v_1, v_2, r_1, r_2, \lambda, t) = S\mathbb{E}^{\mathbb{Q}^S}(1_{\{X_T > k\}}|\mathcal{F}_t) - KP(t,T)\mathbb{E}^{\mathbb{Q}^T}(1_{\{X_T > k\}}|\mathcal{F}_t) \tag{2.14}$$

We define

$$\phi_S(u) := \mathbb{E}^{\mathbb{Q}^S}(e^{iuX_T}|\mathcal{F}_t), \tag{2.15}$$

$$\phi_T(u) := \mathbb{E}^{\mathbb{Q}^T}(e^{iuX_T}|\mathcal{F}_t), \tag{2.16}$$

$$\phi(u) := \mathbb{E}^{\mathbb{Q}}(e^{\int_t^T r_s ds + iuX_T}|\mathcal{F}_t), \tag{2.17}$$

where $\phi_S(u)$ denotes the characteristic function under $\mathbb{Q}^S$, $\varphi_T(u)$ denotes the characteristic function under $\mathbb{Q}^T$, and $\varphi(u)$ denotes the discounted characteristic function under $\mathbb{Q}$. Furthermore, by using Radon–Nikodym derivatives we can have the following expression:

$$C(S,v_1,v_2,r_1,r_2,\lambda,t)=S\Big(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty R\Big(\frac{e^{-iuk}\phi(u-i)}{iu\phi(-i)}\Big)du\Big)-KP(t,T)\Big(\frac{1}{2}+\frac{1}{\pi}\int_0^\infty R\Big(\frac{e^{-iuk}\phi(u)}{iuP(t,T)}\Big)du\Big) \tag{2.18}$$

All we need to do is to derive the formula of $\phi(u)$ to have the pricing formula.

Theorem 2.1. *If the asset price is governed by the dynamic system (1), the discounted characteristic function $\phi(u;X,v_1,v_2,r_1,r_2,\lambda,\tau)$ takes the following form:*

$$\phi(u;X,v_1,v_2,r_1,r_2,\lambda,\tau)=Y(v_1,u,\tau)Z(v_2,u,\tau)e^{C(u,\tau)+E(u,\tau)r_1+F(u,\tau)r_2+G(u,\tau)\lambda+iuX} \tag{2.19}$$

where

$$\begin{aligned}
C(u,\tau) &= (iu-1)\Big((\frac{\alpha_1}{\beta_1}(\beta_1 t-1-e^{-\beta_1 t})+\frac{\alpha_2}{\beta_2}(\beta_2 t-1-e^{-\beta_2 t})\Big)\\
&- \frac{\eta_1^2}{4\beta_1^3}(iu-1)^2\Big(e^{-2\beta_1 t}-4e^{-\beta_1 t}-2\beta_1 t+3\Big)-\frac{\eta_2^2}{4\beta_2^3}(iu-1)^2\Big(e^{-2\beta_2 t}-4e^{-\beta_2 t}-2\beta_2 t+3\Big)\\
&+ \rho_r\eta_1\eta_2(iu-1)^2\frac{1}{\beta_1\beta_2}\Big(t+\frac{1}{\beta_2}e^{-\beta_2 t}+\frac{1}{\beta_1}e^{-\beta_1 t}-\frac{1}{\beta_2+\beta_1}e^{-(\beta_2+\beta_1)t}-\frac{1}{\beta_1}-\frac{1}{\beta_2}\\
&+ \frac{1}{\beta_1+\beta_2}\Big)\frac{2k_\lambda\theta_\lambda}{\sigma^2}\Big[\frac{(k_\lambda-\varsigma)\tau}{2}+ln\frac{2\varsigma}{2\varsigma+(k_\lambda-\varsigma)(1-e^{-\hat{d}\tau})}\Big]
\end{aligned}$$

$$\begin{aligned}
Y(v_1,\tau) =&e^{\frac{-2\theta_1}{\sigma_1^2 v_1\left[e^{\theta_1\tau}-1\right]}}\left(\frac{2\theta_1}{\sigma_1^2 v_2\left[e^{\theta_1\tau}-1\right]}\right)^{\varpi_1}\frac{\Gamma\left(\varpi_1+\frac{2b_1}{\sigma_1^2}+2\right)}{\Gamma\left(2\varpi_1+\frac{2b_1}{\sigma_1^2}+2\right)}\\
&\times M\left(\varpi_1+\frac{2b_1}{\sigma_1^2}+2,2\varpi_1+\frac{2b_1}{\sigma_1^2}+2,\frac{2\theta_1}{\sigma_1^2 v_1\left[e^{\theta_1\tau}-1\right]}\right)
\end{aligned} \tag{2.20}$$

$$\begin{aligned}
Z(v_2,\tau) =&e^{\frac{-2\theta_1}{\sigma_2^2 v_2\left[e^{\theta_2\tau}-1\right]}}\left(\frac{2\theta_1}{\sigma_2^2 v\left[e^{\theta_2\tau}-1\right]}\right)^{\varpi_2}\frac{\Gamma\left(\varpi_2+\frac{2b_2}{\sigma_2^2}+2\right)}{\Gamma\left(2\varpi_2+\frac{2b_2}{\sigma_2^2}+2\right)}\\
&\times M\left(\varpi_2+\frac{2b_2}{\sigma_2^2}+2,2\varpi_2+\frac{2b_2}{\sigma_2^2}+2,\frac{2\theta_2}{\sigma_2^2 v_2\left[e^{\theta_2\tau}-1\right]}\right)
\end{aligned} \tag{2.21}$$

$$G(u,\tau)=2\omega(u)\frac{1-e^{-\varsigma\tau}}{2\varsigma+(k_\lambda-\varsigma)(1-e^{-\varsigma\tau})} \tag{2.22}$$

$$E(u,\tau)=\frac{1}{\beta_1}(iu-1)(1-e^{-\beta_1\tau}) \tag{2.23}$$

$$F(u,\tau)=\frac{1}{\beta_2}(iu-1)(1-e^{-\beta_2\tau}) \tag{2.24}$$

and where

$$L(u)=\frac{p\gamma_1}{\gamma_1-iu}+\frac{q\gamma_2}{\gamma_2+iu}-1,\ w(u)=L(u)-iu\mu_J,\ \varsigma=\sqrt{k_\lambda^2-2\sigma_\lambda^2 w(u)} \tag{2.25}$$

$$\varpi_1 = \frac{-(1+\frac{2b_1}{\sigma_1^2\varepsilon^{2H-1}}) + \sqrt{(1+\frac{2b_1}{\sigma_1^2\varepsilon^{2H-1}}) + \frac{8c}{\sigma_1^2\varepsilon^{2H-1}}}}{2},\ \varpi_2 = \frac{-(1+\frac{2b_2}{\sigma_2^2}) + \sqrt{(1+\frac{2b_2}{\sigma_2^2}) + \frac{8c}{\sigma_2^2}}}{2} \tag{2.26}$$

$$c = \frac{u^2}{2} + \frac{ju}{2},\ b_1 = a_1 - ju\rho_1\varepsilon^{H-\frac{1}{2}}\sigma_1,\ b_2 = a_2 - ju\rho_2\sigma_2, \tag{2.27}$$

and Where M is the Kummer M function.

Proof. $\varphi(u; X, v_1, v_2, r_1, r_2, \lambda, \tau)$ satisfies a PIDE by applying the Feynman–Kac theorem:

$$\begin{aligned}
&-\frac{\partial\phi}{\partial\tau} + (r_1 + r_2 - \lambda\mu_J - \frac{1}{2}(v_1+v_2))\frac{\partial\phi}{\partial x} + \frac{1}{2}(v_1+v_2)\frac{\partial^2\phi}{\partial x^2} + v(\theta_1 - a_1v_1) + (H-\frac{1}{2})\sigma_1\sqrt{v_1})\frac{\partial\phi}{\partial v_1}\\
&+\frac{1}{2}\sigma_1^2\varepsilon^{2H-1}v_1^3\frac{\partial^2\phi}{\partial v^2} + v(\theta_2 - a_2v_2)\frac{\partial\phi}{\partial v_2} + \frac{1}{2}\sigma_2^2v_2^3\frac{\partial v_2^2\phi}{\partial v_2^2} + \rho_1\sigma_1v_1^2\varepsilon^{H-\frac{1}{2}}\frac{\partial^2\phi}{\partial x\partial v_1} + \rho_2\sigma_2v_2^2\frac{\partial^2\phi}{\partial x\partial v_2}\\
&+\beta_1(\alpha_1 - r_1)\frac{\partial\phi}{\partial r_1} + \frac{1}{2}\eta_1^2\frac{\partial^2\phi}{\partial r_1^2} + \beta_2(\alpha_1 - r_2)\frac{\partial\phi}{\partial r_2} + \frac{1}{2}\eta_2^2\frac{\partial^2\phi}{\partial r_2^2} + \eta_1\eta_2\rho_r\frac{\partial^2\phi}{\partial r_1\partial r_2} + k_\lambda(\theta-\lambda)\frac{\partial\phi}{\partial\lambda}\\
&+\frac{1}{2}\sigma_\lambda^2\lambda\frac{\partial^2\phi}{\partial\lambda^2} + \lambda\int_{-\infty}^{+\infty}(\phi(x+y) - \phi(x))h(y)dy - r\phi = 0
\end{aligned} \tag{2.28}$$

If we assume that $\phi(u; X, v_1, v_2, r_1, r_2, \lambda, \tau)$ takes the form of

$$\phi(u; X, v_1, v_2, r_1, r_2, \lambda, \tau) = Y(v_1, u, \tau)Z(v_2, u, \tau)e^{C(u,\tau)+E(u,\tau)r_1+F(u,\tau)r_2+G(u,\tau)\lambda+iuX} \tag{2.29}$$

and substitute into Equation (2.28), we can obtain

$$\begin{cases}
\frac{\partial Y}{\partial\tau} = \frac{\sigma_1^2\varepsilon^{2H-1}v_1^3}{2}\frac{\partial^2 Y}{\partial^{v_1}} + \left[\rho_1\sigma_1v_1^2ju\varepsilon^{H-\frac{1}{2}} + v_1(\theta_1 - a_1v_1)\right]\frac{\partial Y}{\partial v_1} - (\frac{u^2}{2} + \frac{ju}{2})v_1Y\\
\frac{\partial Z}{\partial\tau} = \frac{\sigma_2^2v_2^3}{2}\frac{\partial^2 Z}{\partial_2^v} + \left[\rho_2\sigma_2v_2^2ju + v_2(\theta_2 - a_2v_2)\right]\frac{\partial Z}{\partial v_2} - (\frac{u^2}{2} + \frac{ju}{2})v_2Z\\
\frac{\partial G}{\partial\tau} = \frac{\sigma_\lambda^2\lambda}{2}G^2 - k_\lambda G + (L(u) - \mu_j ju)\\
\frac{\partial C}{\partial\tau} = k_\lambda\theta_\lambda G + \frac{\eta_1^2}{2}E^2 + \beta_1\alpha_1E + \frac{\eta_2^2}{2}F^2 + \beta_2\alpha_2F\\
\frac{\partial E}{\partial\tau} = -k_1E + ju - 1\\
\frac{\partial F}{\partial\tau} = -k_2F + ju - 1
\end{cases} \tag{2.30}$$

with boundary conditions$E(u,0) = F(u,0) = G(u,0) = C(u,0) = 0$, and $Y(v_1,u,0) = Z(v_2,u,0) = 1$ for Y and Z we apply Lie's method as in [18], and by applying some algebraic calculations, we will obtain the result. □

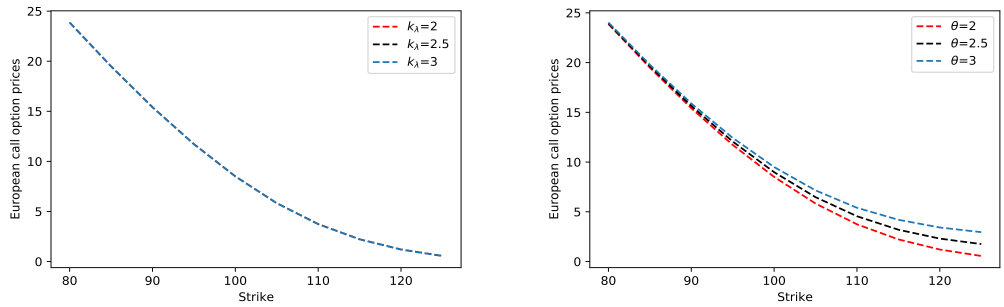


Figure 1. The impact of k_λ and θ on call option prices for $T = 1$.

3. Numerical Discussion

In this section, we analyze European option prices under the Double 3/2-Volatility Jump-Diffusion (DJD) model with a two-factor stochastic interest rate model. The model parameters used in our analysis are listed in Table (1).

Table 1. Values of parameters.

Parameter	Value	Parameter	value
k_1	9.9772k1	k_2	2.3388
θ_1	0.0189	θ_2	0.001
σ_1	0.8379	σ_2	0.9957
ρ_1	-0.9764	ρ_2	-0.8178
v_1	0.0002	v_2	0.0633
ε	0.00005	ρ_r	1
β_1	0.3322	β_2	0.26594
α_1	0.1	α_2	0.1
η_1	0.02	η_2	0.02
r_1	0.001	r_2	0.012
k_λ	2	σ	0.1
θ	0.001	λ	0.001
γ_1	1.0333	γ_2	19.7482
S	100	K	100

Figure (1) illustrates the significant impact of changes in the mean-reversion level θ_λ on call option prices, while changes in the mean-reversion rate k_λ have a relatively minor effect on call option prices.

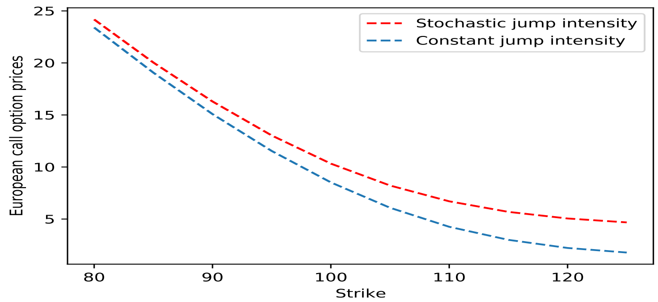
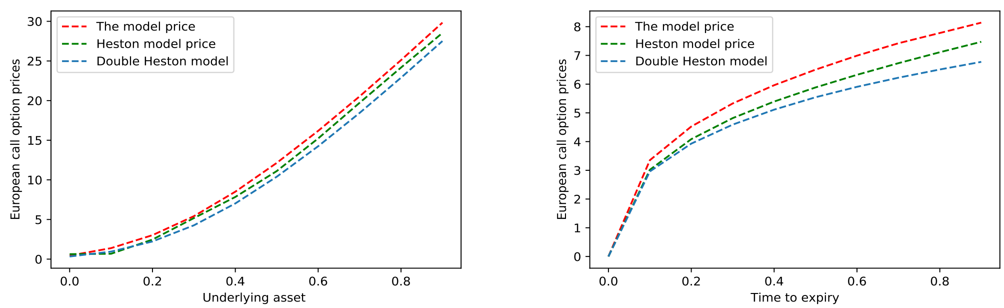


Figure 2. The impact of the existence of the jump intensity process on call option prices for $T = 1$.



(a) (b)

Figure 3. The model price, the Heston price, and double Heston price with respect to the underlying asset price (a) and time to expiry (b).

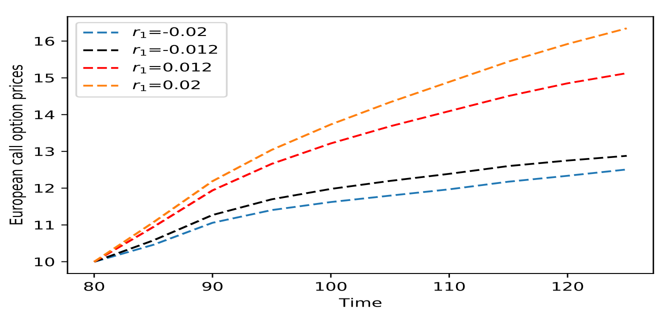
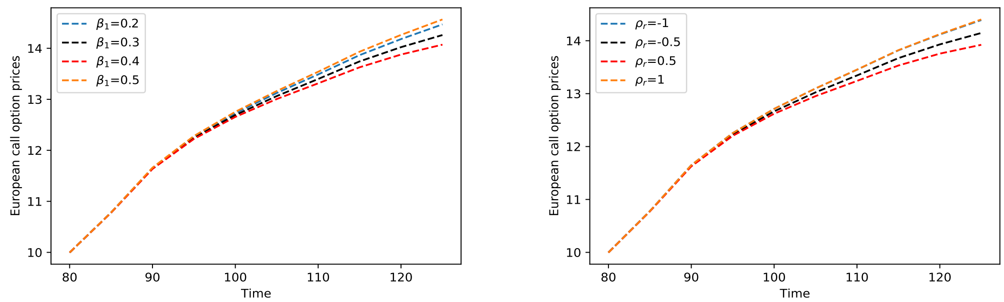


Figure 4. The European call option price with respect to different values of r_1

Our findings indicate that an increase in the value of θ_λ leads to a corresponding increase in the call option price.

Figure (2) presents the effect of the presence of the jump intensity process on call option prices. It clearly demonstrates that the call option price with stochastic jump intensity is higher than that with



(a) (b)

Figure 5. The model price price, The impact of β_1 (a) and ρ_r (b) with respect to the time .

a constant jump intensity. Additionally, the interest rate value plays a crucial role in option pricing (3). As the interest rate r increases, the option price also rises, particularly when β and ρ_r are significant factors (5).

Conversely, we proceed to analyze the impact of incorporating a two-factor stochastic interest rate model with approximate fractional Brownian motion and stochastic intensity using the theoretical conclusions derived from the pricing formula. Evidently, our proposed price model surpasses that of the traditional Heston model. Specifically, Figure (3) displays the option prices with varying time to expiry. Notably, our model's price and the Heston model's price are nearly equivalent when the time to expiry increases, but the discrepancy between the two widens with extended time to expiry. This phenomenon is attributed to the increased time duration for interest rate changes, which allows our model to reflect the expanded divergence more effectively.

4. Conclusion

In conclusion, our research introduces a comprehensive pricing model for European options, combining a two-factor Vasicek model with interest rate dynamics, a stochastic process for jump intensity, and the double 3/2-volatility jump-diffusion model. By leveraging the power of approximate fractional Brownian motion, our approach captures long-range dependence and self-similarity in asset volatility. Numerical results demonstrate the superiority of our model over the double Heston and Heston models in pricing European call options. The closed-form solution enhances efficiency and accuracy, making our approach valuable for option pricing and risk management in real-world financial markets. Further research can explore its performance under diverse market conditions, solidifying its position as a significant contribution to financial modeling.

Appendix

If the risk-free interest rate follows the Two-Vasicek model, then $P(r_1, r_2, t, T)$ should satisfy the following PDE problem:

$$\begin{cases} \frac{\partial P}{\partial t} + \beta_1(\alpha 2 - r_1)\frac{\partial P}{\partial r_1} + \beta_2(\alpha_2 - r_2)\frac{\partial P}{\partial r_1} + \frac{1}{2}\eta_1^2\frac{\partial^2 P}{\partial r_1^2} + \frac{1}{2}\eta_2^2\frac{\partial^2 P}{\partial r_2^2} + \rho_r\eta_1\eta_2\frac{\partial^2 P}{\partial r_1 \partial r_2} - (r_2 + r_2)P = 0 \\ P(r_1, r_2, T, T) = 1 \end{cases} \tag{4.1}$$

If we assume that $P(r_1, r_2, t, T)$ takes the form of

$$P(r_1, r_2, t, T) = e^{[A(\tau) - B_1(\tau)r_1 - B_1(\tau)r2]} \tag{4.2}$$

and substitute it into PDE (4.1), we can obtain :

$$\begin{cases} \frac{\partial B_1}{\partial \tau} = 1 - \beta_1 B_1 \\ \\ \frac{\partial B_2}{\partial \tau} = 1 - \beta_2 B_2 \\ \\ \frac{\partial A}{\partial \tau} = -\beta_1\alpha_1 B_1 - \beta_2\alpha_2 B_2 + \frac{1}{2}\eta_1^2 B_1^2 + \frac{1}{2}\sigma_2^2 B_2^2 + \rho_r\eta_1\eta_2 B_1 B_2 \end{cases} \tag{4.3}$$

with the terminal condition $B_1(0) = B_2(0) = A(0) = 0$ Then we have :

$$B_1(\tau) = \frac{1}{\beta_1}(1 - e^{\beta_1\tau}) \tag{4.4}$$

$$B_2(\tau) = \frac{1}{\beta_2}(1 - e^{\beta_2\tau}) \tag{4.5}$$

$$\begin{aligned} A(\tau) = &-\alpha_1(\tau + \frac{1}{\beta_1}e^{-\beta_1\tau} - \frac{1}{\beta_1}) - \alpha_2(\tau + \frac{2}{\beta_2}e^{-\beta_2\tau} - \frac{1}{\beta_2}) + \frac{\sigma_1^2}{\beta_1^2}(t + \frac{2}{\beta_1}e^{-\beta_1 t} - \frac{1}{2\beta_1}e^{-2\beta_1 t} - \frac{3}{2\beta_1}) \\ &+p_r\sigma_2\sigma_2\frac{1}{\beta_1\beta_2}(t + \frac{1}{\beta_1}e^{-\beta_1 t} + \frac{1}{\beta_2}e^{\beta_2 t} - \frac{1}{\beta_1+\beta_2}e^{-(\beta_1+\beta_2)t} + \frac{1}{\beta_1+\beta_2} - \frac{1}{\beta_1} - \frac{1}{\beta_2}) \\ &+\frac{\sigma_2^2}{\beta_2^2}(t + \frac{2}{\beta_2}e^{-\beta_2 t} - \frac{1}{2\beta_2}e^{-2\beta_2 t} - \frac{3}{2\beta_2}). \end{aligned} \tag{4.6}$$

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