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Common Fixed Point Theorems for Mappings Satisfying $(E.A)$-Property on Cone Normed B-Metric Spaces

K. Maheshwaran[1], R. Jahir Hussain[2], Mohammad Saeed Khan[3], Salvatore Sessa[4,*]

[1]*Jamal Mohamed College (Autonomous) - Affiliated to Bharathidasan University, Tiruchirappllі-620020,Tamil Nadu, India*

[2]*Jamal Mohamed College (Autonomous) - Affiliated to Bharathidasan University, Tiruchirappllі-620020,Tamil Nadu, India*

[3]*Department of Mathematics and Applied Mathematics, Sefako Makgatho Health Sciences University,Gauteng 0208, South Africa*

[4]*Department of Architecture, Federico II Naples University, Via Toledo 402, 80134 Naples, Italy*

**Corresponding author:* salvasessa@gmail.com

Abstract. In this article, we demonstrate the conditions for the existence of common fixed points (CFP) theorems for four self-maps satisfying the common limit range $(E.A)$-property on cone normed B-metric spaces $(CNBMS)$. Furthermore, we have an unique common fixed point for two weakly compatible (WC) pairings.

1. Introduction

In 1989, I. A. Bakhtin [3] introduced b-metric space, an extension of metric space, and established the renowned contraction principle in metric spaces as extension in b-metric spaces. Cone metric space was introduced by Huang, L. G. Zhang [7] in 2007. They also examined various sequence convergence properties and established the fixed point theorems of a contraction mapping for cone metric spaces. Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \leq k < 1$, the inequality $D_b(\xi\varrho, \xi\varsigma) \leq kD_b(\varrho, \varsigma), \forall \varrho, \varsigma \in M$ has a unique fixed point. Cone b-metric space was first discribed by Hussain and Saha [8] in 2011 as an extension of b-metric spaces and cone metric spaces. In 2002, Aamri and Moutawakil [1] introduced $(E.A)$-property and (CFT) in metric spaces. Several researchers [2,12,13] studies various contraction and lot of fixed point theorems are proved in various spaces.

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We provide the sufficient conditions for the existence of (CFP) theorems for four self-maps satisfying the $(E.A)$-property on $(CNbMS)$ in the major of this article.

2. Preliminaries

Definition 2.1. *[[6]] Let $(E, \|.\|)$ be the real Banach space. A subset P of E is called a cone if and only if:*

(b_1) P is closed, non empty and $P \neq \emptyset$
(b_2) $a\varrho + b\varsigma \in P$ for all $\varrho, \varsigma \in P$ and non negative real numbers a, b
(b_3) $P \cap (-P) = \{0\}$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $\varrho \leq \varsigma$ if and only if $\varsigma - \varrho \in P$. We will write $\varrho \prec \varsigma$ to indicate that $\varrho \leq \varsigma$ but $\varrho \neq \varsigma$, while ϱ, ς will stand for $\varsigma - \varrho \in intP$, where $intP$ denotes the interior of P. The cone P is called normal if there is a number $k > 0$ such that $0 \leq \varrho \leq \varsigma$ implies $\|\varrho\| \leq k\|\varsigma\|$ for all $\varrho, \varsigma \in E$. The least positive number k satisfying the above is called the normal constant.

Definition 2.2. *[[6]] Let M be a nonempty set and $b \geq 1$ be a given real number. A mapping $D_b : M \times M \to E$ is said to be cone b-metric if and only if, $\forall \varrho, \varsigma, \upsilon \in M$, the following conditions are satisfied:*

(cb_1) $D_b(\varrho, \varsigma) = 0$ if and only if $\varrho = \varsigma$,
(cb_2) $D_b(\varrho, \varsigma) = D_b(\varsigma, \varrho)$,
(cb_3) $D_b(\varrho, \varsigma) \leq b[D_b(\varrho, \varsigma) + D_b(\varsigma, \upsilon)]$.

Then (M, D_b) is called a cone b-metric space.

Example 2.1. *Let $E = \mathcal{R}^2$*

$$P = \{(\varrho, \varsigma) : \varrho, \varsigma \geq 0\}$$

$M = \mathcal{R}$ and $D_b : M \times M \to E$ such that

$$D_b(\varrho, \varsigma) = (|\varrho - \varsigma|^p, \alpha|\varrho - \varsigma|^p)$$

where $\alpha \geq 0$ and $p > 1$ are two real constants. Then (M, D_b) is a cone b-metric space with the coefficient $b = 2^p > 1$, but not a cone metric space. In fact, we only need to prove (iii) in Definition (2.2) as follows: Let $\varrho, \varsigma, \upsilon \in M$. Set $\mu = \varrho - \upsilon, \nu = \upsilon - \varsigma$, so $\varrho - \varsigma = \mu + \nu$. From the inequality

$$(a + b)^p \leq (2 \max\{a, b\})^p \leq 2^p(a^p + b^p) \text{ for all } a, b \geq 0.$$

we have

$$|\varrho - \varsigma|^p = |\mu + \nu|^p \leq (|\mu| + |\nu|)^p \leq 2^p(|\mu|^p + |\nu|^p) = 2^p(|\varrho - \upsilon|^p + |\upsilon - \varsigma|^p)$$

which implies that $D_b(\varrho, \varsigma) \leq b[D_b(\varrho, \upsilon) + D_b(\upsilon, \varsigma)]$ with $b = 2^p > 1$. But

$$|\varrho - \varsigma|^p \leq |\varrho - \varsigma|^p + |\upsilon - \varsigma|^p$$

is impossible for all $\varrho > \upsilon > \varsigma$. Indeed, taking account of the inequality

$$(a + b)^p > a^p + b^p \text{ for all } a, b \geq 0,$$

$$|\varrho-\varsigma|^p = |\mu+\nu|^p = (\mu+\nu)^p > \mu^p+\nu^p = |\varrho-\upsilon|^p + |\upsilon-\varsigma|^p$$

for all $\varrho > \upsilon > \varsigma$. Thus, $D_b(\varrho,\varsigma) \le D_b(\varrho,\varsigma) + D_b(\varsigma,\upsilon)$ is not satisfied, i.e., (M, D_b) is not a cone metric space.

Definition 2.3. *[[9]] Let M be a vector space over R. Suppose the mapping $||.|| : M \to E$ satisfies*

(i) $||\varrho|| \ge 0$ for all$\varrho \in M$
(ii) $||\varrho|| = 0$ if and only if $\varrho = 0$
(iii) $||\varrho+\varsigma|| \le ||\varrho|| + ||\varsigma||$ for all $\varrho,\varsigma \in M$
(iv) $||k\varrho|| = |k|||\varrho||$ for all $k \in \mathcal{R}$

Then $||.||$ is called a norm on M, and $(M, ||.||)$ is called a cone normed space. Clearly each cone normed space is a cone metric space with metric defined by $D_b(\varrho,\varsigma) = ||\varrho-\varsigma||$

Definition 2.4. *[[6]] Let $(M, ||.||)$ be a $(CNbMS)$, $\varrho \in M$ and $\{\varrho_n\}$ be a sequence in M. Then*

(i) $\{\varrho_n\}$ converges to ϱ whenever, for every $c \in E$ with $0 \ll c$, there is a natural number N such that $||(\varrho_n,\varrho)|| \ll c$ for all $n \ge N$. We denote this by $\lim_{n\to\infty} \varrho_n = \varrho$ or $\varrho_n \to \varrho(n \to \infty)$.
(ii) $\{\varrho_n\}$ is a Cauchy sequence whenever, for every $c \in E$ with $0 \ll c$, there is a natural number N such that $||(\varrho_n,\varrho_m)|| \ll c$ for all $n, m \ge N$
(iii) $(M, ||.||)$ is a complete cone normed b-metric space if every Cauchy sequence is convergent. A complete cone normed space is a cone Banach space.

Lemma 2.1. *[[6]] Let $(M, ||.||)$ be a $(CNbMS)$. P be a normal cone with constant K. Let $\{\mu_p\}$ be a sequence in M. Then*

(i) $\{\mu_p\}$ b-converges to μ if and only if $||\mu_p - \mu|| \to 0$ as $p \to \infty$.
(ii) $\{\mu_p\}$ is a b-Cauchy sequence if and only if $||\mu_p - \mu_m|| \to 0$ as $p, m \to \infty$.
(iii) If the $\{\mu_p\}$ b-converges to μ and $\{\varsigma_n\}$ b-converges to ν then $||\mu_p - \nu_p|| \to ||\mu - \nu||$.

Lemma 2.2. *[[6]] Let $(M, ||.||)$ be a $(CNbMS)$ with $b \ge 1$. Suppose that $\{\varrho_n\}$ and $\{\varsigma_n\}$ are b-convergent to ϱ and ς, respectively. Then we have,*

$$\frac{1}{b^2}||(\varrho-\varsigma)|| \le \lim_{n\to\infty} \inf ||(\varrho_n - \varsigma_n)|| \le \lim_{n\to\infty} sup||(\varrho_n - \varsigma_n)|| \le b^2||(\varrho-\varsigma)||.$$

In particular, if $M = L$, then we have $\lim_{n\to\infty} ||(\varrho_n - \varsigma_n)|| = 0$, moreover for each $\upsilon \in M$, we have

$$\frac{1}{b}||(\varrho-\upsilon)|| \le \lim_{n\to\infty} \inf ||(\varrho_n - \upsilon)|| \le \lim_{n\to\infty} \sup ||(\varrho_n - \upsilon)|| \le b||(\varrho-\upsilon)||.$$

Example 2.2. *Let $M = \mathcal{R}^2, E = \mathcal{R}^2$, and $P = \{(\varrho,\varsigma) \in E | \varrho,\varsigma \ge 0\}$, we define $\parallel (\varrho,\varsigma) \parallel = (|\varrho|^2, |\varsigma|^2)$. Then $(M, ||.||)$ is a $(CNBMS)$.*

Proof. First, according to the definition 2.3 of $\parallel (\varrho,\varsigma) \parallel$, we obtain

$$\parallel (\varrho,\varsigma) \parallel = (|\varrho|^2, |\varsigma|^2) \ge 0,$$

for all $\mu = (\varrho, \varsigma) \in M$. It is obvious that $\| (\varrho, \varsigma) \| = 0 \; iff \; (\varrho, \varsigma) = 0$. Then, we have

$$(\varrho + \varsigma)^p \leq \varrho^p + \varsigma^p, \forall \; \varrho, \varsigma \geq 0, 0 < p \leq 1,$$

$$(\varrho + \varsigma)^p \leq 2^{p-1}(\varrho^p + \varsigma^p), \forall \; \varrho, \varsigma \geq 0, p \geq 1.$$

We check for any $(\varrho_1, \varsigma_1), (\varrho_2, \varsigma_2) \in M$, thus, it follows

$$\begin{aligned}
\| (\varrho_1, \varsigma_1) + (\varrho_2, \varsigma_2, \| &= \| (\varrho_1 + \varrho_2, \; \varsigma_1 + \varsigma_2) \| \\
&= (|\varrho_1 + \varrho_2|^2, \; |\varsigma_1 + \varsigma_2|^2) \\
&\leq 2(|\varrho_1|^2 + |\varrho_2|^2, \; |\varsigma_1|^2 + |\varsigma_2|^2) \\
&= 2(|\varrho_1|^2, |\varrho_2|^2,) + 2(\; |\varsigma_1|^2, |\varsigma_2|^2) \\
&= 2 \| (\varrho_1, \varsigma_1) \| + 2 \| (\varrho_2, \varsigma_2) \| .
\end{aligned}$$

If any $\mu = (\varrho, \varsigma) \in M, k \in \mathcal{R}$, there are

$$\begin{aligned}
\| k\mu \| &= \| k(\varrho, \varsigma) \| \\
&= \| (k\varrho, k\varsigma) \| \\
&= |k\varrho|^2, |k\varsigma|^2 \\
&= |k|^2(|\varrho|^2, |\varsigma|^2) \\
&= |k|^2 \| \mu \| .
\end{aligned}$$

According, by definition2.3, $(M, \|.\|)$ is a $(CNBMS)$ with coefficient $\tau = 2 > 1$. □

Definition 2.5. *Let ω and χ be two self maps defined on a $(CNBMS)$ $(M, \|.\|)$ maps ω and χ are said to be (WC) if they commute at coincidence points, that is if $\omega\varrho = \chi\varrho$ for all $\varrho \in M$ then $\omega\chi\varrho = \chi\omega\varrho$.*

Example 2.3. *Define $\omega, \chi : \mathcal{R} \to \mathcal{R}$ by $\omega(\varrho) = \frac{\varrho}{4}, \forall \varrho \in \mathcal{R}$ and $\chi(\varrho) = \varrho^2, \forall \varrho \in \mathcal{R}$. Here, 0 and $\frac{1}{4}$ are two coincidence points for the maps ω and χ. Note that ω and χ commute at 0, i.e.$\omega\chi(0) = \chi\omega(0) = 0$, but $\omega\chi(\frac{1}{4}) = \omega(\frac{1}{16}) = \frac{1}{64}$ and $\chi\omega(\frac{1}{4}) = \chi(\frac{1}{16}) = \frac{1}{256}$ so ω and χ are not (WC) on $\mathcal{R}$.*

Lemma 2.3. *Let ω and χ be (WC) self-maps of a $(CNBMS)$ $(M, \|.\|)$. If ω and χ have a unique point of coincidence, that is $x = \omega\varrho = \chi\varrho$ then x is the unique (CFP) of ω and χ.*

Lemma 2.4. *Let $(M, \|.\|)$ is a $(CNBMS)$. If there exits two sequences $\{\varrho_n\}$ and $\{\varsigma_n\}$ such that $\lim_{n\to\infty} \|(\varrho_n - \varsigma_n)\| = 0$, whenever $\{\varrho_n\}$ is a sequence in M such that $\lim_{n\to\infty} \varrho_n = \mathcal{T}$ for some $\mathcal{T} \in M$ then $\lim_{n\to\infty} \varsigma_n = \mathcal{T}$.*

Proof. By a triangle inequality in $(CNBMS)$, we have

$$\|(\varsigma_n - \mathcal{T})\| \leq b(\|(\varsigma_n - \varrho_n)\| + \|(\varrho_n - \mathcal{T})\|).$$

Now by taking the upper limit when $n \to \infty$ in the above inequality we get

$$\lim_{n\to\infty} \sup \|(\varsigma_n - \mathcal{T})\| \leq b(\lim_{n\to\infty} \sup \|(\varrho_n - \varsigma_n)\| + \lim_{n\to\infty} \sup \|(\varrho_n - \mathcal{T})\|) = 0.$$

□

Definition 2.6. *Let $(M, \|.\|)$ be a $(CNBMS)$. A pair $\{\omega, \chi\}$ is said to be compatible if $\lim_{n\to\infty} \|(\omega\chi\varrho_n - \chi\omega\varrho_n)\| = 0$, for every sequence $\{\varrho_n\}$ in M with $\lim_{n\to\infty} \omega\varrho_n = \lim_{n\to\infty} \chi\varrho_n = \mathcal{T}$ for some $\mathcal{T} \in M$.*

Definition 2.7. *Let $(M, \|.\|)$ be a $(CNBMS)$ and $\omega, \chi : M \to M$, two mapping ω and χ are said to satisfy the $(E.A)$-property if there exists a sequence $\{\varrho_n\}$ such that*

$$\lim_{n\to\infty} \chi\{\varrho_n\} = \lim_{n\to\infty} \omega\{\varrho_n\} = \mathcal{T} \text{ for some } \mathcal{T} \in M$$

Example 2.4. *Let $M = [0, \infty)$ and define $D_b(\varrho, \varsigma) = \|\varrho - \varsigma\|$ $\forall \varrho, \varsigma \in M$ and $\omega, \chi : M \to M$, defined by $\chi(\varrho) = \varrho + 3$ and $\omega(\varrho) = 4\varrho$, $\forall$ $\varrho \in M$. Consider the sequence $\{\varrho_n\} = \{1 + \frac{1}{n}\}$. Then*

$$\lim_{n\to\infty} \chi\{\varrho_n\} = \lim_{n\to\infty} \chi\{1 + \frac{1}{n}\} = \lim_{n\to\infty} \{4 + \frac{1}{n}\} = 4;$$

$$\lim_{n\to\infty} \omega\{\varrho_n\} = \lim_{n\to\infty} \omega\{1 + \frac{1}{n}\} = \lim_{n\to\infty} \{4 + \frac{4}{n}\} = 4;$$

Thus,

$$\lim_{n\to\infty} \chi\{\varrho_n\} = \lim_{n\to\infty} \omega\{\varrho_n\} = 4 \in M.$$

So ω and χ are satisfy the $(E.A)$-property.

Definition 2.8. *[[11]] A function $\psi : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ is called an altering distance function if the condition hold:*

(i) ψ is continuous and non decreasing.
(ii) $\psi(t) = 0$ iif and only if $t = 0$.

Definition 2.9. *[[11]] An ultra altering distance function is a non decreasing and continuous mapping $\varphi : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ such that $\varphi(t) > 0, t > 0$ and $\varphi(0) = 0$.*

3. Main Results

During this article, we assume the control function $\psi, \phi : [0, \infty) \to [0, \infty)$ are continuous, non-decreasing functions with $\psi(t) = 0$ if and only if $t = 0$.

Lemma 3.1. *Let $(M, \|.\|)$ be a complete $(CNBMS)$ with the coefficient $b \geq 1$. Suppose that the mappings ω, χ, ρ, and ξ are four self-mappings of a $(CNBMS)$ $(M, \|.\|)$ satisfying the following conditions:*

$$\psi(\|(\omega\varrho - \chi\varsigma)\| \leq \psi(\lambda(\varrho, \varsigma)) - \varphi(\lambda(\varrho, \varsigma)) \tag{3.1}$$

for all $\varrho, \varsigma \in M$, where

$\lambda(\varrho, \varsigma) = \max\{\|\rho\varrho - \xi\varsigma\|, \frac{\|\omega\varrho - \rho\varrho\|}{1+\|\omega\varrho - \rho\varrho\|}, \frac{\|\chi\varsigma - \xi\varsigma\|}{1+\|\chi\varsigma - \xi\varsigma\|}, \frac{\|\omega\varrho - \xi\varsigma\| + \|\rho\varrho - \chi\varsigma\|}{2b}, \frac{\|\omega\varrho - \rho\varrho\| + \|\chi\varsigma - \xi\varsigma\|}{2b}\}$. *If*

(1) The pairs (ω, ρ) and (χ, ξ) satisfy the common $(E.A)$-property.
(2) $\omega(M) \subseteq \xi(M)$ and $\chi(M) \subseteq \rho(M)$.
(3) $\rho(M)$ and $\xi(M)$ are closed in M.
(4) $\{\chi\varsigma_n\}$ converges for each sequence $\{\varsigma_n\}$ in M whenever $\{\xi\varsigma_n\}$ converges (respectively $\{\omega\varrho_n\}$ converges for every sequence $\{\varrho_n\}$ in M whenever $\{\rho\varrho_n\}$ converges).
(5) There exist $\varphi \in \Phi$ and $\psi \in \Psi$

Then ω, ρ, χ and ξ have a unique (CFP).

Proof. If the pair (ω,ρ) satisfy the $(E.A)$-property,then there is sequence $\{\varrho_n\}$ in M such that $\lim_{n\to\infty}\omega\{\varrho_n\}=\lim_{n\to\infty}\rho\{\varrho_n\}=\mathcal{T}$ where $\mathcal{T}\in M$. Now, since $\omega(M)\subseteq\xi(M)$, every sequence $\{\varrho_n\}$, there is a sequence $\{\varsigma_n\}$ in M such that $\omega\{\varrho_n\}=\xi\{\varsigma_n\}$. As $\xi(M)$ is closed, so $\lim_{n\to\infty}\xi\varsigma_n=\lim_{n\to\infty}\omega\varrho_n=\mathcal{T}$. So that $\mathcal{T}\in M$. Thus, we get $\omega\varrho_n\to\mathcal{T},\rho\varrho_n\to\mathcal{T},\xi\varsigma_n\to\mathcal{T}$, as $n\to\infty$.
Let us show that $\chi\varsigma_n\to\mathcal{T}$ as $n\to\infty$. On the contrary suppose that $\chi\varsigma_n\to\int(\neq\mathcal{T})$ as $n\to\infty$. Putting $\varrho=\varrho_n$ and $\varsigma=\varsigma_n$ in (3.1), we get

$$\begin{aligned}&\psi(\|\omega\varrho_n-\chi\varsigma_n\|)\\&\leq\psi(\lambda(\varrho_n,\varsigma_n))-\varphi(\lambda(\varrho_n,\varsigma_n))\end{aligned}\tag{3.2}$$

where

$$\begin{aligned}\lambda(\varrho_n,\varsigma_n)&=\max\{\|\rho\varrho_n-\xi\varsigma_n\|,\frac{\|\omega\varrho_n-\rho\varrho_n\|}{1+\|\omega\varrho_n-\rho\varrho_n\|},\frac{\|\chi\varsigma_n-\xi\varsigma_n\|}{1+\|\chi\varsigma_n-\xi\varsigma_n\|},\\&\quad\frac{\|\omega\varrho_n-\xi\varsigma_n\|+\|\rho\varrho_n-\chi\varsigma_n\|}{2b},\frac{\|\omega\varrho_n-\rho\varrho_n\|+\|\chi\varsigma_n-\xi\varsigma_n\|}{2b}\}\\&=\max\{\|\rho\varrho_n-\omega\varrho_n\|,\frac{\|\omega\varrho_n-\rho\varrho_n\|}{1+\|\omega\varrho_n-\rho\varrho_n\|},\frac{\|\chi\varsigma_n-\omega\varrho_n\|}{1+\|\chi\varsigma_n-\omega\varrho_n\|},\\&\quad\frac{\|\omega\varrho_n-\omega\varrho_n\|+\|\rho\varrho_n-\chi\varsigma_n\|}{2b},\frac{\|\omega\varrho_n-\rho\varrho_n\|+\|\chi\varsigma_n-\omega\varrho_n\|}{2b}\}\\&\leq\max\{\|\rho\varrho_n-\omega\varrho_n\|,\|\omega\varrho_n-\rho\varrho_n\|,\|\chi\varsigma_n-\omega\varrho_n\|,\\&\quad\frac{\|\rho\varrho_n-\chi\varsigma_n\|}{2b},\frac{\|\omega\varrho_n-\rho\varrho_n\|+\|\chi\varsigma_n-\omega\varrho_n\|}{2b}\}\\&\leq\max\{\|\rho\varrho_n-\omega\varrho_n\|,\|\omega\varrho_n-\rho\varrho_n\|,\|\chi\varsigma_n-\omega\varrho_n\|,\\&\quad\frac{b[\|\rho\varrho_n-\omega\varrho_n\|+\|\omega\varrho_n-\chi\varsigma_n\|]}{2b},\frac{\|\omega\varrho_n-\rho\varrho_n\|+\|\chi\varsigma_n-\omega\varrho_n\|}{2b}\}\\&=\max\{\|\rho\varrho_n-\omega\varrho_n\|,\|\omega\varrho_n-\rho\varrho_n\|,\|\chi\varsigma_n-\omega\varrho_n\|,\\&\quad\frac{\|\rho\varrho_n-\omega\varrho_n\|+\|\omega\varrho_n-\chi\varsigma_n\|}{2},\frac{\|\omega\varrho_n-\rho\varrho_n\|+\|\chi\varsigma_n-\omega\varrho_n\|}{2b}\}\\&=\max\{\|\mathcal{T}-\textstyle\int\|,\frac{\|\omega\varrho_n-\chi\varsigma_n\|}{2}\}=\|\mathcal{T}-\textstyle\int\|\end{aligned}\tag{3.3}$$

Taking $n\to\infty$, we get

$$\begin{gathered}\|\psi D_b(\mathcal{T},\textstyle\int)\|\leq\|\psi(\lim_{n\to\infty}\lambda(\varrho_n,\varsigma_n))\|\\\|\psi(\lim_{n\to\infty}D_b(\mathcal{T},\textstyle\int))\|\leq K\|\psi(\lim_{n\to\infty}D_b(\mathcal{T},\textstyle\int))\|\\\psi(\|\mathcal{T}-\textstyle\int\|)\leq\psi(\|\mathcal{T}-\textstyle\int\|)-\varphi(\|\mathcal{T}-\textstyle\int\|)\leq\psi(\|\mathcal{T}-\textstyle\int\|)\end{gathered}$$

Hence, $\|\mathcal{T}-\int\|=0$ implies that $\mathcal{T}=\int$, a contradiction. Hence, $\int\to\mathcal{T}$ which shows that the pairs (ω,ρ) and (χ,ξ) share the$(E.A)$-property. Using the above lemma, in the following theorem, we show the existence of unique (CFP). □

Theorem 3.1. *Let $(M,\|.\|)$ be a complete $(CNBMS)$ with the coefficient $\tau\geq1$. suppose that the mappings ω,χ,ρ and ξ are four self-mappings of a $(CNBMS)$ $(M,\|.\|)$ satisfying (3.1). If the pairs (ω,ρ) and (χ,ξ) have a point of coincidence. Moreover if (ω,ρ) and (χ,ξ) are (WC) the ω,χ,ρ and ξ have a unique (CFP).*

Proof. Since the pairs (ω, ρ) and (χ, ξ) satisfying the $(E.A)$-property, so there exists sequence $\{\varrho_n\}$ and $\{\varsigma_n\}$ in M such that

$$\lim_{n\to\infty} \omega\varrho_n = \lim_{n\to\infty} \rho\varrho_n = \lim_{n\to\infty} \chi\varsigma_n = \lim_{n\to\infty} \xi\varsigma_n = \mathcal{T}.$$

where $\mathcal{T} \in M$ since $\rho(M) \subseteq \xi(M)$ so a point $\int \in M$ such that $\rho\int = \mathcal{T}$ we show that $\omega\int = \rho\int$. Putting $\varrho = \int, \varsigma = \varsigma_n$ in (3.1), we get

$$\psi(\|\omega\textstyle\int - \chi\varsigma_n\|) \leq \psi(\lambda(\textstyle\int, \varsigma_n)) - \varphi(\lambda(\textstyle\int, \varsigma_n)) \tag{3.4}$$

where

$$\begin{aligned} \lambda(\textstyle\int, \varsigma_n) = & \max\{\|\rho\textstyle\int - \xi\varsigma_n\|, \frac{\|\omega\int - \rho\int\|}{1 + \|\omega\int - \rho\int\|}, \frac{\|\chi\varsigma_n - \xi\varsigma_n\|}{1 + \|\chi\varsigma_n - \xi\varsigma_n\|}, \\ & \frac{\|\omega\int - \xi\varsigma_n\| + \|\rho\int - \chi\varsigma_n\|}{2b}, \frac{\|\omega\int - \rho\int\| + \|\chi\varsigma_n - \xi\varsigma_n\|}{2b}\} \\ = & \max\{\|\mathcal{T} - \xi\varsigma_n\|, \frac{\|\omega\int - \mathcal{T}\|}{1 + \|\omega\int - \mathcal{T}\|}, \frac{\|\chi\varsigma_n - \xi\varsigma_n\|}{1 + \|\chi\varsigma_n - \xi\varsigma_n\|}, \\ & \frac{\|\omega\int - \xi\varsigma_n\| + \|\mathcal{T} - \chi\varsigma_n\|}{2b}, \frac{\|\omega\int - \mathcal{T}\| + \|\chi\varsigma_n - \xi\varsigma_n\|}{2b}\} \\ \leq & \max\{\|\mathcal{T} - \xi\varsigma_n\|, \|\omega\textstyle\int - \mathcal{T}\|, \|\chi\varsigma_n - \xi\varsigma_n\|, \\ & \frac{\|\omega\int - \xi\varsigma_n\| + \|\mathcal{T} - \chi\varsigma_n\|}{2b}, \frac{\|\omega\int - \mathcal{T}\| + \|\chi\varsigma_n - \xi\varsigma_n\|}{2b}\} \end{aligned}$$

Making $n \to \infty$, we get

$$\begin{aligned} \lim_{n\to\infty} \lambda(\textstyle\int, \varsigma_n) = & \max\{\|\mathcal{T} - \mathcal{T}\|, \|\omega\textstyle\int - \mathcal{T}\|, \|\mathcal{T} - \mathcal{T}\|, \frac{\|\omega\int - \mathcal{T}\| + \|\mathcal{T} - \mathcal{T}\|}{2b}, \\ & \frac{\|\omega\int - \mathcal{T}\| + \|\mathcal{T} - \mathcal{T}\|}{2b}\} \\ = & \max\{\|\omega\textstyle\int - \mathcal{T}\|, \frac{\|\omega\int - \mathcal{T}\|}{2b}\} = \|\omega\textstyle\int - \mathcal{T}\|. \end{aligned}$$

Taking limit as $n \to \infty$ in (3.4) and using definition of φ, we get

$$\psi(\|\omega\textstyle\int - \mathcal{T}\|) \leq \psi(\|\omega\textstyle\int - \mathcal{T}\|) - \varphi(\|\omega\textstyle\int - \mathcal{T}\|) \leq \psi(\|\omega\textstyle\int - \mathcal{T}\|)$$

implies that, $\varphi(\|\omega\int - \mathcal{T}\|) = 0$. Hence $\omega\int = \mathcal{T} = \rho\int$. Therefore, $\int$ is the point of coincidence of the pair (ω, ρ). As $\mathcal{T} \in M$, there exist a point $\gamma \in M$ such that $\xi\gamma = \mathcal{T}$. We assert that $\chi\gamma = \xi\gamma$. Putting $\varrho = \int$ and $\varsigma = \gamma$ in equation (3.1) we get

$$\psi(\|\omega\textstyle\int - \chi\gamma\|) \leq \psi(\lambda(\textstyle\int, \gamma)) - \varphi(\lambda(\textstyle\int, \gamma)) \tag{3.5}$$

where

$$\begin{aligned}\lambda(\int,\gamma) = & \max\{\|\rho\int-\xi\gamma\|, \frac{\|\omega\int-\rho\int\|}{1+\|\omega\int-\rho\int\|}, \frac{\|\chi\gamma-\xi\gamma\|}{1+\|\chi\gamma-\xi\gamma\|},\\ & \frac{\|\omega\int-\xi\gamma\|+\|\rho\int-\chi\gamma\|}{2b}, \frac{\|\omega\int-\rho\int\|+\|\chi\gamma-\xi\gamma\|}{2b}\}\\ = & \max\{\|\rho\int-\mathcal{T}\|, \frac{\|\omega\int-\rho\int\|}{1+\|\omega\int-\rho\int\|}, \frac{\|\chi\gamma-\mathcal{T}\|}{1+\|\chi\gamma-\mathcal{T}\|},\\ & \frac{\|\omega\int-\mathcal{T}\|+\|\rho\int-\chi\gamma\|}{2b}, \frac{\|\omega\int-\rho\int\|+\|\chi\gamma-\mathcal{T}\|}{2b}\}\\ \le & \max\{\|\rho\int-\mathcal{T}\|, \|\omega\int-\rho\int\|, \|\chi\gamma-\mathcal{T}\|,\\ & \frac{\|\omega\int-\mathcal{T}\|+\|\rho\int-\chi\gamma\|}{2b}, \frac{\|\omega\int-\rho\int\|+\|\chi\gamma-\mathcal{T}\|}{2b}\}\\ = & \max\{\|\mathcal{T}-\mathcal{T}\|, \|\mathcal{T}-\mathcal{T}\|, \|\chi\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\mathcal{T}\|+\|\mathcal{T}-\chi\gamma\|}{2b},\\ & \frac{\|\mathcal{T}-\mathcal{T}\|+\|\chi\gamma-\mathcal{T}\|}{2b}\}\\ = & \max\{\|\chi\gamma-\mathcal{T}\|, \frac{\|\mathcal{T}-\chi\gamma\|}{2b}\} = \|\chi\gamma-\mathcal{T}\|.\end{aligned}$$

Thus, from (3.5), and using definition of φ , we get

$$\psi(\|\chi\gamma-\mathcal{T}\|) \le \psi(\|\chi\gamma-\mathcal{T}\|) - \varphi(\|\chi\gamma-\mathcal{T}\|) \le \psi(\|\chi\gamma-\mathcal{T}\|)$$

which gives that, $\varphi(\|\chi\gamma-\mathcal{T}\|) = 0$, *i.e.* $\chi\gamma = \mathcal{T} = \xi\gamma$.
Since the pair (ω,ρ) is weakly compatible and $\omega\mathcal{T} = \rho\mathcal{T}$. Therefore, $\omega\mathcal{T} = \omega\rho\int = \rho\omega\int = \rho\mathcal{T}$.
Now we show that $\mathcal{T}$ is a (CFP) of the pair (ω,ρ). Putting $\varrho = \int, \varsigma = \gamma$ in (3.1), we get

$$\psi(\|\omega\mathcal{T}-\chi\gamma\|) \le \psi(\lambda(\mathcal{T},\gamma)) - \varphi(\lambda(\mathcal{T},\gamma)) \tag{3.6}$$

where

$$\begin{aligned}\lambda(\mathcal{T},\gamma) = & \max\{\|\rho\mathcal{T}-\xi\gamma\|, \frac{\|\omega\mathcal{T}-\rho\mathcal{T}\|}{1+\|\omega\mathcal{T}-\rho\mathcal{T}\|}, \frac{\|\chi\gamma-\xi\gamma\|}{1+\|\chi\gamma-\xi\gamma\|},\\ & \frac{\|\omega\mathcal{T}-\xi\gamma\|+\|\rho\mathcal{T}-\chi\gamma\|}{2b}, \frac{\|\omega\mathcal{T}-\rho\mathcal{T}\|+\|\chi\gamma-\xi\gamma\|}{2b}\}\\ = & \max\{\|\omega\mathcal{T}-\xi\gamma\|, \frac{\|\omega\mathcal{T}-\omega\mathcal{T}\|}{1+\|\omega\mathcal{T}-\omega\mathcal{T}\|}, \frac{\|\chi\gamma-\xi\gamma\|}{1+\|\chi\gamma-\xi\gamma\|},\\ & \frac{\|\omega\mathcal{T}-\xi\gamma\|+\|\omega\mathcal{T}-\chi\gamma\|}{2b}, \frac{\|\omega\mathcal{T}-\omega\mathcal{T}\|+\|\chi\gamma-\mathcal{T}\|}{2b}\}\\ \le & \max\{\|\omega\mathcal{T}-\xi\gamma\|, \|\omega\mathcal{T}-\omega\mathcal{T}\|, \|\chi\gamma-\xi\gamma\|,\\ & \frac{\|\omega\mathcal{T}-\xi\gamma\|+\|\omega\mathcal{T}-\chi\gamma\|}{2b}, \frac{\|\omega\mathcal{T}-\omega\mathcal{T}\|+\|\chi\gamma-\xi\gamma\|}{2b}\}\end{aligned}$$

$$
\begin{aligned}
&= \max\{\|\omega\mathcal{T}-\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\|\mathcal{T}-\mathcal{T}\|,\frac{\|\omega\mathcal{T}-\mathcal{T}\|+\|\omega\mathcal{T}-\mathcal{T}\|}{2b},\\
&\quad \frac{\|\omega\mathcal{T}-\omega\mathcal{T}\|+\|\mathcal{T}-\mathcal{T}\|}{2b}\}\\
&= \max\{\|\omega\mathcal{T}-\mathcal{T}\|,\frac{\|\omega\mathcal{T}-\mathcal{T}\|}{2b}\}=\|\omega\mathcal{T}-\mathcal{T}\|.
\end{aligned}
$$

From (3.6) and using property of φ we get,

$$\psi(\|\omega\mathcal{T}-\mathcal{T}\|)\leq\psi(\|\omega\mathcal{T}-\mathcal{T}\|)-\varphi(\|\omega\mathcal{T}-\mathcal{T}\|)\leq\psi(\|\omega\mathcal{T}-\mathcal{T}\|),$$

Which gives that, $\varphi(\|\omega\mathcal{T}-\mathcal{T}\|)=0, i.e, \omega\mathcal{T}=\mathcal{T}=\rho\mathcal{T}$. Which show that $\mathcal{T}$ is a (CFP) of the pair (ω,ρ). Similarly, one can show that $\chi\mathcal{T}=\xi\mathcal{T}=\mathcal{T}$. Hence $\mathcal{T}$ is a (CFP) of ω,χ,ρ and ξ.
To prove that $\mathcal{T}$ is the unique (CFP), let β be another (CFP) of ω,χ,ρ and ξ. Using (3.1), $\varrho=\mathcal{T},\varsigma=\beta$, we get

$$\psi(\|\omega\mathcal{T}-\chi\beta\|)\leq\psi(\lambda(\mathcal{T},\beta))-\varphi(\lambda(\mathcal{T},\beta) \tag{3.7}$$

where

$$
\begin{aligned}
\lambda(\mathcal{T},\beta) &= \max\{\|\rho\mathcal{T}-\xi\beta\|,\frac{\|\omega\mathcal{T}-\rho\mathcal{T}\|}{1+\|\omega\mathcal{T}-\rho\mathcal{T}\|},\frac{\|\chi\beta-\xi\beta\|}{1+\|\chi\beta-\xi\beta\|},\\
&\quad \frac{\|\omega\mathcal{T}-\xi\beta\|+\|\rho\mathcal{T}-\chi\beta\|}{2b},\frac{\|\omega\mathcal{T}-\rho\mathcal{T}\|+\|\chi\beta-\xi\beta\|}{2b}\}\\
&= \max\{\|\mathcal{T}-\beta\|,\frac{\|\mathcal{T}-\mathcal{T}\|}{1+\|\mathcal{T}-\mathcal{T}\|},\frac{\|\beta-\beta\|}{1+\|\beta-\beta\|},\\
&\quad \frac{\|\mathcal{T}-\beta\|+\|\mathcal{T}-\beta\|}{2b},\frac{\|\mathcal{T}-\mathcal{T}\|+\|\beta-\mathcal{T}\|}{2b}\}\\
&\leq \max\{\|\mathcal{T}-\beta\|,\|\mathcal{T}-\mathcal{T}\|,\|\beta-\beta\|,\frac{\|\mathcal{T}-\beta\|+\|\mathcal{T}-\beta\|}{2b},\frac{\|\mathcal{T}-\mathcal{T}\|+\|\beta-\mathcal{T}\|}{2b}\}\\
&= \max\{\|\mathcal{T}-\beta\|,\|\mathcal{T}-\mathcal{T}\|,\|\beta-\beta\|,\frac{\|\mathcal{T}-\beta\|\|}{b},\frac{\|\beta-\mathcal{T}\|\|}{2b}\}\\
&= \|\mathcal{T}-\beta\|.
\end{aligned}
$$

Thus, from equation (3.7), we get

$$\psi(\|\mathcal{T}-\beta\|)\leq\psi(\|\mathcal{T}-\beta\|)-\varphi(\|\mathcal{T}-\beta\|)$$

implies that $\varphi(\|\mathcal{T}-\beta\|)=0$ *i.e,* $\mathcal{T}=\beta$.
Hence, $\mathcal{T}$ is the unique (CFP) of ω,χ,ρ and ξ. □

Corollary 3.1. *Let $(M,\|.\|))$ be a complete CNBMS with the coefficient $\tau\geq1$. Suppose that the mappings ω,ξ be self-mapping of a $(CNBMS)$ $(M,\|.\|)$ satisfy:*

$$\psi(\|\omega\varrho-\omega\varsigma\|)\leq\psi(\lambda(\varrho,\varsigma))-\varphi(\lambda(\varrho,\varsigma))$$

for all ϱ,ς, where $\lambda(\varrho,\varsigma)=\max\{\|\rho\varrho-\xi\varsigma\|,\|\omega\varrho-\xi\varrho\|,\|\omega\varsigma-\xi\varsigma\|,\frac{\|\omega\varrho-\xi\varsigma\|+\xi\varrho-\omega\varsigma\|}{2b}\}$. If the pairs (ω,ξ) satisfies $(E.A)$-property, ω and ξ have a common point of coincidence. Moreover if ω and ξ are (WC), then the pair (ω,ξ) has a unique (CFP).

Corollary 3.2. *Let $(M, \|.\|)$ be a complete $(CNBMS)$ with the coefficient $\tau \geq 1$. Suppose that the mappings ω, χ, ρ, and ξ be a self mapping of a $(CNBMS)$ $(M, \|.\|)$ satisfy*

$$\psi(\|\omega\varrho - \chi\varsigma\|) \leq \psi(\lambda(\varrho, \varsigma)) - \varphi((\lambda(\varrho, \varsigma))),$$

for all ϱ, ς, where $\lambda(\varrho, \varsigma) = \max\{\|\rho\varrho - \xi\varsigma\|, \|\omega\varrho - \rho\varrho\|, \|\chi\varsigma - \xi\varsigma\|, \frac{\|\omega\varrho-\xi\varsigma\|+\|\rho\varrho-\chi\varsigma\|}{2b}\}$. If the pairs (ω, ρ) and (χ, ξ) satisfy the $(E.A)$-property, then (ω, ρ) and (χ, ξ) have a point of coincidence. Moreover, if (ω, ρ) and (χ, ξ) are compatible, then ω, χ, ρ and ξ have a unique (CFP).

Example 3.1. *Let $M = (0, 7)$, $E = \mathcal{R}$ and $P = \{(\varrho, \varsigma) \in E : \varrho, \varsigma \geq 0\}$ and $D_b : M \times M \to E$ as $D_b(\varrho, \varsigma) = \| \varrho - \varsigma \|^2$, $\forall\ \varrho, \varsigma \in M$ and $\omega,\ \chi, \rho, \xi : M \to M$. Then $(M, \| \,.\, \|)$ is a complete $(CNBMS)$ with co-efficient $\tau = 2$.*

$$\omega(\varrho) = \begin{cases} 5 \ \ if \ \varrho \in (0, 3] \\ 1 \ \ if \ \varrho \in (3, 7) \end{cases}, \qquad \chi(\varrho) = \begin{cases} 5 \ \ if \ \varrho \in (0, 3] \\ \frac{1}{2} \ \ if \ \varrho \in (3, 7) \end{cases}$$

$$\rho(\varrho) = \begin{cases} 5 \ \ if \ \varrho \in (0, 3] \\ 2 \ \ if \ \varrho \in (3, 7) \end{cases}, \qquad \xi(\varrho) = \begin{cases} 5 \ \ if \ \varrho \in (0, 3] \\ 4 \ \ if \ \varrho \in (3, 7) \end{cases}$$

First we verify condition (1) of Lemma (3.1)

Let $\{\varrho_n\} = \left\{\frac{3n}{n+1}\right\}_{n\geq 1}$ and $\{\varsigma_n\} = \left\{\frac{3}{n+1}\right\}_{n\geq 1}$ be two sequence in M. Then

$$\lim_{n\to\infty} \omega(\varrho_n) = \lim_{n\to\infty} \omega\left(\frac{3n}{n+1}\right) = 5$$

$$\lim_{n\to\infty} \rho(\varrho_n) = \lim_{n\to\infty} \rho\left(\frac{3n}{n+1}\right) = 5$$

$$\lim_{n\to\infty} \chi(\varsigma_n) = \lim_{n\to\infty} \chi\left(\frac{3}{n+1}\right) = 5$$

$$\lim_{n\to\infty} \xi(\varsigma_n) = \lim_{n\to\infty} \xi\left(\frac{3}{n+1}\right) = 5$$

Thus

$$\lim_{n\to\infty} \omega(\varrho_n) = \lim_{n\to\infty} \rho(\varrho_n) = \lim_{n\to\infty} \chi(\varsigma_n) = \lim_{n\to\infty} \xi(\varsigma_n) = 5$$

That is, (ω, ρ) and (χ, ξ) satisfies the common $(E.A)$-property.

Next, to verify inequality (3.1). Let us define the function $\psi, \varphi : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ by $\psi\,(t) = t$ and $\varphi\,(t) = \frac{t}{3}$.

Case (i): *Let $\varrho, \varsigma \in (0, 3]$. Then $\omega(\varrho) = \rho(\varrho) = \chi(\varsigma) = \xi(\varsigma) = 5$ and from the inequality (3.1)*

$L.H.S = \psi(\| \omega\varrho - \chi\varsigma \|^2) = \psi(\| 5 - 5 \|^2) = \psi(0) = 0$

$R.H.S = \psi\,(\lambda(\varrho, \varsigma)) - \varphi\,(\lambda(\varrho, \varsigma))$

$$\lambda(\varrho, \varsigma) = \max\left\{\| \rho\varrho - \xi\varsigma \|^2, \frac{\|\omega\varrho-\rho\varrho\|^2}{1+\|\omega\varrho-\rho\varrho\|^2}, \frac{\|\chi\varsigma-\xi\varsigma\|^2}{1+\|\chi\varsigma-\xi\varsigma\|^2}, \frac{\|\omega\varrho-\xi\varsigma\|^2+\|\rho\varrho-\chi\varsigma\|^2}{2b}, \frac{\|\omega\varrho-\rho\varrho\|^2+\|\chi\varsigma-\xi\varsigma\|^2}{2b}\right\}$$

$$= \max\left\{\| 5 - 5 \|^2, \frac{\|5-5\|^2}{1+\|5-5\|^2}, \frac{\|5-5\|^2}{1+\|5-5\|^2}, \frac{\|5-5\|^2+\|5-5\|^2}{4}, \frac{\|5-5\|^2+\|5-5\|^2}{4}\right\}$$

$$= \max\{0, 0, 0, 0, 0\}$$

$\lambda(\varrho, \varsigma) = 0$

$R.H.S = \psi(0) - \varphi(0) = 0 - 0 = 0$. *Therefore, $L.H.S = R.H.S$*

Case (ii): *Let $\varrho, \varsigma \in (3, 7)$*

$L.H.S = \psi(\| \omega\varrho - \chi\varsigma \|^2) = \psi\left(\| 1 - \frac{1}{2} \|^2\right) = \psi(\frac{1}{4}) = \frac{1}{4}$
$R.H.S = \psi\left(\lambda(\varrho, \varsigma)\right) - \varphi\left(\lambda(\varrho, \varsigma)\right)$
$\lambda(\varrho, \varsigma) = \max\left\{\| \rho\varrho - \xi\varsigma \|^2, \frac{\|\omega\varrho - \rho\varrho\|2}{1+\|\omega\varrho-\rho\varrho\|^2}, \frac{\|\chi\varsigma-\xi\varsigma\|^2}{1+\|\chi\varsigma-\xi\varsigma\|^2}, \frac{\|\omega\varrho-\xi\varsigma\|^2+\|\rho\varrho-\chi\varsigma\|^2}{2b}, \frac{\|\omega\varrho-\rho\varrho\|^2+\|\chi\varsigma-\xi\varsigma\|^2}{2b}\right\}$
$= \max\left\{\| 2 - 4 \|^2, \frac{\|1-2\|^2}{1+\|1-2\|^2}, \frac{\|\frac{1}{2}-4\|^2}{1+\|\frac{1}{2}-4\|^2}, \frac{\|1-4\|^2+\|2-\frac{1}{2}\|^2}{4}, \frac{\|1-2\|^2+\|\frac{1}{2}-4\|^2}{4}\right\}$
$= \max\left\{4, \frac{1}{2}, \frac{49}{4} \times \frac{4}{53}, \frac{45}{4} \times \frac{1}{4}, \frac{53}{4} \times \frac{1}{4}\right\}$
$= \max\left\{4, \frac{1}{2}, \frac{49}{53}, \frac{45}{16}, \frac{53}{16}\right\}$
$R.H.S = \psi(4) - \varphi(4) = 4 - \frac{4}{3} = \frac{8}{3}$
Therefore, the inequality (3.1) holds. $\frac{1}{2} < \frac{8}{3}$. $L.H.S < R.H.S$

$$\psi(\| \omega\varrho - \chi\varsigma \|) \leq \psi\left(\lambda(\varrho, \varsigma)\right) - \varphi\left(\lambda(\varrho, \varsigma)\right)$$

By corollary 3.2, then the mappings ω, χ, ρ *and* ξ *have a unique common fixed point* $\varrho = 3$.

Example 3.2. *Let* $(M, \| . \|)$ *be a* $(CNBMS)$ *with coefficient* $b = 2$ *and* $E = \mathcal{R}^2$, $P = \{(\varrho, \varsigma) \in E \big| \varrho, \varsigma > 0\} \subset \mathcal{R}^2$, $M = [0, \infty)$, $D_b : M \times M \to E$, *such that* $D_b(\varrho, \varsigma) = \left(\| \varrho - \varsigma \|^2, \alpha \| \varrho - \varsigma \|^2\right), \forall\ \varrho, \varsigma \in M$, *where* $\alpha \geq 0$ *is a constant and* ω, $\chi, \rho, \xi : M \to M$. *Let us define the function* $\psi, \varphi : \mathcal{R}_0^+ \to \mathcal{R}_0^+$ *by* $\psi(t) = t$ *and* $\varphi(t) = \frac{t}{3}$.

$$\omega(\varrho) = \chi(\varrho) = \begin{cases} \frac{\varrho}{16}, & if\ \varrho \in [0,1] \\ \frac{\varrho+11}{16}, & if\ \ \varrho \in (1, \infty) \end{cases} \qquad \rho(\varrho) = \xi(\varrho) = \begin{cases} \frac{\varrho}{4}, & if\ \varrho \in [0,1] \\ \frac{\varrho+3}{4}, & if\ \ \varrho \in (1, \infty) \end{cases}$$

First we verify that condition (1) of Lemma (3.1)
Let $\{\varrho_n\} = \left\{\frac{1}{n}\right\}_{n\geq 1}$ *and* $\{\varsigma_n\} = \left\{\frac{1}{n+1}\right\}_{n\geq 1}$ *be two sequence in* M. *Then we have*

$$\begin{aligned}
\lim_{n\to\infty} \omega\{\varrho_n\} &= \lim_{n\to\infty} \omega\left\{\frac{1}{n}\right\} = \lim_{n\to\infty}\left\{\frac{1}{16n}\right\} = 0 \\
\lim_{n\to\infty} \rho\{\varrho_n\} &= \lim_{n\to\infty} \rho\left\{\frac{1}{n}\right\} = \lim_{n\to\infty}\left\{\frac{1}{4n}\right\} = 0 \\
\lim_{n\to\infty} \chi\{\varsigma_n\} &= \lim_{n\to\infty} \chi\left\{\frac{1}{n+1}\right\} = \lim_{n\to\infty}\left\{\frac{1}{16(n+1)}\right\} = 0 \\
\lim_{n\to\infty} \xi\{\varsigma_n\} &= \lim_{n\to\infty} \xi\left\{\frac{1}{n+1}\right\} = \lim_{n\to\infty}\left\{\frac{1}{4(n+1)}\right\} = 0
\end{aligned}$$

Since $\varrho(0) = \xi(0) = 0$ *we have* $0 \in \rho(M) \cap \xi(M)$. *Therefore there exists sequence* $\{\varrho_n\}$ *and* $\{\varsigma_n\}$ *in* M *such that*

$$\lim_{n\to\infty} \omega\{\varrho_n\} = \lim_{n\to\infty} \rho\{\varrho_n\} = \lim_{n\to\infty} \chi\{\varsigma_n\} = \lim_{n\to\infty} \xi\{\varsigma_n\}$$

That is, (ω, ρ) *and* (χ, ξ) *satisfies the common* $(E.A)$*-property.*
Next, to verify inequality (3.1).

$$\psi(\| \omega\varrho - \chi\varsigma \|) \leq \psi\left(\lambda(\varrho, \varsigma)\right) - \varphi\left(\lambda(\varrho, \varsigma)\right)$$

$$\lambda(\varrho,\varsigma) = \max\left\{\| \rho\varrho - \xi\varsigma \|, \frac{\|\omega\varrho-\rho\varrho\|}{1+\|\omega\varrho-\rho\varrho\|}, \frac{\|\chi\varsigma-\xi\varsigma\|}{1+\|\chi\varsigma-\xi\varsigma\|}, \frac{\|\omega\varrho-\xi\varsigma\|+\|\rho\varrho-\chi\varsigma\|}{2b}, \frac{\|\omega\varrho-\rho\varrho\|+\|\chi\varsigma-\xi\varsigma\|}{2b}\right\}$$

Case (i): *When* $\varrho,\ \varsigma \in [0,1]$

L.H.S

$$= \psi\left(\| \omega\varrho - \chi\varsigma \|^2,\ \alpha \| \omega\varrho - \chi\varsigma \|^2\right)$$

$$= \psi\left(\| \tfrac{\varrho}{16} - \tfrac{\varsigma}{16} \|^2,\ \alpha \| \tfrac{\varrho}{16} - \tfrac{\varsigma}{16} \|^2\right)$$

$$= \psi\left(\tfrac{1}{256} \| \varrho - \varsigma \|^2, \tfrac{1}{256}\ \alpha \| \varrho - \varsigma \|^2\right)$$

$$= \left(\tfrac{1}{256}, \tfrac{1}{256}\alpha\right)\psi\left(\| \varrho - \varsigma \|^2\right)$$

R.H.S

$$\begin{aligned}
\lambda(\varrho,\varsigma) &= \max\left\{\begin{array}{c}
\left(\| \tfrac{\varrho}{4} - \tfrac{\varsigma}{4} \|^2,\ \alpha \| \tfrac{\varrho}{4} - \tfrac{\varsigma}{4} \|^2\right), \frac{\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)}{1+\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)},\\
\frac{\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)}{1+\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)},\\
\frac{\left(\left(\|\frac{\varrho}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varsigma}{4}\|^2\right)+\left(\|\frac{\varrho}{4}-\frac{\varsigma}{16}\|^2,\ \alpha\|\frac{\varrho}{4}-\frac{\varsigma}{16}\|^2\right)\right)}{4},\\
\frac{\left(\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)+\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)\right)}{4}
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
\left(\tfrac{1}{16} \| \varrho - \varsigma \|^2, \tfrac{1}{16}\ \alpha \| \varrho - \varsigma \|^2\right), \frac{\left(\frac{1}{256}\|\varrho-4\varrho\|^2,\frac{1}{256}\ \alpha\|\varrho-4\varrho\|^2\right)}{1+\left(\frac{1}{256}\|\varrho-4\varrho\|^2,\frac{1}{256}\alpha\|\varrho-4\varrho\|^2\right)},\\
\frac{\left(\frac{1}{256}\|\varsigma-4\varsigma\|^2,\frac{1}{256}\ \alpha\|\varsigma-4\varsigma\|^2\right)}{1+\left(\frac{1}{256}\|\varsigma-4\varsigma\|^2,\frac{1}{256}\ \alpha\|\varsigma-4\varsigma\|^2\right)},\\
\frac{\left(\left(\frac{1}{256}\|\varrho-4\varsigma\|^2,\frac{1}{256}\alpha\|\varrho-4\varsigma\|^2\right)+\left(\frac{1}{256}\|4\varrho-\varsigma\|^2,\frac{1}{256}\ \alpha\|4\varrho-\varsigma\|^2\right)\right)}{4},\\
\frac{\left(\left(\frac{1}{256}\|\varrho-4\varrho\|^2,\frac{1}{256}\alpha\|\varrho-4\varrho\|^2\right)+\left(\frac{1}{256}\|\varsigma-4\varsigma\|^2,\frac{1}{256}\alpha\|\varsigma-4\varsigma\|^2\right)\right)}{4}
\end{array}\right\}\\
&= \max\left\{\begin{array}{c}
\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| \varrho - \varsigma \|^2\right), \frac{\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|\varrho-4\varrho\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4\varrho\|^2\right)},\\
\frac{\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|\varsigma-4\varsigma\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|\varsigma-4\varsigma\|^2\right)},\\
\frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4\varsigma\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|4\varrho-\varsigma\|^2\right)}{4},\\
\frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4\varrho\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varsigma-4\varsigma\|^2\right)}{4}
\end{array}\right\}\\
\lambda(\varrho,\varsigma) &= \left\{\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| \varrho - \varsigma \|^2\right\}
\end{aligned}$$

$$R.H.S = \psi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| \varrho - \varsigma \|^2\right) - \varphi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| \varrho - \varsigma \|^2\right)$$

$$= \left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\left(\psi\left(\|\,\varrho - \varsigma\,\|^2\right) - \varphi\left(\|\,\varrho - \varsigma\,\|^2\right)\right)$$

$$\left(\frac{1}{256}, \frac{1}{256}\alpha\right)\psi\left(\|\,\varrho - \varsigma\,\|^2\right) \leq \left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\left(\psi\left(\|\,\varrho - \varsigma\,\|^2\right) - \varphi\left(\|\,\varrho - \varsigma\,\|^2\right)\right)$$

Therefore, L.H.S $\leq$ R.H.S. Clearly inequality (3.1) is satisfied for each $\varrho, \varsigma \in [0,1]$ and $\alpha \geq 0$.

Case (ii): *When $\varrho \in [0,1]$ and $\varsigma \in (1,\infty)$*

L.H.S

$$= \psi\left(\|\,\omega\varrho - \chi\varsigma\,\|^2,\ \alpha\,\|\,\omega\varrho - \chi\varsigma\,\|^2\,\right)$$

$$= \psi\left(\|\,\frac{\varrho}{16} - \frac{\varsigma+11}{16}\,\|^2,\ \alpha\,\|\,\frac{\varrho}{16} - \frac{\varsigma+11}{16}\,\|^2\,\right)$$

$$= \psi\left(\frac{1}{256}\,\|\,\varrho - (\varsigma+11)\,\|^2, \frac{1}{256}\,\alpha\,\|\,\varrho - (\varsigma+11)\,\|^2\,\right)$$

$$= \left(\frac{1}{256}, \frac{1}{256}\alpha\right)\psi\left(\|\,\varrho - (\varsigma+11)\,\|^2\right)$$

R.H.S

$$\lambda(\varrho,\varsigma) = \max\left\{\begin{matrix} \left(\|\,\frac{\varrho}{4} - \frac{\varsigma+3}{4}\,\|^2,\ \alpha\,\|\,\frac{\varrho}{4} - \frac{\varsigma+3}{4}\,\|^2\right), \frac{\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)}{1+\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)}, \\ \frac{\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)}{1+\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)}, \\ \frac{\left(\left(\|\frac{\varrho}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varsigma+3}{4}\|^2\right)+\left(\|\frac{\varrho}{4}-\frac{\varsigma+11}{16}\|^2,\ \alpha\|\frac{\varrho}{4}-\frac{\varsigma+11}{16}\|^2\right)\right)}{4}, \\ \frac{\left(\left(\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2,\ \alpha\|\frac{\varrho}{16}-\frac{\varrho}{4}\|^2\right)+\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)\right)}{4} \end{matrix}\right\}$$

$$= \max\left\{\begin{matrix} \left(\left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\|\,\varrho - (\varsigma+3)\,\|^2\right), \frac{\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|\varrho-4\varrho\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4\varrho\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4(\varsigma+3)\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|4\varrho-(\varsigma+11)\|^2\right)}{4}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varrho-4\varrho\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}{4} \end{matrix}\right\}$$

$$\lambda(\varrho,\varsigma) = \left\{\left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\|\,\varrho - (\varsigma+3)\,\|^2\right\}$$

$$R.H.S = \psi\left(\left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\|\,\varrho - (\varsigma+3)\,\|^2\right) - \varphi\left(\left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\|\,\varrho - (\varsigma+3)\,\|^2\right)$$

$$= \left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\left(\psi\left(\|\,\varrho - (\varsigma+3)\,\|^2\right) - \varphi\left(\|\,\varrho - (\varsigma+3)\,\|^2\right)\right)$$

$$\left(\frac{1}{256}, \frac{1}{256}\alpha\right)\psi\left(\|\,\varrho - (\varsigma+11)\,\|^2\right) \leq \left(\frac{1}{16}, \frac{1}{16}\,\alpha\right)\left(\psi\left(\|\,\varrho - (\varsigma+3)\,\|^2\right) - \varphi\left(\|\,\varrho - (\varsigma+3)\,\|^2\right)\right)$$

Therefore, L.H.S $\leq$ R.H.S. Calculating the same as in case (i), we conclude that inequality (3.1) is satisfied for each $\varrho \in [0,1]$ and $\varsigma \in (1,\infty)$ and $\alpha \geq 0$.

Case (iii): *When $\varrho \in (1,\infty)$ and $\varsigma \in [0,1]$*
L.H.S

$$= \psi\left(\| \,\omega\varrho - \chi\varsigma\, \|^2,\ \alpha \,\| \,\omega\varrho - \chi\varsigma\, \|^2\right)$$

$$= \psi\left(\| \,\tfrac{\varrho+11}{16} - \tfrac{\varsigma}{16}\, \|^2,\ \alpha \,\| \,\tfrac{\varrho+11}{16} - \tfrac{\varsigma}{16}\, \|^2\right)$$

$$= \psi\left(\tfrac{1}{256}\,\| \,(\varrho+11) - \varsigma\, \|^2, \tfrac{1}{256}\,\alpha \,\| \,(\varrho+11) - \varsigma\, \|^2\right)$$

$$= \left(\tfrac{1}{256}, \tfrac{1}{256}\alpha\right)\psi\left(\| \,(\varrho+11) - \varsigma\, \|^2\right)$$

R.H.S

$$\lambda(\varrho,\varsigma) = \max\left\{\begin{array}{c} \left(\| \,\frac{\varrho+3}{4} - \frac{\varsigma}{4}\, \|^2,\ \alpha \,\| \,\frac{\varrho+3}{4} - \frac{\varsigma}{4}\, \|^2\right), \frac{\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)}{1+\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)}, \\ \frac{\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)}{1+\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)}, \\ \frac{\left(\left(\|\frac{\varrho+11}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varsigma}{4}\|^2\right)+\left(\|\frac{\varrho+3}{4}-\frac{\varsigma}{16}\|^2,\ \alpha\|\frac{\varrho+3}{4}-\frac{\varsigma}{16}\|^2\right)\right)}{4}, \\ \frac{\left(\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)+\left(\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2,\ \alpha\|\frac{\varsigma}{16}-\frac{\varsigma}{4}\|^2\right)\right)}{4} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \left(\left(\frac{1}{16}, \frac{1}{16}\,\alpha\right) \| \,(\varrho+3) - \varsigma\, \|^2\right), \frac{\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|(\varrho+11)-4(\varrho+3)\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+11)-4(\varrho+3)\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|\varsigma-4\varsigma\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|\varsigma-4\varsigma\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+11)-4\varsigma\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\,\alpha\right)\|4(\varrho+3)-\varsigma\|^2\right)}{4}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+15)-4(\varrho+3)\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|\varsigma-4\varsigma\|^2\right)}{4} \end{array}\right\}$$

$$\lambda(\varrho,\varsigma) = \left\{\left(\tfrac{1}{16}, \tfrac{1}{16}\,\alpha\right) \| \,(\varrho+3) - \varsigma\, \|^2\right\}$$

$$R.H.S = \psi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\,\alpha\right) \| \,(\varrho+3) - \varsigma\, \|^2\right) - \varphi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\,\alpha\right) \| \,(\varrho+3) - \varsigma\, \|^2\right)$$

$$= \left(\tfrac{1}{16}, \tfrac{1}{16}\,\alpha\right)\left(\psi\left(\| \,(\varrho+3) - \varsigma\, \|^2\right) - \varphi\left(\| \,(\varrho+3) - \varsigma\, \|^2\right)\right)$$

$$\left(\tfrac{1}{256}, \tfrac{1}{256}\alpha\right)\psi\left(\| \,(\varrho+11) - \varsigma\, \|^2\right) \leq \left(\tfrac{1}{16}, \tfrac{1}{16}\,\alpha\right)\left(\psi\left(\| \,(\varrho+3) - \varsigma\, \|^2\right) - \varphi\left(\| \,(\varrho+3) - \varsigma\, \|^2\right)\right)$$

Therefore, L.H.S $\leq$ R.H.S. Inequality (3.1) is satisfied for each $\varrho \in (1,\infty)$ and $\varsigma \in [0,1]$ and $\alpha \geq 0$.

Case (iv): *When $\varrho, \varsigma \in (1,\infty)$*

L.H.S

$$= \psi\left(\| \omega\varrho - \chi\varsigma \|^2,\ \alpha \| \omega\varrho - \chi\varsigma \|^2 \right)$$

$$= \psi\left(\| \tfrac{\varrho+11}{16} - \tfrac{\varsigma+11}{16} \|^2,\ \alpha \| \tfrac{\varrho+11}{16} - \tfrac{\varsigma+11}{16} \|^2 \right)$$

$$= \psi\left(\tfrac{1}{256} \| (\varrho+11) - (\varsigma+11) \|^2, \tfrac{1}{256}\ \alpha \| (\varrho+11) - (\varsigma+11) \|^2 \right)$$

$$= \left(\tfrac{1}{256}, \tfrac{1}{256}\alpha\right)\psi\left(\| (\varrho+11) - (\varsigma+11) \|^2\right)$$

R.H.S

$$\lambda(\varrho,\varsigma) = \max\left\{\begin{array}{c} \left(\| \frac{\varrho+3}{4} - \frac{\varsigma+3}{4} \|^2,\ \alpha \| \frac{\varrho+3}{4} - \frac{\varsigma+3}{4} \|^2\right), \frac{\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)}{1+\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)}, \\ \frac{\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)}{1+\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)}, \\ \frac{\left(\left(\|\frac{\varrho+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varsigma+3}{4}\|^2\right)+\left(\|\frac{\varrho+3}{4}-\frac{\varsigma+11}{16}\|^2,\ \alpha\|\frac{\varrho+3}{4}-\frac{\varsigma+11}{16}\|^2\right)\right)}{4}, \\ \frac{\left(\left(\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2,\ \alpha\|\frac{\varrho+11}{16}-\frac{\varrho+3}{4}\|^2\right)+\left(\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2,\ \alpha\|\frac{\varsigma+11}{16}-\frac{\varsigma+3}{4}\|^2\right)\right)}{4} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \left(\left(\frac{1}{16}, \frac{1}{16}\ \alpha\right) \| (\varrho+3) - (\varsigma+3) \|^2\right), \frac{\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|(\varrho+11)-4(\varrho+3)\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+11)-4(\varrho+3)\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}{1+\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+11)-4(\varsigma+3)\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\ \alpha\right)\|4(\varrho+3)-(\varsigma+11)\|^2\right)}{4}, \\ \frac{\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varrho+15)-4(\varrho+3)\|^2\right)+\left(\left(\frac{1}{256},\frac{1}{256}\alpha\right)\|(\varsigma+11)-4(\varsigma+3)\|^2\right)}{4} \end{array}\right\}$$

$$\lambda(\varrho,\varsigma) = \left\{\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| (\varrho+3) - (\varsigma+3) \|^2\right\}$$

$$R.H.S\ = \psi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| (\varrho+3) - (\varsigma+3) \|^2\right) - \varphi\left(\left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right) \| (\varrho+3) - (\varsigma+3) \|^2\right)$$

$$= \left(\frac{1}{16}, \frac{1}{16}\ \alpha\right)\left(\psi\left(\| (\varrho+3) - (\varsigma+3) \|^2\right) - \varphi\left(\| (\varrho+3) - (\varsigma+3) \|^2\right)\right)$$

$$\left(\tfrac{1}{256}, \tfrac{1}{256}\alpha\right)\psi\left(\| (\varrho+11) - (\varsigma+11) \|^2\right) \le \left(\tfrac{1}{16}, \tfrac{1}{16}\ \alpha\right)\begin{pmatrix} \psi\left(\| (\varrho+3) - (\varsigma+3) \|^2\right) \\ -\varphi\left(\| (\varrho+3) - (\varsigma+3) \|^2\right) \end{pmatrix}$$

Therefore, L.H.S ≤ R.H.S. Inequality (3.1) is satisfied for each $\varrho, \varsigma \in (1, \infty)$ *and* $\alpha \ge 0$. *Thus all the conditions of Theorem (3.2) are satisfied and 0 remains the unique common fixed point of the mappings* $\omega,\ \chi, \rho$ *and* ξ.

4. Conclusion

This article illustrates the work establishing the $(E.A)$- properties for four self-maps via ultra-altering and altering distance mapping within the framework of common fixed points theorems for $(CNBMS)$. The result's existence and uniqueness are described in this article. Our result might serve as inspiration for future writers to extend and improve several results in such spaces and applications to other related areas.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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