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Derivations of Hilbert Algebras

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Abstract. In this paper, we introduce the notions of (l, r)-derivations, (r, l)-derivations, and derivations of Hilbert algebras and investigate some related properties. In addition, we define two subsets for a derivation d of a Hilbert algebra X, $\operatorname{Ker} d(X)$ and $\operatorname{Fix} d(X)$, and we also take a look at some of their characteristics.

1. Introduction and Preliminaries

Logic algebras are a significant class of algebras among several other algebraic structures. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [9] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [7] from algebraic point of view. Diego [7] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [4, 5] and Jun [12] and some of their filters forming deductive systems were recognized.

The study of derivations has continued, for example, in 2021, Muangkarn et al. [14] studied f_q-derivations, and Bantaojai et al. [3] studied derivations induced by an endomorphism of B-algebras. In 2022, Bantaojai et al. [1, 2] studied derivations on d-algebras and B-algebras, and Muangkarn et al. [13, 15] studied derivations induced by an endomorphism of BG-algebras and d-algebras. Iampan et al. [10, 16, 17] studied derivations on UP-algebras.

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The concepts of (l, r)-derivations, (r, l)-derivations, and derivations of Hilbert algebras are introduced in this work along with several related features. In addition, we define two subsets for a derivation d of a Hilbert algebra X, $\operatorname{Ker} d(X)$ and $\operatorname{Fix} d(X)$, and we also take a look at some of their characteristics.

Let's go through the idea of Hilbert algebras as it was introduced by Diego [7] in 1966 before we start.

Definition 1.1. *[7] A Hilbert algebra is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:*

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [8], the following conclusion was established.

Lemma 1.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 1.2. *[18] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a subalgebra of X if $x \cdot y \in D$ for all $x, y \in D$.*

Definition 1.3. *[6] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an ideal of X if the following conditions hold:*

(1) $1 \in D$,
(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,
(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

For any x, y in a Hilbert algebra $X = (X, \cdot, 1)$, we define $x \vee y$ by $(y \cdot x) \cdot x$. Note that $x \vee y$ is an upper bound of x and y for all $x, y \in X$. A Hilbert algebra $X = (X, \cdot, 1)$ is said to be *commutative* [11] if for all $x, y \in X$, $(y \cdot x) \cdot x = (x \cdot y) \cdot y$, that is, $x \vee y = y \vee x$. From [11], we know that

$$(\forall x \in X)(x \vee x = x),$$
$$(\forall x \in X)(x \vee 1 = 1 \vee x = 1).$$

2. Main Results

In this section, we introduce the notions of an (l, r)-derivation, an (r, l)-derivation and a derivation of a Hilbert algebra and study some of their basic properties. Finally, we define two subsets $\operatorname{Ker} d(X)$ and $\operatorname{Fix} d(X)$ for a derivation d of a Hilbert algebra X, and we consider some properties of these as well.

Definition 2.1. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. A self-map $d : X \to X$ is called an (l, r)-derivation of X if it satisfies the identity $d(x \cdot y) = (d(x) \cdot y) \vee (x \cdot d(y))$ for all $x, y \in X$. Similarly, a self-map $d : X \to X$ is called an (r, l)-derivation of X if it satisfies the identity $d(x \cdot y) = (x \cdot d(y)) \vee (d(x) \cdot y)$ for all $x, y \in X$. Moreover, if d is both an (l, r)-derivation and an (r, l)-derivation of X, it is called a derivation of X.*

Example 2.1. *Let $X = \{1, 2, 3, 4\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table:*

$\cdot$	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	1	2	1	4
4	1	2	3	1

Define a self-map $d : X \to X$ by for any $x \in X$,

$$d(x) = \begin{cases} 1 & \text{if } x \neq 2 \\ 2 & \text{if } x = 2. \end{cases}$$

Then d is a derivation of X.

Definition 2.2. *An (l, r)-derivation (resp., (r, l)-derivation, derivation) d of a Hilbert algebra $X = (X, \cdot, 1)$ is said to be regular if $d(1) = 1$.*

Theorem 2.1. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) every (l, r)-derivation of X is regular,
(2) every (r, l)-derivation of X is regular.

Proof. (1) Assume that d is an (l, r)-derivation of X. Then $d(1) = d(1 \cdot 1) = (d(1) \cdot 1) \vee (1 \cdot d(1)) = 1 \vee d(1) = 1$. Hence d is regular.

(2) Assume that d is an (r,l)-derivation of X. Then $d(1) = d(1\cdot 1) = (1\cdot d(1)) \vee (d(1)\cdot 1) = d(1)\vee 1 = 1$. Hence d is regular. □

Corollary 2.1. *Every derivation of a Hilbert algebra $X = (X,\cdot,1)$ is regular.*

Theorem 2.2. *In a Hilbert algebra $X = (X,\cdot,1)$, the following statements hold:*

(1) if d is an (l,r)-derivation of X, then $d(x) = x \vee d(x)$ for all $x \in X$,

(2) if d is an (r,l)-derivation of X, then $d(x) = d(x)\vee x$ for all $x \in X$.

Proof. (1) Assume that d is an (l,r)-derivation of X. Then for all $x \in X$, $d(x) = d(1\cdot x) = (d(1)\cdot x)\vee(1\cdot d(x)) = (1\cdot x)\vee d(x) = x\vee d(x)$.

(2) Assume that d is an (r,l)-derivation of X. Then for all $x\in X$, $d(x) = d(1\cdot x) = (1\cdot d(x))\vee (d(1)\cdot x) = d(x)\vee(1\cdot x) = d(x)\vee x$. □

Corollary 2.2. *If d is a derivation of a Hilbert algebra $X = (X,\cdot,1)$, then $d(x)\vee x = x\vee d(x)$ for all $x\in X$.*

Definition 2.3. *Let d be an (l,r)-derivation (resp., (r,l)-derivation, derivation) of a Hilbert algebra $X = (X,\cdot,1)$. We define a subset* $\operatorname{Ker} d(X)$ *of X by* $\operatorname{Ker} d(X) = \{x\in X : d(x) = 1\}$.

Proposition 2.1. *Let d be an (l,r)-derivation of a Hilbert algebra $X = (X,\cdot,1)$. Then the following properties hold: for any $x,y\in X$,*

(1) $x\le d(x)$,

(2) $d(x)\cdot y\le d(x\cdot y)$,

(3) $d(x\cdot d(x)) = 1$,

(4) $d(d(x)\cdot x) = 1$,

(5) $x\le d(d(x))$.

Proof. (1) For all $x\in X$, $x\cdot d(x) = x\cdot(x\vee d(x)) = x\cdot((d(x)\cdot x)\cdot x) = 1$. Hence $x\le d(x)$.

(2) For all $x,y\in X$, $(d(x)\cdot y)\cdot d(x\cdot y) = (d(x)\cdot y)\cdot((d(x)\cdot y)\vee(x\cdot d(y))) = (d(x)\cdot y)\cdot(((x\cdot d(y))\cdot(d(x)\cdot y))\cdot(d(x)\cdot y)) = 1$. Hence $d(x)\cdot y\le d(x\cdot y)$.

(3) For all $x\in X$, $d(x\cdot d(x)) = (d(x)\cdot d(x))\vee(x\cdot d(d(x))) = 1\vee(x\cdot d(d(x))) = 1$.

(4) For all $x\in X$, $d(d(x)\cdot x) = (d(d(x))\cdot x)\vee(d(x)\cdot d(x)) = (d(d(x))\cdot x)\vee 1 = 1$.

(5) For all $x\in X$, $d(d(x)) = d(x\vee d(x)) = d((d(x)\cdot x)\cdot x) = (d(d(x)\cdot x)\cdot x)\vee((d(x)\cdot x)\cdot d(x)) = (1\cdot x)\vee((d(x)\cdot x)\cdot d(x)) = x\vee((d(x)\cdot x)\cdot d(x)) = (((d(x)\cdot x)\cdot d(x))\cdot x)\cdot x$. Thus $x\cdot d(d(x)) = x\cdot((((d(x)\cdot x)\cdot d(x))\cdot x)\cdot x) = 1$. Hence $x\le d(d(x))$. □

Proposition 2.2. *Let d be an (r,l)-derivation of a Hilbert algebra $X = (X,\cdot,1)$. Then the following properties hold: for any $x,y\in X$,*

(1) $x\cdot d(y)\le d(x\cdot y)$,

(2) $d(x\cdot d(x)) = 1$,

(3) $d(d(x) \cdot x) = 1$.

Proof. (1) For all $x, y \in X$, $(x \cdot d(y)) \cdot d(x \cdot y) = (x \cdot d(y)) \cdot ((x \cdot d(y)) \vee (d(x) \cdot y)) = (x \cdot d(y)) \cdot (((d(x) \cdot y) \cdot (x \cdot d(y))) \cdot (x \cdot d(y))) = 1$. Hence $x \cdot d(y) \leq d(x \cdot y)$.

(2) For all $x \in X$, $d(x \cdot d(x)) = (x \cdot d(d(x))) \vee (d(x) \cdot d(x)) = (x \cdot d(d(x))) \vee 1 = 1$.

(3) For all $x \in X$, $d(d(x) \cdot x) = (d(x) \cdot d(x)) \vee (d(d(x)) \cdot x) = 1 \vee (d(d(x)) \cdot x) = 1$. $\square$

Theorem 2.3. *Let $d_1, d_2, \ldots, d_n$ be (l, r)-derivations of a Hilbert algebra $X = (X, \cdot, 1)$ for all $n \in \mathbb{N}$. Then $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))$ for all $x \in X$. In particular, if d is an (l, r)-derivation of X, then $x \leq d_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$.*

Proof. For $n = 1$, it follows from Proposition 2.1 (1) that $x \leq d_1(x)$ for all $x \in X$. Let $n \in \mathbb{N}$ and assume that $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))$ for all $x \in X$. Let $D_n = d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))$. Then

$$\begin{aligned} d_{n+1}(D_n) &= d_{n+1}(1 \cdot D_n) \\ &= (d_{n+1}(1) \cdot D_n) \vee (1 \cdot d_{n+1}(D_n)) \\ &= (1 \cdot D_n) \vee (1 \cdot d_{n+1}(D_n)) \\ &= D_n \vee d_{n+1}(D_n) \\ &= (d_{n+1}(D_n) \cdot D_n) \cdot D_n. \end{aligned}$$

Thus

$$D_n \cdot d_{n+1}(D_n) = D_n \cdot ((d_{n+1}(D_n) \cdot D_n) \cdot D_n) = 1.$$

Therefore, $D_n \leq d_{n+1}(D_n)$. By assumption, we get

$$x \leq D_n \leq d_{n+1}(D_n) = d_{n+1}(d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)))$$

for all $x \in X$. Hence $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots))$ for all $n \in \mathbb{N}$ and $x \in X$. In particular, put $d = d_n$ for all $n \in \mathbb{N}$. Hence $x \leq d_n(d_{n-1}(\ldots(d_2(d_1(x)))\ldots)) = d_n(x)$ for all $n \in \mathbb{N}$ and $x \in X$. $\square$

Definition 2.4. *An ideal D of a Hilbert algebra $X = (X, \cdot, 1)$ is said to be invariant (with respect to an (l, r)-derivation (resp., (r, l)-derivation, derivation) d of X) if $d(D) \subseteq D$.*

Theorem 2.4. *Every ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is invariant with respect to any (l, r)-derivation of X.*

Proof. Let D be an ideal of X and d an (l, r)-derivation of X. Let $y \in d(D)$. Then $y = d(x)$ for some $x \in D$. It follows that $y \cdot x = d(x) \cdot x = 1 \in D$, which implies $y \in D$. Thus $d(D) \subseteq D$. Hence D is invariant with respect to an (l, r)-derivation d of X. $\square$

Corollary 2.3. *Every ideal of a Hilbert algebra $X = (X, \cdot, 1)$ is invariant with respect to any derivation of X.*

Theorem 2.5. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X, then $y \vee x \in \operatorname{Ker} d(X)$ for all $y \in \operatorname{Ker} d(X)$ and $x \in X$,

(2) if d is an (r, l)-derivation of X, then $y \vee x \in \operatorname{Ker} d(X)$ for all $y \in \operatorname{Ker} d(X)$ and $x \in X$.

Proof. (1) Assume that d is an (l, r)-derivation of X. Let $y \in \operatorname{Ker} d(X)$ and $x \in X$. Then $d(y) = 1$. Thus $d(y \vee x) = d((x \cdot y) \cdot y) = (d(x \cdot y) \cdot y) \vee ((x \cdot y) \cdot d(y)) = (d(x \cdot y) \cdot y) \vee ((x \cdot y) \cdot 1) = (d(x \cdot y) \cdot y) \vee 1 = 1$. Hence $y \vee x \in \operatorname{Ker} d(X)$.

(2) Assume that d is an (r, l)-derivation of X. Let $y \in \operatorname{Ker} d(X)$ and $x \in X$. Then $d(y) = 1$. Thus $d(y \vee x) = d((x \cdot y) \cdot y) = ((x \cdot y) \cdot d(y)) \vee (d(x \cdot y) \cdot y) = ((x \cdot y) \cdot 1) \vee (d(x \cdot y) \cdot y) = 1 \vee (d(x \cdot y) \cdot y) = 1$. Hence $y \vee x \in \operatorname{Ker} d(X)$. □

Corollary 2.4. *If d is a derivation of a Hilbert algebra $X = (X, \cdot, 1)$, then $y \vee x \in \operatorname{Ker} d(X)$ for all $y \in \operatorname{Ker} d(X)$ and $x \in X$.*

Theorem 2.6. *In a commutative Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X and for any $x, y \in X$ is such that $y \leq x$ and $y \in \operatorname{Ker} d(X)$, then $x \in \operatorname{Ker} d(X)$,

(2) if d is an (r, l)-derivation of X and for any $x, y \in X$ is such that $y \leq x$ and $y \in \operatorname{Ker} d(X)$, then $x \in \operatorname{Ker} d(X)$.

Proof. (1) Assume that d is an (l, r)-derivation of X. Let $x, y \in X$ be such that $y \leq x$ and $y \in \operatorname{Ker} d(X)$. Then $y \cdot x = 1$ and $d(y) = 1$. Thus $d(x) = d(1 \cdot x) = d((y \cdot x) \cdot x) = d((x \cdot y) \cdot y) = (d(x \cdot y) \cdot y) \vee ((x \cdot y) \cdot d(y)) = (d(x \cdot y) \cdot y) \vee ((x \cdot y) \cdot 1) = (d(x \cdot y) \cdot y) \vee 1 = 1$. Hence $x \in \operatorname{Ker} d(X)$.

(2) Assume that d is an (r, l)-derivation of X. Let $x, y \in X$ be such that $y \leq x$ and $y \in \operatorname{Ker} d(X)$. Then $y \cdot x = 1$ and $d(y) = 1$. Thus $d(x) = d(1 \cdot x) = d((y \cdot x) \cdot x) = d((x \cdot y) \cdot y) = ((x \cdot y) \cdot d(y)) \vee (d(x \cdot y) \cdot y) = ((x \cdot y) \cdot 1) \vee (d(x \cdot y) \cdot y) = 1 \vee (d(x \cdot y) \cdot y) = 1$. Hence $x \in \operatorname{Ker} d(X)$. □

Corollary 2.5. *If d is a derivation of a commutative Hilbert algebra $X = (X, \cdot, 1)$ and for any $x, y \in X$ is such that $y \leq x$ and $y \in \operatorname{Ker} d(X)$, then $x \in \operatorname{Ker} d(X)$.*

Theorem 2.7. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X, then $y \cdot x \in \operatorname{Ker} d(X)$ for all $x \in \operatorname{Ker} d(X)$ and $y \in X$,

(2) if d is an (r, l)-derivation of X, then $y \cdot x \in \operatorname{Ker} d(X)$ for all $x \in \operatorname{Ker} d(X)$ and $y \in X$.

Proof. (1) Assume that d is an (l, r)-derivation of X. Let $x \in \operatorname{Ker} d(X)$ and $y \in X$. Then $d(x) = 1$. Thus $d(y \cdot x) = (d(y) \cdot x) \vee (y \cdot d(x)) = (d(y) \cdot x) \vee (y \cdot 1) = (d(y) \cdot x) \vee 1 = 1$. Hence $y \cdot x \in \operatorname{Ker} d(X)$.

(2) Assume that d is an (r, l)-derivation of X. Let $x \in \operatorname{Ker} d(X)$ and $y \in X$. Then $d(x) = 1$. Thus $d(y \cdot x) = (y \cdot d(x)) \vee (d(y) \cdot x) = (y \cdot 1) \vee (d(y) \cdot x) = 1 \vee (d(y) \cdot x) = 1$. Hence $y \cdot x \in \operatorname{Ker} d(X)$. □

Corollary 2.6. *If d is a derivation of a Hilbert algebra $X = (X, \cdot, 1)$, then $y \cdot x \in \operatorname{Ker} d(X)$ for all $x \in \operatorname{Ker} d(X)$ and $y \in X$.*

Theorem 2.8. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X, then $\operatorname{Ker} d(X)$ is a subalgebra of X,

(2) if d is an (r, l)-derivation of X, then $\operatorname{Ker} d(X)$ is a subalgebra of X.

Proof. (1) Assume that d is an (l, r)-derivation of X. By Theorem 2.1 (1), we have $d(1) = 1$ and so $1 \in \operatorname{Ker} d(X) \neq \emptyset$. Let $x, y \in \operatorname{Ker} d(X)$. Then $d(x) = 1$ and $d(y) = 1$. Thus $d(x \cdot y) = (d(x) \cdot y) \vee (x \cdot d(y)) = (1 \cdot y) \vee (x \cdot 1) = y \vee 1 = 1$. Hence $x \cdot y \in \operatorname{Ker} d(X)$, so $\operatorname{Ker} d(X)$ is a subalgebra of X.

(2) Assume that d is an (r, l)-derivation of X. By Theorem 2.1 (2), we have $d(1) = 1$ and so $1 \in \operatorname{Ker} d(X) \neq \emptyset$. Let $x, y \in \operatorname{Ker} d(X)$. Then $d(x) = 1$ and $d(y) = 1$. Thus $d(x \cdot y) = (x \cdot d(y)) \vee (d(x) \cdot y) = (x \cdot 1) \vee (1 \cdot y) = 1 \vee y = 1$. Hence $x \cdot y \in \operatorname{Ker} d(X)$, so $\operatorname{Ker} d(X)$ is a subalgebra of X. □

Corollary 2.7. *If d is a derivation of a Hilbert algebra $X = (X, \cdot, 1)$, then $\operatorname{Ker} d(X)$ is a subalgebra of X.*

Definition 2.5. *Let d be an (l, r)-derivation (resp., (r, l)-derivation, derivation) of a Hilbert algebra $X = (X, \cdot, 1)$. We define a subset $\operatorname{Fix} d(X)$ of X by $\operatorname{Fix} d(X) = \{x \in X : d(x) = x\}$.*

Theorem 2.9. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X, then $\operatorname{Fix} d(X)$ is a subalgebra of X,

(2) if d is an (r, l)-derivation of X, then $\operatorname{Fix} d(X)$ is a subalgebra of X.

Proof. (1) Assume that d is an (l, r)-derivation of X. By Theorem 2.1 (1), we have $d(1) = 1$ and so $1 \in \operatorname{Fix} d(X) \neq \emptyset$. Let $x, y \in \operatorname{Fix} d(X)$. Then $d(x) = x$ and $d(y) = y$. Thus $d(x \cdot y) = (d(x) \cdot y) \vee (x \cdot d(y)) = (x \cdot y) \vee (x \cdot y) = x \cdot y$. Hence $x \cdot y \in \operatorname{Fix} d(X)$, so $\operatorname{Fix} d(X)$ is a subalgebra of X.

(2) Assume that d is an (r, l)-derivation of X. By Theorem 2.1 (2), we have $d(1) = 1$ and so $1 \in \operatorname{Fix} d(X) \neq \emptyset$. Let $x, y \in \operatorname{Fix} d(X)$. Then $d(x) = x$ and $d(y) = y$. Thus $d(x \cdot y) = (x \cdot d(y)) \vee (d(x) \cdot y) = (x \cdot y) \vee (x \cdot y) = x \cdot y$. Hence $x \cdot y \in \operatorname{Fix} d(X)$, so $\operatorname{Fix} d(X)$ is a subalgebra of X. □

Corollary 2.8. *If d is a derivation of a Hilbert algebra $X = (X, \cdot, 1)$, then $\operatorname{Fix} d(X)$ is a subalgebra of X.*

Theorem 2.10. *In a Hilbert algebra $X = (X, \cdot, 1)$, the following statements hold:*

(1) if d is an (l, r)-derivation of X, then $x \vee y \in \operatorname{Fix} d(X)$ for all $x, y \in \operatorname{Fix} d(X)$,

(2) if d is an (r, l)-derivation of X, then $x \vee y \in \operatorname{Fix} d(X)$ for all $x, y \in \operatorname{Fix} d(X)$.

Proof. (1) Assume that d is an (l, r)-derivation of X. Let $x, y \in \operatorname{Fix} d(X)$. Then $d(x) = x$ and $d(y) = y$. By Theorem 2.9 (1), we get $d(y \cdot x) = y \cdot x$. Thus $d(x \vee y) = d((y \cdot x) \cdot x) = (d(y \cdot x) \cdot x) \vee ((y \cdot x) \cdot d(x)) = ((y \cdot x) \cdot x) \vee ((y \cdot x) \cdot x) = (y \cdot x) \cdot x = x \vee y$. Hence $x \vee y \in \operatorname{Fix} d(X)$.

(2) Assume that d is an (r,l)-derivation of X. Let $x, y \in \operatorname{Fix} d(X)$. Then $d(x) = x$ and $d(y) = y$. By Theorem 2.9 (2), we get $d(y \cdot x) = y \cdot x$. Thus $d(x \vee y) = d((y \cdot x) \cdot x) = ((y \cdot x) \cdot d(x)) \vee (d(y \cdot x) \cdot x) = ((y \cdot x) \cdot x) \vee ((y \cdot x) \cdot x) = (y \cdot x) \cdot x = x \vee y$. Hence $x \vee y \in \operatorname{Fix} d(X)$. □

Corollary 2.9. *If d is a derivation of a Hilbert algebra $X = (X, \cdot, 1)$, then $x \vee y \in \operatorname{Fix} d(X)$ for all $x, y \in \operatorname{Fix} d(X)$.*

3. Conclusion

In this article, we introduced the ideas of (l,r)-derivations, (r,l)-derivations, and derivations of Hilbert algebras, and deduced their significant features. Additionally, two subsets $\operatorname{Ker} d(X)$ and $\operatorname{Fix} d(X)$ for a derivation d of a Hilbert algebra X are defined. As a result, we have found that $\operatorname{Ker} d(X)$ and $\operatorname{Fix} d(X)$ are subalgebras of X.

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