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The Möbius Invariant $\mathcal{Q}_H^T$ Spaces

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Abstract. In this article, we introduce a new space of harmonic mappings that is an extension of the well known space $\mathcal{Q}^T$ in the unit disk $\mathbb{D}$ in term of non decreasing function. Several characterizations of the space $\mathcal{Q}_H^T$ are investigated. We also define the little subspace of $\mathcal{Q}_H^T$. Finally, the boundedness of the composition operators C_φ mapping into the space $\mathcal{Q}_H^T$ and $\mathcal{Q}_{H,0}^T$ are considered.

1. Introduction

A harmonic mapping on a simply connected domain ψ is a complex-valued function k such that the Laplace's equation satisfied

$$\Delta k := 4k_{\eta\overline{\eta}} \equiv 0, \qquad \text{on } \psi,$$

where $k_{\eta\overline{\eta}}$ represents the mixed complex derivative of k.

The harmonic mapping k admits a representaion of the form $f + \overline{g}$, where f and g are analytic functions. This representaion is unique up to an additive constant. In this work, we consider all the functions defined on the open unit disk $\mathbb{D} := \{\eta \in \mathbb{C} : |\eta| < 1\}$ so, the representaion of k is given by $k = f + \overline{g}$ and $g(0) = 0$.

Let $H(\mathbb{D})$ denotes the collection of all analytic functions on $\mathbb{D}$ and $\mathcal{H}(\mathbb{D})$ be the collection of harmonic mappings on $\mathbb{D}$.

The operator theory of spaces of analytic functions on a various settings on the unit disk has been completely analyzed and a enormous amount of research papers on this matter have appeared in the literature, but the study of a similarly coverage in the harmonic setting is still limited.

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In recent years, some papers have concentrated on the study of harmonic mappings. Besides [2], for characterization of Bloch type spaces of harmonic mapping, see [6], for harmonic zygmund spaces. In [18], the authors investigate the compactness and boundedness of C_φ mapping into weighted Banach spaces of harmonic mappings. We also encourage the reader to see the additional references related to the harmonic mappings such as [[21] [5], [16], [14], [15], [17], [13], [7], [8], [10], [11], [12], [17], [9]].

The results carried out in [19] bring the interesting question for whether we can extend the space $\mathcal{Q}^T$ to the harmonic setting and study the operator theoretic properties of C_φ.

2. preliminaries and background

We start this section with several preliminaries facts on the spaces that will be used in this work.

Harmonic Bloch space $\mathcal{B}_H$ can be seen as the collection of $k \in \mathcal{H}(\mathbb{D})$ and the a semi-norm b_k satisfies the following condition

$$b_k := \sup_{\eta\in\mathbb{D}}(1-|\eta|^2)(|f'(\eta)|+|g'(\eta)|) < \infty. \tag{2.1}$$

$\mathcal{B}_H$ is a Banach space when it is equipped with the harmonic Bloch norm defined as

$$\|k\|_{\mathcal{B}_H} := |k(0)| + b_k.$$

$\mathcal{B}_H$ space extends the well known Bloch space $\mathcal{B}$. An analytic function $f \in \mathcal{B}$ if and only if

$$b_f = \sup_{\eta\in\mathbb{D}}(1-|\eta|^2)|f'(\eta)| < \infty, \tag{2.2}$$

with norm

$$\|f\|_{\mathcal{B}} = |f(0)| + b_f.$$

In [3], the author obtains that the Bloch constant of k can be written as follows

$$b_k := \sup_{\eta\in\mathbb{D}}(1-|\eta|^2)(|k_\eta(\eta)|+|k_{\bar\eta}(\eta)|) < \infty. \tag{2.3}$$

and

$$\max\{b_f, b_g\} \le b_k \le b_f + b_g. \tag{2.4}$$

Consequently, a harmonic mapping k belongs to the harmonic Bloch space if and only if the functions $f, g \in H(\mathbb{D})$ such that $k = f + \bar g$ with $g(0) = 0$ are in the classical Bloch space. For more details, see [2].

The little harmonic Bloch space $\mathcal{B}_{H,0}$ is the subspace of $\mathcal{B}_H$ such that

$$\mathcal{B}_{H,0} := \{k \in \mathcal{B}_H : \lim_{|\eta|\to 1}(1-|\eta|^2)\big(|k_\eta(\eta)|+|k_{\bar\eta}(\eta)|\big) = 0\}.$$

and the little Bloch spaces $\mathcal{B}_0$ defined as

$$\mathcal{B}_0 := \{f \in \mathcal{B} : \lim_{|\eta|\to 1}(1-|\eta|^2)|f'(\eta)| = 0\}.$$

Consider nondecreasing function $T : [0,+\infty) \to [0,+\infty)$. The logarithmic order of $T(r)$ is given by

$$\lambda = \overline{\lim_{r\to\infty}} \frac{\log^* \log^* T(r)}{\log r},$$

where $\log^* \gamma = \max\{0, \log \gamma\}$

If $\lambda > 0$, the logarithmic type of the function $T(r)$ is given by

$$\Gamma = \overline{\lim_{r\to\infty}} \frac{\log^* T(r)}{r^\lambda},$$

The space $\mathcal{Q}^T$ is the collection of analytic functions f defined on $\mathbb{D}$ and

$$q^T(f) = \sup_{\nu\in\mathbb{D}} \left(\int_{\mathbb{D}} (|f'(\eta)|^2 T(g(\eta,\nu)) dA(\eta) \right)^{\frac{1}{2}} < \infty,$$

where $dA(\eta)$ represents the area measure on the unit disk and $g(\eta,\nu) = -\log|\sigma_\nu(\eta)|$ is the Green function of $\mathbb{D}$ with pole at $\nu \in \mathbb{D}$ and $\sigma_\nu(\eta) = \dfrac{(\nu-\eta)}{(1-\bar{\nu}\eta)}$ be a Möbius transformation of $\mathbb{D}$.

3. The Möbius invariant $\mathcal{Q}_H^T$ spaces

We now introduce the harmonic $\mathcal{Q}_H^T$ space of harmonic mapping by a nondecreasing function $T(r)$ on $r \in [0,\infty)$.

Definition 3.1. *For nondecreasing function $T : [0,+\infty) \to [0,+\infty)$. A harmonic mapping $k \in \mathcal{H}(\mathbb{D})$ is said to be in the class $\mathcal{Q}_H^T$ if*

$$[q^T(k)]^2 = \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(g(\eta,\nu)) dA(\eta) < \infty,$$

and the norm of $\mathcal{Q}_H^T$ is defined as:

$$\|k\|_{\mathcal{Q}_H^T} := |k(0)| + q^T(k). \tag{3.1}$$

The little harmonic $\mathcal{Q}_{H,0}^T$ is the subspace of $\mathcal{Q}_H^T$ such that

$$\mathcal{Q}_{H,0}^T := \Big\{ k \in \mathcal{H}(\mathbb{D}) : \lim_{|\eta|\to 1} \int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(g(\eta,\nu)) dA(\eta) = 0 \Big\}.$$

Remark 3.1. *As a special case when $k \in H(\mathbb{D})$, the functions f, g in the canonical decomosition of k are given by $k = f$ and $g \equiv 0$. Moreover, the collections of analytic function on the unit disk in the $\mathcal{Q}_H^T$ is just the space $\mathcal{Q}^T$.*

Corollary 3.1. *For $T : [0, +\infty) \to [0, +\infty)$ be non-decreasing function. Let $f \in H(\mathbb{D})$, if $k \in \mathcal{H}(\mathbb{D})$ be the real part of f or imaginary part of f then*

$$q^T(k) = q^T(f)$$

Proof. Assume $f = Re(k)$. Then we have,

$$k = \frac{1}{2}(f + \bar{f}).$$

Therefore,

$$\begin{aligned} q^T(k) &= \left(\sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} (\frac{1}{2}|f'(\eta)| + \frac{1}{2}|f'(\eta)|)^2 T(g(\eta,\nu)) dA(\eta)\right)^{\frac{1}{2}} \\ &= \left(\sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} |f'(\eta)|^2 T(g(\eta,\nu)) dA(\eta)\right)^{\frac{1}{2}} = q^T(f) \end{aligned}$$

In a similar way, assume $f = Im(k)$, then we have

$$k = \frac{1}{2i} f - \frac{1}{2i}\bar{f}.$$

Thus,

$$\begin{aligned} q^T(k) &= (\sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} (\frac{1}{2}|f'(\eta)| + \frac{1}{2}|f'(\eta)|)^2 T(g(\eta,\nu)) dA(\eta))^{\frac{1}{2}} \\ &= (\sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} |f'(\eta)|^2 T(g(\eta,\nu)) dA(\eta))^{\frac{1}{2}} \\ &= q^T(f) \end{aligned}$$

Theorem 3.1. *For $T : [0, +\infty) \to [0, +\infty)$ be non-decreasing function. Let $k = f + \bar{g} \in \mathcal{H}(\mathbb{D})$ where $f, g \in H(\mathbb{D})$.Then $f, g \in \mathcal{Q}^T$ if and only if $k \in \mathcal{Q}^T_H$. Moreover, if $g(0) = 0$, then*

$$\frac{1}{2}(\|f\|_{\mathcal{Q}^T} + \|g\|_{\mathcal{Q}^T}) \leq \|k\|_{\mathcal{Q}^T_H} \leq 2((\|f\|_{\mathcal{Q}^T} + \|g\|_{\mathcal{Q}^T})).$$

Proof. Consider $f, g \in \mathcal{Q}^T$ and let $k = f + \bar{g}$. Then

$$f' = k_\eta \quad \text{and} \quad g' = k_{\bar{\eta}}.$$

Therefore,

$$(|k_\eta(\eta)| + |h_{\bar{\eta}}(\eta)|)^2 < 2^2(|k_\eta(\eta)|^2 + |k_{\bar{\eta}}(\eta)|^2)$$

The above inequality follows from the fact that for $c_1, c_2 \geq 0$,

$$\left(\frac{c_1 + c_2}{2}\right)^2 \leq [\max\{c_1, c_2\}]^2 = \max\{c_1^2, c_2^2\} \leq c_1^2 + c_2^2,$$

we have

$$\begin{aligned} q^T(k)^2 &= \sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_\eta(\eta)|+|k_{\bar\eta}(\eta)|)^2T(g(\eta,\nu))dA(\eta)\\ &\leq 2^2\Big[\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_\eta(\eta)|)^2T(g(\eta,\nu))dA(\eta)+\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_{\bar\eta}(\eta)|)^2T(g(\eta,\nu))dA(\eta)\Big]<\infty. \end{aligned}$$

Therefore $k\in\mathcal{Q}_H^T$ and,

$$q^T(f+\bar g)^2\leq 4(q^T(f)^2+q^T(g)^2). \tag{3.2}$$

Taking the square root, we get

$$q^T(k)\leq 2\sqrt{\left(q^T(f)^2+q^T(g)^2\right)}<2\left(q^T(f)+q^T(g)\right).$$

Moreover, using $|k(0)|\leq|f(0)|+|g(0)|$, the upper estimate holds

Conversely, let $k\in\mathcal{Q}_H^T$ and note that

$$|f'(\eta)|^2+|g'(\eta)|^2\leq(|f'(\eta)|+|g'(\eta)|)^2,$$

Thus

$$\begin{aligned} &\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_\eta(\eta)|)^2T(g(\eta,\nu))dA(\eta)+\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_{\bar\eta}(\eta)|)^2T(g(\eta,\nu))dA(\eta))\\ \leq\ &\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_\eta(\eta)|+|k_{\bar\eta}(\eta)|)^2T(g(\eta,\nu))dA(\eta)<\infty. \end{aligned}$$

Therefore, both f and g are in the space $\mathcal{Q}^T$ and

$$q^T(f)^2+q^T(g)^2\leq q^T(k)^2.$$

Hence, by 3.2

$$\frac{1}{2}[q^T(f)+q^T(g)]\leq\sqrt{q^T(f)^2+q^T(g)^2}.$$

Then, we combine these two inequalities to get

$$\frac{1}{2}[q^T(f)+q^T(g)]\leq q^T(k).$$

By the assumption $g(0)=0$, we have

$$\frac{1}{2}|f(0)|\leq|f(0)|=|k(0)|.$$

Therefore,

$$\frac{1}{2}[\|f\|_{\mathcal{Q}^T}+\|g\|_{\mathcal{Q}^T}]\leq\|k\|_{\mathcal{Q}_H^T},$$

We deduce the lower estimate.

Lemma 3.1. *For $T : [0, +\infty) \to [0, +\infty)$ be non-decreasing function. Then $k \in \mathcal{Q}_H^T$ if and only if*

$$\sup_{\nu\in\mathbb{D}} \Big(\int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(1 - |\sigma_\nu(\eta)|^2) dA(\eta) \Big)^{\frac{1}{2}} < \infty, \tag{3.3}$$

Proof. Recall that for $s \in (0, 1]$, we have

$$-2 \log s \geq 1 - s^2$$

and for $s \in (\frac{1}{4}, 1)$ we have

$$-\log s \leq 4(1 - s^2)$$

Assume $k \in \mathcal{Q}_H^T$ then we have,

$$\begin{aligned} q^T(k) &= \sup_{\nu\in\mathbb{D}} \Big(\int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(g(\eta, \nu)) dA(\eta) \Big)^{\frac{1}{2}} & (3.4) \\ &\leq \sup_{\nu\in\mathbb{D}} \Big(\int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(1 - |\sigma_\nu(\eta)|^2) dA(\eta) \Big)^{\frac{1}{2}} & (3.5) \end{aligned}$$

Since $\int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 |d\eta|$ is increasing function on $\delta \in (0, 1)$, we have

$$\int_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 |d\eta| \leq \int_{\mathbb{D}/\mathbb{D}(0,\frac{1}{4})} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(1 - |\sigma_\nu(\eta)|^2) dA(\eta) \leq (q^T(k))^2.$$

This inequality with 3.4, prove the theorem. □

We now study the relationship between $k \in \mathcal{Q}_H^T$ and the associated real and imaginary parts.

Proposition 3.1. *For $T : [0, +\infty) \to [0, +\infty)$ be non-decreasing function. Let $k \in \mathcal{H}(\mathbb{D})$ and assume that τ be the real part of k and θ is the imaginary part of k such that*

$$\tau = Re(k) \quad \text{and } \theta = Im(k).$$

Then $k \in \mathcal{Q}_H^T$, if and only if $\tau, \theta \in \mathcal{Q}_H^T$. Moreover

$$\frac{1}{4}\big(\|\tau\|_{\mathcal{Q}_H^T} + \|\theta\|_{\mathcal{Q}_H^T}\big) \leq \|k\|_{\mathcal{Q}_H^T} \leq \|\tau\|_{\mathcal{Q}_H^T} + \|\theta\|_{\mathcal{Q}_H^T}.$$

Proof. Assume $\tau, \theta \in \mathcal{Q}_H^T$. Due to linearity, $k \in \mathcal{Q}_H^T$ and the upper estimate hold directly by the property of the norm (triangle inequality) .

Let $k \in \mathcal{Q}_H^T$ and recall that

$$J(\tau, \theta) = \tau_x \theta_y - \theta_x \tau_y$$

We have

$$2|J(\tau, \theta)| \leq \|\nabla\tau\|^2 + \|\nabla\theta\|^2, \tag{3.6}$$

where $\nabla\tau = (\tau_x, \tau_y)$, and $\nabla\theta = (\theta_x, \theta_y)$.
From this, we get

$$(\|\nabla\tau\|^2 + \|\nabla\theta\|^2 + 2J(\tau,\theta))^{\frac{1}{2}} + (\|\nabla\tau\|^2 + \|\nabla\theta\|^2 - 2J(\tau,\theta))^{\frac{1}{2}} \geq \sqrt{2}(\|\nabla\tau\|^2 + \|\nabla\theta\|^2)^{\frac{1}{2}} \tag{3.7}$$

By squaring (3.7), the left-hand side becomes

$$\|\nabla\tau\|^2 + \|\nabla\theta\|^2 + 2J(\tau,\theta) + \|\nabla\tau\|^2 + \|\nabla\theta\|^2 - 2J(\tau,\theta) + 2\Big(\|\nabla\tau\|^2 + \|\nabla\theta\|^2)^2 - 4(J(\tau,\theta)^2)\Big)^{\frac{1}{2}},$$

Thus, by neglecting the last term and simple calculation, we obtain

$$2(\|\nabla\tau\|^2 + \|\nabla\theta\|^2).$$

Now, we may find $|k_\eta| + |k_{\bar\eta}|$ with respect to τ and θ by using the partials with respect to η and $\bar\eta$, then calculating the modulus, after that applying (3.7)

$$\begin{aligned}
|k_\eta| + |k_{\bar\eta}| &= |\tau_\eta + i\theta_\eta| + |\tau_{\bar\eta} + i\theta_{\bar\eta}| \\
&= \frac{1}{2}\big|\tau_x + \theta_y + i(\theta_x - \tau_y)\big| + \frac{1}{2}\big|\tau_x - \theta_y + i(\theta_x + \tau_y)\big| \\
&= \frac{1}{2}\sqrt{\Big((\tau_x + \theta_y)^2 + (\theta_x - \tau_y)^2\Big)} + \frac{1}{2}\sqrt{\Big((\tau_x - \theta_y)^2 + (\theta_x + \tau_y)^2\Big)} \\
&= \frac{1}{2}\sqrt{\Big(\|\nabla\tau\|^2 + \|\nabla\theta\|^2 + 2J(\tau,\theta)\Big)} + \frac{1}{2}\sqrt{\Big(\|\nabla\tau\|^2 + \|\nabla\theta\|^2 - 2J(\tau,\theta)\Big)} \\
&\geq \frac{1}{\sqrt{2}}\sqrt{\|\nabla\tau\|^2 + \|\nabla\theta\|^2} \\
&\geq \frac{1}{2}\big(\|\nabla\tau\| + \|\nabla\theta\|\big),
\end{aligned}$$

In the last step, we apply the following inequality

$$\|(\eta_1, \eta_2)\| \geq \frac{|\eta_1| + |\eta_2|}{\sqrt{2}} \quad for \;\; \eta_1, \eta_2 \in \mathbb{C}. \tag{3.8}$$

Therefore,

$$\begin{aligned}
(q^T(k))^2 &\geq \frac{1}{2}\sup_{\eta\in\mathbb{D}}\int_{\mathbb{D}}(\|\nabla\tau(\eta)\| + \|\nabla\theta\eta\|)^2 T(g(\eta,\nu)dA(\eta) \\
&\geq \frac{1}{2}\max\{q_\tau^T, q_\theta^T\} \\
&\geq \frac{1}{4}(q_\tau^T + q_\theta^T)
\end{aligned} \tag{3.9}$$

Therefore, by using inequality (3.8) one more time, we obtain

$$|k(0)| \geq \frac{1}{\sqrt{2}}(|\tau(0)| + |\theta(0)|) \tag{3.10}$$

Now, combine (3.9) and (3.10) to get

$$\|k\|_{\mathcal{Q}_H^T} \geq \frac{1}{4}(\|\tau\|_{\mathcal{Q}_H^T} + \|\theta\|_{\mathcal{Q}_H^T})$$

Thus, τ and θ are in $\mathcal{Q}_H^T$, and that the other estimate is hold.

Theorem 3.2. *$(\mathcal{Q}_H^T, \|\cdot\|_{\mathcal{Q}_H^T})$ is a Banach space.*

Proof. Obviously, $\mathcal{Q}_H^T$ is a normed linear space, we only wish to show completeness.

For each $n \in \mathbb{N}$, let $\{k_n\}$ be a Cauchy sequence in $\mathcal{Q}_H^T$. By Theorem 3.1, the analytic functions $\{f_n\}$ and $\{g_n\}$ such that $k_n = f_n + \bar{g}_n$ with $g_n(0) = 0$ are in $\mathcal{Q}^T$ and $\{f_n\}$ and $\{g_n\}$ are Cauchy sequence in $\mathcal{Q}^T$. By proposition 2.2 in [4], $\mathcal{Q}^T$ is complete. Thus, $\{f_n\}$ and $\{g_n\}$ converge to f and g, respectively in the $\mathcal{Q}^T$ norm.

Define $k = f + \bar{g}$. Then, $k \in \mathcal{Q}_H^T$ by the estimates in Theorem 3.1, and

$$\|k_n - k\|_{\mathcal{Q}_H^T} \leq 2(\|f_n - f\|_{\mathcal{Q}^T} + \|g_n - g\|_{\mathcal{Q}^T}) \to 0, \text{ as } n \to \infty.$$

We ends up with $k_n \to k$ in $\mathcal{Q}_H^T$.

Theorem 3.3. *For nondecreasing function $T : [0, +\infty) \to [0, +\infty)$. The space $\mathcal{Q}_H^T$ is a subset of $\mathcal{B}_H$. Moreover, for $k \in \mathcal{Q}_H^T$ we have*

$$\|k\|_{\mathcal{B}_H} \leq m\|k\|_{\mathcal{Q}_H^T},$$

for some constant $m > 0$.

Proof. Assume $k \in \mathcal{Q}_H^T$ and let

$$\sup_{\nu \in \mathbb{D}} \int\limits_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(g(\eta, \nu)) dA(\eta) = M < \infty,$$

For $\delta \in (0, 1)$ define $\mathbb{D}(\succneqq, \npreceq) := \{\eta \in \mathbb{D} : |\sigma_\nu(\eta)| < \delta\}$. Since T is nondecreasing function and by the change of variable $w = \sigma_\nu(\eta)$ we have

$$\begin{aligned}
M &\geq \int\limits_{\mathbb{D}} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T(g(\eta, \nu)) dA(\eta) \\
&\geq \int\limits_{\mathbb{D}(\succneqq, \npreceq)} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 T\Big(\log \frac{1}{\sigma_\nu(\eta)}\Big) dA(\eta) \\
&\geq T\Big(\log \frac{1}{\delta}\Big) \int\limits_{\mathbb{D}(\succneqq, \npreceq)} (|k_\eta(\eta)| + |k_{\bar{\eta}}(\eta)|)^2 dA(\eta) \\
&= T\Big(\log \frac{1}{\delta}\Big) \int\limits_{|w| < \delta} (|(k \circ \sigma_\nu)_w(w)| + |(k \circ \sigma_\nu)_{\bar{w}}(w)|)^2 dA(w) \\
&\geq \pi\delta^2 T\Big(\log \frac{1}{\delta}\Big)(|(k \circ \sigma_\nu)_\nu(0)| + |(k \circ \sigma_\nu)_{\bar{\nu}}(0)|)^2 \\
&= \pi\delta^2 T\Big(\log \frac{1}{\delta}\Big)(|(k_\nu(\nu)| + |(k_{\bar{\nu}}(\nu)|)^2 (1 - |\nu|^2)^2
\end{aligned}$$

Fix $\delta_0 \in (0,1)$. Thus

$$\sup_{\nu\in\mathbb{D}}(1-|\nu|^2)[|(k_\nu(\nu)|+|(k_{\bar{\nu}}(\nu)|] \le \sqrt{\frac{M}{\pi\delta_0^2 T\big(\log\frac{1}{\delta_0}\big)}}$$

Therefore,

$$b_k \le \frac{q^T(k)}{\delta_0\sqrt{\pi T\big(\log\frac{1}{\delta_0}\big)}} \tag{3.11}$$

We obtained that $k \in \mathcal{B}_H$ and $\mathcal{Q}_H^T \subset \mathcal{B}_H$.

□

Theorem 3.4. *If the logarithmic type Γ and the logarithmic order λ of $T(r)$ satisfying one of the following cases,*

(1) $\lambda > 1$,

(2) $\Gamma > 2$ *and* $\lambda = 1$,

then the space $\mathcal{Q}_H^T$ has only constant functions(trivial space).

Proof. By theorem 3.3, it is sufficient to prove that for each non constant harmonic Bloch function k can not be in the space $\mathcal{Q}_H^T$. Indeed, if either $\lambda > 1$ or $\Gamma > 2$ and $\lambda = 1$, there is a sequence $\{r_j\}$ as $j \to \infty$, the sequence $\{r_j\} \to \infty$ as follows

$$\lim_{j\to\infty}\frac{\log^*\log^* T(r_j)}{\log r_j} = \lambda > 1, \tag{3.12}$$

or

$$\lim_{j\to\infty}\frac{\log^* T(r_j)}{r_j} = \Gamma > 2, \tag{3.13}$$

In the case 3.12 or 3.13, we get

$$\lim_{j\to\infty}\frac{T(r_j)}{e^{2r_j}} = \infty. \tag{3.14}$$

Set $h_j = e^{-r_j}$, for $j \in \mathbb{N}$, then

$$\lim_{j\to\infty} h_j^2 T\Big(\log\frac{1}{h_j}\Big) = \infty. \tag{3.15}$$

Assume $k \in \mathcal{B}_H$ be a non-constant. Then it is clear that the semi-norm $b_k \neq 0$.

However, by 3.11, and 3.15, as $j \to \infty$ we obtain

$$\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}(|k_\eta(\eta)|+|k_{\bar{\eta}}(\eta)|)^2 T(g(\eta,\nu))\,dA(\eta) \ge \pi\, b_k^2\, h_j^2\, T(\log\frac{1}{h_j}) \to \infty.$$

That implies $k \notin \mathcal{Q}_H^T$ which proves the theorem. □

The next theorem shows that the Möbius invariance of $\mathcal{Q}^T$ space extends to the harmonic setting.

Theorem 3.5. *For $T : [0,+\infty) \to [0,+\infty)$ be non-decreasing function. $\mathcal{Q}_H^T$ is a Möbius invariant space.*

Proof. It is obvious that rotations have no effect on the semi-norm $q^T(k)$. We wish to show $q^T(k \circ \varphi_\nu) = q^T(k)$, for $\nu \in \mathbb{D}$ and $k \in \mathcal{Q}_H^T$.

For $\nu \in \mathbb{D}$, and since φ_ν is its own inverse, we have

$$(1-|\eta|^2)|\varphi^{'}(\eta)| = 1 - |\varphi_\nu(\eta)|^2$$

and

$$\varphi_\nu^{'}(\varphi_\nu(\eta)) = \frac{1}{\varphi_\nu^{'}(\eta)}$$

By change of variables $\xi = \varphi_\nu(\eta)$, we get

$$\begin{aligned}
q^T(k\circ\varphi_\nu)^2 &= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\varphi_\nu(\eta)|^2)[|(k\circ\varphi_\nu)_\eta(\eta)| + |(k\circ\varphi_\nu)_{\bar{\eta}}(\eta)|]^2 dA(\eta) \\
&= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\varphi_\nu(\eta)|^2)[|k_\eta(\varphi_\nu(\eta))\varphi_\nu^{'}(\eta)| + |(k_{\bar{\eta}}(\varphi_\nu(\eta))\overline{\varphi_\nu^{'}(\eta))}|]^2 dA(\eta) \\
&= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\varphi_\nu(\eta)|^2)|\varphi_\nu^{'}(\eta)|^2[|k_\eta(\varphi_\nu(\eta))| + |k_{\bar{\eta}}(\varphi_\nu(\eta))|]^2 dA(\eta) \\
&= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\xi|^2)|\varphi_\nu^{'}(\varphi_\nu(\xi))|^2[|k_\eta(\xi)| + |(k_{\bar{\eta}}(\xi))|]^2|\varphi_\nu^{'}(\xi)|^2 dA(\xi) \\
&= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\xi|^2)\frac{1}{|\varphi_\nu^{'}(\xi)|^2}[|h_\eta(\xi)| + |h_{\bar{\eta}}(\eta)|]^2|\varphi_\nu^{'}(\xi)|^2 dA(\xi) \\
&= \sup_{\nu\in\mathbb{D}} \int_{\mathbb{D}} T(1-|\xi|^2)[|k_\eta(\xi)| + |k_{\bar{\eta}}(\xi)|]^2 dA(\xi) \\
&= q^T(k)^2
\end{aligned}$$

as desired.

Finally, we move our attention to study the boundedness of composition operator C_φ from the harmonic Bloch space $\mathcal{B}_H$ to $\mathcal{Q}_H^T$ and $\mathcal{Q}_{H,0}^T$.

4. Boundedness

Due to the representation of the harmonic mapping, the composition operator C_φ induced by analytic or a conjugate analytic self-maps of $\mathbb{D}$ is given by

$$C_\varphi k = k \circ \varphi,$$

for all k belonging to a class of harmonic mappings.

The following is a basic property of the harmonic Bloch space was introduced in [20].

Lemma 4.1. *For $\eta \in \mathbb{D}$. If k_1 , $k_2 \in \mathcal{B}_H$ we have*

$$(1-|\eta|^2)^{-1} \leq (k_1)_\eta(\eta)| + |(k_1)_{\bar{\eta}}(\eta)| + |(k_2)_\eta(\eta)| + |(k_2)_{\bar{\eta}}(\eta)|.$$

The next result which will be used in the proof of the main theorem of this section is a special case of Theorem 3.6 in [1]

Lemma 4.2. *For $k \in \mathcal{B}_H$ and $\varphi : \mathbb{D} \to \mathbb{D}$,*

$$|k(\varphi(0))| \leq |k(0)| + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\, b_k.$$

Theorem 4.1. *For $T : [0,+\infty) \to [0,+\infty)$ be non-decreasing function. Let φ be analytic function such that $\varphi : \mathbb{D} \to \mathbb{D}$. Then $C_\varphi : \mathcal{B}_H \to \mathcal{Q}_H^T$ is bounded operator if and only if*

$$\sup_{\nu\in\mathbb{D}} \int\limits_{\mathbb{D}} \frac{|\varphi'(\eta)|^2}{(1-|\varphi(\eta)|^2)^2}\, T(g(\eta,\nu))dA(\eta) < \infty. \tag{4.1}$$

Proof. Let us assume 4.1 holds and let ρ_1^2 be the supremum in 4.1. Let $\eta \in \mathbb{D}$ and $k \in \mathcal{B}_H$, then

$$\begin{aligned}
&\int_{\mathbb{D}} T(g(\eta,\nu))[|(k\circ\varphi)_\eta(\eta)| + |(k\circ\varphi)_{\bar{\eta}}(\eta)|]^2 dA(\eta)\\
&= \int_{\mathbb{D}} T(g(\eta,\nu))|\varphi^{'}(\eta)|^2[|k_\eta(\varphi(\eta))| + |k_{\bar{\eta}}(\varphi(\eta))|]^2 dA(\eta)\\
&\leq b_k^2 \int_{\mathbb{D}} T(g(\eta,\nu))\frac{|\varphi_z^{'}(\xi)|^2}{(1-|\varphi(\eta)|^2)^2} dA(\eta)\\
&\leq \rho_1^2 b_k^2.
\end{aligned}$$

Therefore, $q^T(k\circ\varphi) \leq \rho_1\, b_k$. Since $k \in \mathcal{B}_H$ we have

$$\begin{aligned}
\|C_\varphi k\|^2_{\mathcal{Q}_H^T} &= \big(|k\circ\varphi(0)| + q^T(C_\varphi k)\big)^2\\
&\leq \big(|k(0)| + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|} b_k + \rho_1\, b_k)\big)^2\\
&\leq \rho^2\big(|k(0)| + b_k\big)^2 = \rho^2\|k\|^2_{\mathcal{B}_H}.
\end{aligned}$$

where $\rho = \max\{1, \rho_1 + \frac{1}{2}\log\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\}$.

Therefore, $\|C_\varphi k\|_{\mathcal{Q}_H^T} \leq \rho\|k\|_{\mathcal{B}_H}$ which implies that $C_\varphi : \mathcal{B}_H \to \mathcal{Q}_H^T$ is bounded. Conversely, Assume the boundedness of $C_\varphi : \mathcal{B}_H \to \mathcal{Q}_H^T$ holds, then there is a positive constant $\rho > 0$ for all $k \in \mathcal{B}_H$, we have

$$\|C_\varphi k\|_{\mathcal{Q}_H^T} \leq \rho\|k\|_{\mathcal{B}_H}.$$

On the other hand, by Lemma 4.1 for all $\eta \in \mathbb{D}$, there exist k_1 , $k_2 \in \mathcal{B}_H$ such that

$$(1-|\eta|^2)^{-1} \leq |(k_1)_\eta(\eta)| + |(k_1)_{\bar{\eta}}(\eta)| + |(k_2)_\eta(\eta)| + |(k_2)_{\bar{\eta}}(\eta)|$$

Therefore,

$$\begin{aligned}\frac{|\varphi(\eta)'|^2}{\left[1-|\varphi(\eta)|^2\right]^2} &\le 2|(k_1\circ\varphi)_\eta(\eta)|^2+2|(k_1\circ\varphi)_{\bar\eta}(\eta)|^2+2|(k_2\circ\varphi)_\eta(\eta)|^2+2|(k_2\circ\varphi)_{\bar\eta}(\eta)|^2\\ &\le 2[|(k_1\circ\varphi)_\eta(\eta)|+|(k_1\circ\varphi)_{\bar\eta}(\eta)|]^2+2[|(k_2\circ\varphi)_\eta(\eta)|+|(k_2\circ\varphi)_{\bar\eta}(\eta)|]^2\end{aligned}$$

where the last inequity follows from the fact that for $c_1, c_2 \ge 0$ and $m > 1$ we have

$$c_1^m + c_2^m \le (c_1 + c_2)^m$$

Moreover,

$$\begin{aligned}&\int_{\mathbb{D}} T(g(\eta,\nu))\frac{|\varphi(\eta)'|^2}{\left(1-|\varphi(\eta)|^2\right)^2}dA(\eta)\\ &\le 2\int_{\mathbb{D}}\left[[|(k_1\circ\varphi)_\eta(\eta)|+|(k_1\circ\varphi)_{\bar\eta}(\eta)|]^2+[|(k_2\circ\varphi)_\eta(\eta)|+|(k_2\circ\varphi)_{\bar\eta}(\eta)|]^2\right]T(g(\eta,\nu))dA(\eta)\\ &\le 2\rho^2\left(\|k_1\|^2_{\mathcal{B}_H}+\|k_2\|^2_{\mathcal{B}_H}\right),\end{aligned}$$

Thus, take the supremum over all $\eta \in \mathbb{D}$, the quantity 4.1 holds since ρ is a constant and $k \in \mathcal{B}_H$. $\square$

Theorem 4.2. *For nondecreasing function $T : [0, +\infty) \to [0, +\infty)$. Let φ be analytic function such that $\varphi : \mathbb{D} \to \mathbb{D}$. Then $C_\varphi : \mathcal{B}_H \to \mathcal{Q}^T_{H,0}$ is bounded operator if and only if*

$$\lim_{|\nu|\to 1}\int_{\mathbb{D}}\frac{|\varphi'(\eta)|^2}{(1-|\varphi(\eta)|^2)^2}T(g(\eta,\nu))dA(\eta) = 0. \tag{4.2}$$

Proof. By theorem 4.1, we know that $C_\varphi : \mathcal{B}_H \to \mathcal{Q}^T_H$ is bounded since the condition 4.2 implies the following

$$\sup_{\nu\in\mathbb{D}}\int_{\mathbb{D}}\frac{|\varphi'(\eta)|^2}{(1-|\varphi(\eta)|^2)^2}T(g(\eta,\nu))dA(\eta) < \infty.$$

We only wish to show that $C_\varphi k \in \mathcal{Q}^T_{H,0}$ for each $k \in \mathcal{B}_H$ and this comes from the inequality

$$\begin{aligned}&\int_{\mathbb{D}} T(g(\eta,\nu))[|(k\circ\varphi)_\eta(\eta)|+|(k\circ\varphi)_{\bar\eta}(\eta)|]^2dA(\eta)\\ &=\int_{\mathbb{D}} T(g(\eta,\nu))|\varphi'(\eta)|^2[|k_\eta(\varphi(\eta))|+|k_{\bar\eta}(\varphi_z(\eta))|]^2dA(\eta)\\ &\le b_k^2\int_{\mathbb{D}} T(g(\eta,\nu))\frac{|\varphi_z'(\eta)|^2}{(1-|\varphi(\eta)|^2)^2}dA(\eta)\end{aligned}$$

Thus, $C_\varphi k \in \mathcal{Q}^T_{H,0}$.

Conversely, consider $C_\varphi : \mathcal{B}_H \to \mathcal{Q}^T_{H,0}$ is bounded. By Lemma 4.1 there exist k_1, $k_2 \in \mathcal{B}_H$ such that

$$(1-|\eta|^2)^{-1} \le |(k_1)_\eta(\eta)|+|(k_1)_{\bar\eta}(\eta)|+|(k_2)_\eta(\eta)|+|(k_2)_{\bar\eta}(\eta)|$$

Then $C_\varphi k_1$, $C_\varphi k_2 \in \mathcal{Q}^T_{H,0}$.

Therefore,

$$\lim_{|\nu|\to 1}\int_{\mathbb{D}} T(g(\eta,\nu))\frac{|\varphi(\eta)'|^2}{\left[1-|\varphi(\eta)|^2\right]^2}dA(\eta)$$

$$\leq 2\lim_{|\nu|\to 1}\int_{\mathbb{D}} T(g(\eta,\nu))\big([|(k_1\circ\varphi)_\eta(\eta)|+|(k_1\circ\varphi)_{\bar\eta}(\eta)|]^2+[|(k_2\circ\varphi)_\eta(\eta)|+|(k_2\circ\varphi)_{\bar\eta}(\eta)|]^2\big)\,dA(\eta)=0$$

Then 4.2 holds and this complete the proof. □

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