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Quasi-Ideals and Bi-Ideals of Near Left Almost Rings

Thiti Gaketem, Tanaphong Prommai*

Fuzzy Algebras and Decision-Making Problems Research Unit, Department of Mathematics School of Science, University of Phayao, Phayao 56000, Thailand

* *Corresponding author:* tanaphong.pr@up.ac.th

Abstract. In this paper, we define quasi-ideal, bi-ideal, and weak bi-ideal of nLA-ring, and investigate it properties.

1. Introduction

M.A. Kazim and MD. Naseeruddin defined LA-semigroup as the following; a groupoid S is called a left almost semigroup, abbreviated as LA-semigroup if

$$(ab)c = (cb)a, \quad \forall\, a, b, c \in S$$

M.A. Kazim and MD. Naseeruddin [2] asserted that, in every LA-semigroups G a *medial law* hold

$$(a \cdot b) \cdot (c \cdot d) = (a \cdot c) \cdot (b \cdot d), \quad \forall\, a, b, c, d \in G.$$

Q. Mushtaq and M. Khan [4] asserted that, in every LA-semigroups G with left identity

$$(a \cdot b) \cdot (c \cdot d) = (d \cdot b) \cdot (c \cdot a), \quad \forall\, a, b, c, d \in G.$$

Further M. Khan, Faisal, and V. Amjid [3], asserted that, if an LA-semigroup G with left identity the following law holds

$$a \cdot (b \cdot c) = b \cdot (a \cdot c), \quad \forall\, a, b, c \in G.$$

M. Sarwar (Kamran) [6] defined LA-group as the following; a groupoid G is called a left almost group, abbreviated as LA-group, if (i) there exists $e \in G$ such that $ea = a$ for all $a \in G$, (ii) for every $a \in G$ there exists $a' \in G$ such that, $a'a = e$, (iii) $(ab)c = (cb)a$ for every $a, b, c \in G$.

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A non-empty subset A of an LA-group G is called an LA-subgroup of G if A is itself an LA-group under the same operation as defined in G.

S.M. Yusuf in [8] introduced the concept of a left almost ring (LA-ring). That is, a non-empty set R with two binary operations "+" and "$\cdot$" is called a left almost ring, if $\langle R, +\rangle$ is an LA-group, $\langle R, \cdot\rangle$ is an LA-semigroup and distributive laws of "$\cdot$" over "+" holds. T. Shah and I. Rehman [8] asserted that a commutative ring $\langle R, +, \cdot\rangle$, we can always obtain an LA-ring $\langle R, \oplus, \cdot\rangle$ by defining, for $a, b, c \in R$, $a \oplus b = b - a$ and $a \cdot b$ is same as in the ring. We can not assume the addition to be commutative in an LA-ring. An LA-ring $\langle R, +, \cdot\rangle$ is said to be an LA-integral domain if for all $a, b \in R$, with $a \cdot b = 0$, then $a = 0$ or $b = 0$. Let $\langle R, +, \cdot\rangle$ be an LA-ring and S be a non-empty subset of R and S is itself and LA-ring under the binary operation induced by R, then S is called an LA-subring of $\langle R, +, \cdot\rangle$. If S is an LA-subring of an LA-ring $\langle R, +, \cdot\rangle$, then S is called a left ideal of R if $RS \subseteq S$. Right and two-sided ideals are defined in the usual manner.

By [5] a near-ring is a non-empty set N together with two binary operations "+" and "$\cdot$" such that $\langle N, +\rangle$ is a group (not necessarily abelian), $\langle N, \cdot\rangle$ is a semigroup and one sided distributive (left or right) of "$\cdot$" over "+" holds.

By [1] If a subgroup Q of $\langle N, +\rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a quasi-ideal of N.

By [9] If a subgroup B of $\langle N, +\rangle$ is said to be a bi-ideal of N if $BNB \cap (BN) * B \subseteq B$. If N has a zero symmetric near-ring a subgroup B of $\langle N, +\rangle$ is a bi-ideal if and only if $BNB \subseteq B$.

By [10] a subgroup B of $\langle N, +\rangle$ is said to be a weak bi-ideal of N if $B^3 \subseteq B$. In this paper we will define bi-ideal of near-ring has a zero symmetric.

2. Near Left Almost Rings

T. Shah, F. Rehman and M. Raees [7] introduces the concept of a near left almost ring (nLA-ring).

Definition 2.1. *[7]. A non-empty set N with two binary operation "+" and "$\cdot$" is called a near left almost ring (or simply an nLA-ring) if and only if*

(1) *$\langle N, +\rangle$ is an LA-group.*
(2) *$\langle N, \cdot\rangle$ is an LA-semigroup.*
(3) *Left distributive property of $\cdot$ over $+$ holds, that is $a \cdot (b + c) = a \cdot b + a \cdot c$ for all $a, b, c \in N$.*

Definition 2.2. *[7]. An nLA-ring $\langle N, +\rangle$ with left identity 1, such that $1 \cdot a = a$ for all $a \in N$, is called an nLA-ring with left identity.*

Definition 2.3. *[7]. A non-empty subset S of an nLA-ring N is said to be an nLA-subring if and only if S is itself an nLA-ring under the same binary operations as in N.*

Definition 2.4. *[7]. An nLA-subring I of an nLA-ring N is called a left ideal of N if $NI \subseteq I$, and I is called a right ideal if for all $n, m \in N$ and $i \in I$ such that $(i+n)m - nm \in I$, and is called two sided ideal or simply ideal if it is both left and right ideal.*

Definition 2.5. *[7]. Let $\langle N, +, \cdot \rangle$ be an nLA-ring. A non-zero element a of N is called a left zero divisor if there exists $0 \neq b \in N$ such that $a \cdot b = 0$. Similarly a is a right zero divisor if $b \cdot a = 0$. If a is both a left and a right zero divisor, then a is called a zero divisor.*

Definition 2.6. *[7]. An nLA-ring $\langle D, +, \cdot \rangle$ with left identity 1, is called an nLA-ring integral domain if it has no left zero divisor.*

Definition 2.7. *[7]. An nLA-ring $\langle F, +, \cdot \rangle$ with left identity 1, is called a near almost field (n-almost field) if and only if each non-zero element of F has inverse under "$\cdot$"*

3. Quasi-ideals of Near Left Almost rings

Definition 3.1. *If an LA-subgroup Q of $\langle N, + \rangle$ has the property $QN \cap NQ \subseteq Q$, then it is called a* quasi-ideal *of N.*

Lemma 3.1. *Let N be a nLA-ring and Q_1, Q_2 are quasi-ideals of N. Then $Q_1 \cap Q_2$ is a quasi-ideal of N.*

Proof. Since Q_1, Q_2 are LA-subgroups of $\langle N, + \rangle$ we have $Q_1 \cap Q_2$ is a LA-subgroup of $\langle N, + \rangle$. We must show that $(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) \subseteq Q_1 \cap Q_2$. Then

$$\begin{aligned}(Q_1 \cap Q_2)N \cap N(Q_1 \cap Q_2) &\subseteq Q_1N \cap Q_2N \cap NQ_1 \cap NQ_2 \\ &= (Q_1N \cap NQ_1) \cap (Q_2N \cap NQ_2) \\ &\subseteq Q_1 \cap Q_2.\end{aligned}$$

Thus $Q_1 \cap Q_2$ is a quasi-ideal of N. □

Theorem 3.1. *Each quasi-ideal of an nLA-ring N is an nLA-subring.*

Proof. Let Q be a quasi-ideal an nLA-ring N. Then Q is a nLA-subring of $\langle N, + \rangle$. Let $a, b \in Q \subseteq N$. Then $ab \in NQ \subseteq NQ$ and $ab \in QN \subseteq QN$. Thus $ab \in NQ \cap QN \subset Q$, since Q is a quasi-ideal of N. Hence $ab \in Q$. Therefore Q is a nLA-subring of N. □

Theorem 3.2. *The set of all quasi-ideal of nLA-ring.*

Proof. Let $\{Qi\}_{i \in I}$ be a set of quasi-ideal in N and $Q = \cap_{i \in I} Qi$. Then

$$QN \cap NQ \subseteq \bigcap_{i \in I} QiN \cap N \bigcap_{i \in I} Qi \subseteq Qi$$

for every $i \in I$. Thus Q is a quasi-ideal of N. □

4. Bi-ideals and Weak Bi-ideals of Near Left Almost Rings

Next we defined of a bi-ideal and weak bi-ideal in nLA-ring is defines the same as a bi-ideal and weak bi-ideal in near-ring in [9] and [10].

Definition 4.1. *Let N be an nLA-ring. An LA-subgroup B of $\langle N, +\rangle$ is a bi-ideal if $(BN)B \subseteq B$.*

Theorem 4.1. *If B be a bi-ideal of a nLA-ring N and S is an nLA-subring of N. Then $B \cap S$ is a bi-ideal of S.*

Proof. Since B is a bi-ideal of N we have $(BN)B \subseteq B$. Assume that $C := B \cap S$. Then $(CS)C \subseteq (SS)S \subseteq S$, since S is a nLA-subring of N and $C \subseteq S$.

On the other hand $(CS)C \subseteq (BS)B \subseteq (BN)B \subseteq B$. Hence $(CS)C \subseteq B \cap S = C$. Therefore C is a bi-ideal of S. □

Theorem 4.2. *Let N be an nLA-ring and A, B be bi-ideals of an nLA-ring N. Then $A \cap B$ is a bi-ideal of N.*

Proof. Since A, B is bi-ideals of an nLA-ring N, we have $A \cap B$ is an LA-subgroup of $\langle N, +\rangle$. Thus $[(A \cap B)N](A \cap B) \subseteq (AN)(A \cap B) = [(A \cap B)N]A \subseteq (AN)A \subseteq A$ and $[(A \cap B)N](A \cap B) \subseteq (BN)(A \cap B) = [(A \cap B)N]B \subseteq (BN)B \subseteq B$. It following that $A \cap B$ is a bi-ideal of N. □

Theorem 4.3. *The set of all bi-ideal of nLA-ring.*

Proof. Let $\{B_i\}_{i \in I}$ be a set of bi-ideal in N and $B := \cap_{i \in I} B_i$. Then $(BN)B \subseteq (\bigcap_{i \in I} B_i N) \bigcap_{i \in I} B_i \subseteq B_i$ for every $i \in I$. Thus B is a bi-ideal of N. □

Definition 4.2. *Let N be an nLA-ring. An element d of N is called* distributive *if $(n + n')d = nd + n'd$ for all $n, n' \in N$.*

Theorem 4.4. *Let N be an nLA-ring. If B is a bi-ideal of N then Bn and $n'B$ are bi-ideals of N where $n, n' \in N$ and n' is a distributive element in N.*

Proof. Since B is a bi-ideal we have Bn and $n'B$ are LA-subgroup $\langle N, +\rangle$. Thus

$$((Bn)N)(Bn) \subseteq (BN)(Bn) = ((BN)B)n \subseteq Bn.$$

Hence Bn is a bi-ideal of N.

Again

$$((n'B)N)(n'B) \subseteq ((n'B)N)B = (n'BN)B \subseteq n'B.$$

Thus $n'B$ are bi-ideal of N. □

Corollary 4.1. *Let B be a bi-ideal of nLA-ring. For $b, c \in B$, if b is a distributive element in N, then bBc is a bi-ideal of N.*

Proof. Let B be a bi-ideal of nLA-ring and b is a distributive element in N. Then $(n+n')b = nb+n'b$ for all $n, n' \in N$. Since B is a bi-ideal we have bBc is an LA-subgroup $\langle N, +\rangle$ then $((bBc)N)(bBc) \subseteq (BN)B \subseteq B$. □

Definition 4.3. *An nLA-ring N is said to be B-*simple *if it has no proper bi-ideals.*

Theorem 4.5. *Let N be an nLA-ring with more than one element. If N is a near almost field. Then N is a B-simple.*

Proof. Let N be a near almost field then $\{0\}$ and N are the only bi-ideals of N. For if $0 \neq B$ is a bi-ideal of N, then for $0 \neq b \in B$ we get $N = Nb$ and $N = bN$.

Now $N = N^2 = (bN)(Nb) \subseteq bNb \subseteq B$, since B is a bi-ideal of N. Then $N = B$. Thus N is a B-simple. □

The following we defined weak bi-ideal and study properties it.

Definition 4.4. *An LA-subgroup B of $\langle N, +\rangle$ is said to be a* weak bi-ideal *of N if $B^3 \subseteq B$.*

Theorem 4.6. *Every bi-ideal of an nLA-ring is a weak bi-ideal.*

Proof. Since $B^3 = (BB)B \subseteq (BN)B \subseteq B$ we have every bi-ideal is a weak bi-ideal. □

Theorem 4.7. *If B is a weak bi-ideal of a nLA-ring N and S is a nLA-subring of N. Then $B \cap S$ is a weak bi-ideal of N.*

Proof. Assume that $C := B \cap S$. Then

$$\begin{aligned}
C^3 &= ((B\cap S)(B\cap S))(B\cap S) \\
&= ((B\cap S)(B\cap S))B \cap ((B\cap S)(B\cap S))S \\
&\subseteq (BB)B \cap SSS \\
&= B^3 \cap SSS \\
&\subseteq B^3 \cap SS \\
&\subseteq B^3 \cap S \\
&\subseteq B \cap S \\
&= C.
\end{aligned}$$

Thus $C^3 \subseteq C$. Hence C is a weak bi-ideal of N. □

Theorem 4.8. *Let N be an nLA-ring. If B is a weak bi-ideal of N then Bn and $n'B$ are weak bi-ideal of N where $n, n' \in N$ and n' is a distributive element in N.*

Proof. Since B is a weak bi-ideal we have Bn and $n'B$ an LA-subgroup of $\langle N, +\rangle$. Thus

$$(Bn)^3 = (BnBn)Bn \subseteq (BB)Bn \subseteq B^3n \subseteq Bn.$$

Hence Bn is a weak bi-ideal of N.

Again

$$(n'B)^3 = (n'Bn'B)n'B \subseteq (n'BB)B = n'B^3 \subseteq n'B.$$

Thus $n'B$ is a weak bi-ideal of N. □

Corollary 4.2. *Let B be a weak bi-ideal of nLA-ring. For $b, c \in B$, if b is a distributive element in N, then bBc is a weak bi-ideal of N.*

5. Conclusion

In this article, we give the concept of a quasi-ideals and bi- ideals in nLA-rings. We study properties of quasi-ideals and bi- ideals. In the future we study primary and quasi-primary in nLA-ring.

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