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Hypersurfaces With a Common Geodesic Curve in 4D Euclidean space $\mathbb{E}^4$

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Abstract. In this paper, we attain the problem of constructing hypersurfaces from a given geodesic curve in 4D Euclidean space $\mathbb{E}^4$. Using the Serret–Frenet frame of the given geodesic curve, we express the hypersurface as a linear combination of this frame and analyze the necessary and sufficient conditions for that curve to be geodesic. We illustrate this method by presenting some examples.

1. Introduction

In differential geometry, geodesic curves representing in some sense the shortest distance (arc) amidst two points in a surface, or more in general in a Riemannian manifold [7–9]. From this explicitness we can immediately see that the geodesic among two points on a sphere is a great circle. But there are two arcs of a great circle amid two of their points, and only one of them gives the short distance, with the exclusion of the two points are the end points of a diameter. This model indicates that there may exist more than one geodesic among two points. Therefore, for example, the passage of a verticil orbiting about a star is the projection of a geodesic of the curved 4D space-time geometry about the star onto 3D space. Nowadays, numerous research results have concentrated on surfaces family having a common geodesic curve in a diversity of applications, such as the tent manufacturing, designing industry of shoes, cutting and painting path. In general, the goal of mainly works on geodesics is to define a family of surfaces with a given geodesic curve and express it as a linear combination of the Serret–Frenet frame (See for example [1, 2, 4, 5, 11, 12, 14, 16]).

However, there is little written works on differential geometry of parametric surface family in Euclidean, and non-Euclidean 4-spaces [3, 6, 10, 13, 15]. Thus, the current study hopes to serve such a need. In this paper, we consider the parametric representation of hypersurface family passing a given

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isogeodesic curve, that is, both a geodesic and a parameter curve in $\mathbb{E}^4$. Then, we insert three types of the marching-scale functions, and give some examples for the purpose of clarity of our method.

2. Preliminaries

In this section we list some formulas and conclusions for space curves, and surfaces in Euclidean 4-space $\mathbb{E}^4$ which can be found in [7-9, 17]: A curve is smooth if it admits a tangent vector at whole point of the curve. In the following argumentations, all curves are assumed to be regular. Let $\boldsymbol{\alpha} = \boldsymbol{\alpha}(s)$ be a unit speed curve in 4D Euclidean space $\mathbb{E}^4$. We set up $\boldsymbol{\alpha}^{'}(s) \neq 0$ for all $s \in [0, L]$; since this would give us a straight line. In this paper, $\boldsymbol{\alpha}^{'}(s)$ indicate to the derivatives of $\boldsymbol{\alpha}(s)$ with respect to arc-length parameter s. For whole point of $\boldsymbol{\alpha}(s)$, if the set $\{\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}_1(s), \mathbf{b}_2(s)\}$ is the Serret–Frenet frame along $\boldsymbol{\alpha}(s)$, then:

$$\begin{pmatrix} \mathbf{t}^{'}(s) \\ \mathbf{n}^{'}(s) \\ \mathbf{b}_1^{'}(s) \\ \mathbf{b}_2^{'}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & -\kappa_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{t}(s) \\ \mathbf{n}(s) \\ \mathbf{b}_1(s) \\ \mathbf{b}_2(s) \end{pmatrix}, \tag{2.1}$$

where $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ are the tangent, the principal normal, the first binormal, and the second binormal vector fields; $\kappa_i(s)$ $(i = 1, 2, 3)$ are the ith curvature functions $(\kappa_1, \kappa_2 > 0)$ of the curve $\boldsymbol{\alpha}(s)$. For any three vectors $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z} \in \mathbb{E}^4$, the vectorial product is defined by

$$\mathbf{x} \wedge \mathbf{y} \wedge \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{2.2}$$

where $\mathbf{e}_i$ $(i = 1, 2, 3, 4)$ are the standard base vectors of $\mathbb{E}^4$.

Theorem 2.1. Let $\boldsymbol{\alpha}$: $I \mapsto \mathbb{E}^4$ be a unit-speed curve. Then the Serret–Frenet vectors of the curve are given by

$$\mathbf{t}(s) = \boldsymbol{\alpha}^{'}(s),\ \mathbf{n}(s) = \frac{\boldsymbol{\alpha}^{''}(s)}{\left\|\boldsymbol{\alpha}^{''}(s)\right\|},\ \mathbf{b}_2(s) = -\frac{\boldsymbol{\alpha}^{'}(s) \wedge \boldsymbol{\alpha}^{''}(s) \wedge \boldsymbol{\alpha}^{'''}(s)}{\left\|\boldsymbol{\alpha}^{'}(s) \wedge \boldsymbol{\alpha}^{''}(s) \wedge \boldsymbol{\alpha}^{'''}(s)\right\|},\ \mathbf{b}_1(s) = \mathbf{b}_2(s) \wedge \mathbf{t}(s) \wedge \mathbf{n}(s).$$

Theorem 2.2. Let $\boldsymbol{\alpha}$: $I \mapsto \mathbb{E}^4$ be a unit-speed curve. Then the curvatures of the curve are given by:

$$\kappa_2(s) = \frac{< \mathbf{b}_1, \boldsymbol{\alpha}^{'''} >}{\kappa_1},\ \text{and}\ \kappa_3(s) = \frac{< \mathbf{b}_2, \boldsymbol{\alpha}^{(4)} >}{\kappa_1 \kappa_2}.$$

We indicate a surface M in $\mathbb{E}^4$ by

$$M : \mathbf{P}(s, t, r) = (x_1\,(s, t, r)\,, x_2\,(s, t, r)\,, x_3\,(s, t, r)\,,)\, x_4\,(s, t, r)\,,\quad (s, t, r) \in D \subseteq \mathbb{R}^3. \tag{2.3}$$

If $\mathbf{P}_j(s,t,r)=\frac{\partial \mathrm{P}}{\partial j}$, the normal vector field of M is defined as follows [12]

$$\mathbf{N}(s,t,r)=\mathbf{P}_s\wedge\mathbf{P}_t\wedge\mathbf{P}_r, \tag{2.4}$$

which is orthogonal to each of the vectors $\mathbf{P}_s$, $\mathbf{P}_t$, and $\mathbf{P}_r$. Similar to the Euclidean 3-space $\mathbb{E}^3$, the following definition can be given:

Definition 2.1 Let $\boldsymbol{\alpha}$: $I\mapsto\mathbb{E}^4$ be a unit-speed curve. Then the hyperplanes which correspond to the subspaces Sp$\{\mathbf{t},\mathbf{b}_1,\mathbf{b}_2\}$, Sp$\{\mathbf{t},\mathbf{n},\mathbf{b}_1\}$, Sp$\{\mathbf{t},\mathbf{n},\mathbf{b}_2\}$, and Sp$\{\mathbf{n},\mathbf{b}_1,\mathbf{b}_2\}$, respectively, are named the rectifying hyperplane, first osculating hyperplane, second osculating hyperplane, and normal hyperplane.

The projection of a hypersurface into 3-space generally leads to a 3-dimensional volume. If we fix whole of the three variables, one at a time, we obtain three distinguished families of 2-spaces in 4-space. The projections of these 2-surfaces into 3-space are surfaces in 3-space. Thus, they can be displayed by 3D rendering methods [12]. Take $x_4=0$ subspace and assuming $r=$constant for example, then the surface is parametrized as

$$M:\mathbf{P}_{x_4}(s,t)=\left(x_1\left(s,t\right),x_2\left(s,t\right),x_3\left(s,t\right)\right),\quad (s,t)\in D\subseteq\mathbb{R}^2. \tag{2.5}$$

3. Hypersurfaces with a common geodesic curve

In this section, we consider a new approach for constructing a hypersurface family with a common geodesic curve $\boldsymbol{\alpha}(s)$, $0\leq s\leq L$, in which the hypersurface tangent plane is coincident with the rectifying hyperplane Sp$\{\mathbf{t},\mathbf{b}_1,\mathbf{b}_2\}$. Then, the construction of the surface over $\boldsymbol{\alpha}(s)$ is:

$$M:\mathbf{P}(s,t,r)=\boldsymbol{\alpha}(s)+u(s,t,r)\mathbf{t}(s)+v(s,t,r)\mathbf{b}_1(s)+w(s,t,r)\mathbf{b}_2(s), \tag{3.1}$$

where $u(s,t,r)$, $v(s,t,r)$, and $w(s,t,r)$ are all regular functions; $0\leq t\leq T$, $0\leq r\leq H$. These functions are named the marching-scale functions. From now on, we shall often not write the parameters s, t, and r explicitly in the functions $u(s,t,r)$ $v(s,t,r)$, and $w(s,t,r)$.

Our aim is to find necessary and sufficient conditions for which the given $\boldsymbol{\alpha}(s)$ is an iso-parametric and geodesic (*geodesic for short*) on the hypersurface $\mathbf{P}(s,t,r)$. The $\mathbf{P}'s$ tangent vectors are:

$$\left.\begin{array}{l}\mathbf{P}_s=(1+u_s)\mathbf{t}+(u\kappa_1-v\kappa_2)\mathbf{n}+\left(v_s-w\right)\mathbf{b}_1+\left(w_s+v\kappa_3\right)\mathbf{b}_2,\\ \mathbf{P}_t=u_t\mathbf{t}+v_t\mathbf{b}_1+w_t\mathbf{b}_2,\\ \mathbf{P}_r=u_r\mathbf{t}+v_r\mathbf{b}_1+w_r\mathbf{b}_2.\end{array}\right\} \tag{3.2}$$

The normal vector field is

$$\mathbf{N}(s,t,r):=\mathbf{P}_s\wedge\mathbf{P}_t\wedge\mathbf{P}_r=\eta_1\mathbf{t}(s)+\eta_2\mathbf{n}(s)+\eta_3\mathbf{b}_1(s)+\eta_4\mathbf{b}_2(s), \tag{3.3}$$

where

$$\eta_1(s,t,r) = \begin{vmatrix} 0 & v_s & w_s \\ 0 & v_t & w_t \\ 0 & v_r & w_r \end{vmatrix} = 0,\ \eta_2(s,t,r) = \begin{vmatrix} 1+u_s & v_s & w_s \\ u_t & v_t & w_t \\ u_r & v_r & w_r \end{vmatrix},$$

$$\eta_3(s,t,r) = \begin{vmatrix} 1+u_s & 0 & v_s \\ u_t & 0 & v_t \\ u_r & 0 & v_r \end{vmatrix} = 0,\ \eta_4(s,t,r) = \begin{vmatrix} 1+u_s & 0 & v_s \\ u_t & 0 & v_t \\ u_r & 0 & v_r \end{vmatrix} = 0.$$

Since the $\boldsymbol{\alpha(s)}$ is an iso-parametric curve on the hypersurface there exists $t = t_0 \in [0, T]$, and $r = r_0 \in [0, H]$ such that $\mathbf{P}(s, t_0, r_0) = \boldsymbol{\alpha(s)}$; that is,

$$\left.\begin{array}{l} u(s,t_0,r_0) = v(s,t_0,r_0) = w(s,t_0,r_0) = 0, \\ u_s(s,t_0,r_0) = v_s(s,t_0,r_0) = w_s(s,t_0,r_0) = 0. \end{array}\right\} \tag{3.4}$$

Therefore, when $t = t_0$, and $r = r_0$—i.e., along the curve $\boldsymbol{\alpha}(s)$—the hypersurface normal is

$$\mathbf{N}(s,t_0,r_0) = (v_t(s,t_0,r_0)w_r(s,t_0,r_0) - w_t(s,t_0,r_0)v_r(s,t_0,r_0))\,\mathbf{n}(s). \tag{3.5}$$

Coincidence of the hypersurface normal $\mathbf{N}$ with the principal normal $\mathbf{n}(s)$ identifies the curve as a geodesic curve.

Then, we can state the following theorem:

Theorem 3.1. The given spatial curve $\boldsymbol{\alpha}(s)$ is a geodesic curve on the hypersurface $\mathbf{P}(s, t, r)$ iff

$$\left.\begin{array}{l} u(s,t_0,r_0) = v(s,t_0,r_0) = w(s,t_0,r_0) = 0, \\ u_s(s,t_0,r_0) = v_s(s,t_0,r_0) = w_s(s,t_0,r_0) = 0, \\ v_t(s,t_0,r_0)w_r(s,t_0,r_0) - w_t(s,t_0,r_0)v_r(s,t_0,r_0) \neq 0, \end{array}\right\} \tag{3.6}$$

where $0 \le t \le T,\ 0 \le r \le H$.

Evidently, Eqs. (3.6) is further elegant and simple for applications (Compare with [5], eqs. (9)). We call the set of hypersurfaces given by Eqs. (3.1) and satisfying Eqs. (3.6) a geodesic hypersurface family. For get better the conditions in Theorem 3.1, the marching-scale functions $u(s, t, r)$ $v(s, t, r)$, and $w(s, t, r)$ can be formed into three the following types:

Type (a). Let

$$\begin{aligned} u(s,t,r) &= l(s)U(t,r), \\ v(s,t,r) &= m(s)V(t,r), \\ w(s,t,r) &= n(s)W(t,r), \end{aligned} \tag{3.7}$$

where $U(t, r)$, $V(t, r)$, $W(t, r) \in C^1$, and $l(s)$, $m(s)$, $n(s)$ are not identically zero. Then, $\boldsymbol{\alpha(s)}$ being a geodesic curve on the hypersurface $\mathbf{P}(s, t, r)$ iff

$$\left.\begin{array}{l} U(t_0, r_0) = V(t_0, r_0) = W(t_0, r_0) = 0, \\ (V_t W_r - W_t V_r)(t_0, r_0) \neq 0, \\ m(s) \neq 0, \text{ and } n(s) \neq 0;\ 0 \leq t_0 \leq T, \quad 0 \leq r \leq H. \end{array}\right\} \tag{3.8}$$

Type (b). Let

$$\begin{aligned} u(s,t,r) &= l(s,t)U(r), \\ v(s,t,r) &= m(s,t)V(r), \\ w(s,t,r) &= n(s,t)W(r), \end{aligned} \tag{3.9}$$

where $U(t,r)$, $V(t,r)$, $W(t,r) \in C^1$, and $l(s)$, $m(s)$, $n(s)$ are not identically zero. Then, $\boldsymbol{\alpha(s)}$ being a geodesic curve on the hypersurface $\mathbf{P}(s,t,r)$ iff

$$\left.\begin{array}{l} l(s,t_0)U(r_0) = m(s,t_0)V(r_0) = n(s,t_0)W(r_0) = 0, \\ V(r_0)m_t(s,t_0)n(s,t_0)\frac{dW(r_0)}{dr} - W(r_0)n_t(s,t_0)m(s,t_0)\frac{dV(r_0)}{dr} \neq 0, \\ 0 \leq t_0 \leq T, \quad 0 \leq r \leq H. \end{array}\right\} \tag{3.10}$$

Type (c). Let

$$\begin{aligned} u(s,t,r) &= l(s,r)U(t), \\ v(s,t) &= m(s,r)V(t), \\ w(s,t) &= n(s,r)W(t), \end{aligned} \tag{3.11}$$

where $U(t)$, $V(t)$, $W(t) \in C^1$, and $l(s,r)$, $m(s,r)$, $n(s,r)$ are not identically zero. Hence, $\boldsymbol{\alpha}(s)$ being a geodesic curve on the hypersurface $\mathbf{P}(s,t,r)$ iff

$$\left.\begin{array}{l} l(s,r_0)U(t_0) = m(s,r_0)V(t_0) = n(s,r_0)W(t_0) = 0, \\ m(s,r_0)\frac{dV(r_0)}{dt}n_r(s,r_0)W(t_0) - n(s,r_0)\frac{dW}{dt}m_r(s,t_0)V(t_0) \neq 0, \\ 0 \leq t_0 \leq T, \quad 0 \leq r \leq H. \end{array}\right\} \tag{3.12}$$

3.1. **Example.** Now, we are interesting with an example to emphasize the method.

Example 3.1. Let the curve $\boldsymbol{\alpha(s)}$ be

$$\boldsymbol{\alpha}(s) = \left(\frac{1}{2}\cos s, \frac{1}{2}\sin s, \frac{1}{2}s, \frac{1}{\sqrt{2}}s\right),\ 0 \leq s \leq 2\pi.$$

Then,

$$\left.\begin{array}{l} \mathbf{t}(s) = (-\frac{1}{2}\sin s, \frac{1}{2}\cos s, \frac{1}{2}, \frac{1}{\sqrt{2}}), \\ \mathbf{n}(s) = (-\cos s, -\sin s, 0, 0), \\ \mathbf{b}_2(s) = \left(0, 0, \frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right) \\ \mathbf{b}_1(s) = (-\frac{\sqrt{3}}{2}\sin s, \frac{\sqrt{3}}{2}\cos s, -\frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{6}). \end{array}\right\}$$

Thus, the hypersurface family with a common geodesic curve $\boldsymbol{\alpha(s)}$ can be expressed as

$$M:\mathbf{P}(s,t,r)=\begin{pmatrix} \frac{1}{2}\cos s-\frac{1}{2}u(s,t,r)\sin s-\frac{\sqrt{3}}{2}v(s,t,r)\sin s \\ \frac{1}{2}\sin s+\frac{1}{2}u(s,t,r)\cos s+\frac{\sqrt{3}}{2}v(s,t,r)\cos s \\ \frac{1}{2}s+\frac{1}{2}u(s,t,r)-\frac{\sqrt{3}}{6}v(s,t,r)+\frac{\sqrt{6}}{3}w(s,t,r) \\ \frac{1}{\sqrt{2}}s+\frac{1}{\sqrt{2}}u(s,t,r)-\frac{1}{\sqrt{6}}v(s,t,r)-\frac{1}{\sqrt{3}}w(s,t,r) \end{pmatrix}, \tag{3.13}$$

where $0\le s\le 2\pi$, $0\le t_0\le T$, and $0\le r\le H$. A thorough treatment on $\mathbf{P}(s,t,r)$ will be given in the following:

Marching-scale functions of Type (a).

Taking $l(s)=m(s)=n(s)=1$, and

$$U(t,r)=(t-t_0)(r-r_0),\ V(t,r)=t-t_0,\ W(t,r)=r-r_0, \text{ with } 0\le r,t\le 1.$$

Then, we obtain

$$u(s,t,r)=(t-t_0)(r-r_0),\ v(s,t)=t-t_0,\ w(s,t)=r-r_0,$$

where $0\le r,t\le 1$, and with $0\le s\le 2\pi$. Thereby, Eq. (3.13) become:

$$M:\mathbf{P}(s,t,r)=\begin{pmatrix} \frac{1}{2}\cos s-\frac{1}{2}(t-t_0)(r-r_0)\sin s-\frac{\sqrt{3}}{2}(t-t_0)\sin s \\ \frac{1}{2}\sin s+\frac{1}{2}(t-t_0)(r-r_0)\cos s+\frac{\sqrt{3}}{2}(t-t_0)\cos s \\ \frac{1}{2}s+\frac{1}{2}(t-t_0)(r-r_0)-\frac{\sqrt{3}}{6}(t-t_0)+\frac{\sqrt{6}}{3}(r-r_0) \\ \frac{1}{\sqrt{2}}s+\frac{1}{\sqrt{2}}(t-t_0)(r-r_0)-\frac{1}{\sqrt{6}}(t-t_0)-\frac{1}{\sqrt{3}}(r-r_0) \end{pmatrix},$$

where $0\le r,\ t\le 1$, $0\le t_0,\ r_0\le 1$, and $0\le s\le 2\pi$. The position of the curve $\boldsymbol{\alpha}(s)$ can be set on the hypersurface by changing the parameters t_0 and r_0. Setting $t_0=1$ and $r_0=0$. Then, the hypersurface $\mathbf{P}(s,t,r)$ becomes

$$M:\mathbf{P}(s,t,r)=\begin{pmatrix} \frac{1}{2}\cos s-\frac{1}{2}r(t-1)\sin s-\frac{\sqrt{3}}{2}(t-1)\sin s \\ \frac{1}{2}\sin s+\frac{1}{2}r(t-1)\cos s+\frac{\sqrt{3}}{2}(t-1)\cos s \\ \frac{1}{2}s+\frac{1}{2}r(t-1)-\frac{\sqrt{3}}{6}(t-1)+\frac{\sqrt{6}}{3}r \\ \frac{1}{\sqrt{2}}s+\frac{1}{\sqrt{2}}r(t-1)-\frac{1}{\sqrt{6}}(t-1)-\frac{1}{\sqrt{3}}r \end{pmatrix}$$

Depending on the 3D rendering methods, if we (parallel) project the hypersurface $\mathbf{P}(s,t,r)$ into the $x_4=0$ subspace and fixing $r=\frac{1}{2}$ the hypersurface is

$$M:\mathbf{P}_{x_4}(s,t,\frac{1}{2})=\begin{pmatrix} \frac{1}{2}\cos s-\frac{1}{2}(t-1)\left(\frac{1}{2}+\sqrt{3}\right)\sin s \\ \frac{1}{2}\sin s+\frac{1}{2}(t-1)\left(\frac{1}{2}+\sqrt{3}\right)\cos s \\ \frac{1}{2}s+\frac{1}{2}(t-1)\left(\frac{1}{2}+\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}} \end{pmatrix}$$

where $0\le t\le 1$, and $0\le s\le 2\pi$, in 3-space drawn in Figure 1-Type (a).

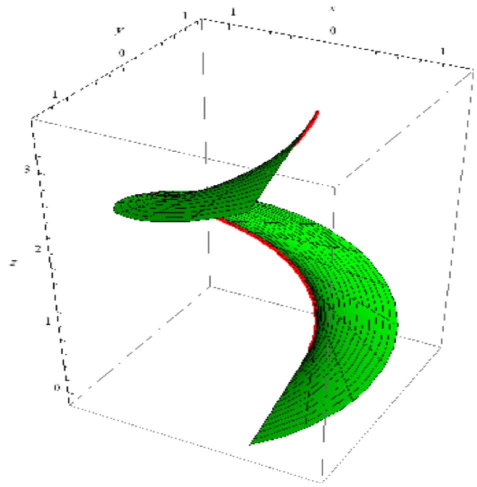

Figure 1. Projection of a member of the hypersurface family and its geodesic.

Let

$$\begin{aligned} m(s,t) &= s+t+1,\ n(s,t)=(s+1)(t-t_0), \\ U(r) &= 0,\ V(r)=r-r_0,\ W(r)=1. \end{aligned}$$

Then,

$$u(s,t,r)=0,\ v(s,t)=(s+t+1)\,(r-r_0)\,,\ w(s,t)=(s+1)\,(t-t_0).$$

Thus, the Eq. (3.13) become:

$$M:\mathbf{P}(s,t,r)=\left(\begin{array}{c} \frac{1}{2}\cos s-\frac{\sqrt{3}}{2}\,(s+t+1)\,(r-r_0)\sin s \\ \frac{1}{2}\sin s+\frac{\sqrt{3}}{2}\,(s+t+1)\,(r-r_0)\cos s \\ \frac{1}{2}s+-\frac{\sqrt{3}}{6}\,(s+t+1)\,(r-r_0)+\frac{\sqrt{6}}{3}\,(s+1)\,(t-t_0) \\ \frac{1}{\sqrt{2}}s-\frac{1}{\sqrt{6}}\,(s+t+1)\,(r-r_0)-\frac{1}{\sqrt{3}}\,(s+1)\,(t-t_0) \end{array}\right).$$

Similarly, we may choose $t_0=1/2$ and $r_0=0$, so that

$$M:\mathbf{P}(s,t,r)=\left(\begin{array}{c} \frac{1}{2}\cos s-\frac{\sqrt{3}}{2}r\,(s+t+1)\sin s \\ \frac{1}{2}\sin s+\frac{\sqrt{3}}{2}r\,(s+t+1)\cos s \\ \frac{1}{2}s+-\frac{\sqrt{3}}{6}r\,(s+t+1)+\frac{\sqrt{6}}{3}\,(s+1)\,(t-\frac{1}{2}) \\ \frac{1}{\sqrt{2}}s-\frac{1}{\sqrt{6}}r\,(s+t+1)-\frac{1}{\sqrt{3}}\,(s+1)\,(t-\frac{1}{2}) \end{array}\right),$$

Hence, if we (parallel) project the hypersurface $\mathbf{P}(s,t,r)$ into the $x_3=0$ subspace, and taking $t=\frac{1}{2}$ we get

$$M:\mathbf{P}_{x_3}(s,\frac{1}{2},r)=\left(\frac{1}{2}\cos s,\frac{1}{2}\sin s,\frac{1}{\sqrt{2}}s-\frac{1}{\sqrt{3}}r\,(s+1)\right)$$

where $0\leq r\leq 1$, and $0\leq s\leq 2\pi$, in 3-space drawn in Figure 2-Type (b).

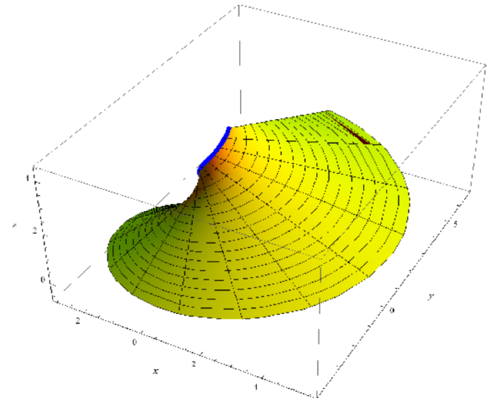

Figure 2. Projection of a member of the hypersurface family and its geodesic.

$$\begin{aligned} m(s,r) &= (r-r_0)\sin s,\ n(s,r) = sr^2, \\ U(t) &= 0,\ V(t) = 1,\ W(r) = t - t_0. \end{aligned}$$

Then, we obtain

$$u(s,t,r) = 0,\ v(s,t,r) = (r-r_0)\sin s,\ \ w(s,r) = sr^2\,(t-t_0)\,.$$

The Eq. (3.13) become:

$$M : \mathbf{P}(s,t,r) = \begin{pmatrix} \frac{1}{2}\cos s - \frac{\sqrt{3}}{2}(r-r_0)\sin s \sin s \\ \frac{1}{2}\sin s + \frac{\sqrt{3}}{2}(r-r_0)\sin s \cos s \\ \frac{1}{2}s - \frac{\sqrt{3}}{2}(r-r_0)\sin s + \frac{\sqrt{6}}{3}(r-r_0) \\ \frac{1}{\sqrt{2}}s - \frac{1}{\sqrt{6}}(r-r_0)\sin s - \frac{1}{\sqrt{3}}sr^2\,(t-t_0) \end{pmatrix}.$$

Similarly, we can choose $t_0 = 1$ and $r_0 = 1$, so that

$$M : \mathbf{P}(s,t,r) = \begin{pmatrix} \frac{1}{2}\cos s - \frac{\sqrt{3}}{2}(r-1)\sin s \sin s \\ \frac{1}{2}\sin s + \frac{\sqrt{3}}{2}(r-1)\sin s \cos s \\ \frac{1}{2}s - \frac{\sqrt{3}}{2}(r-1)\sin s + \frac{\sqrt{6}}{3}(r-1) \\ \frac{1}{\sqrt{2}}s - \frac{1}{\sqrt{6}}(r-1)\sin s - \frac{1}{\sqrt{3}}sr^2\,(t-1) \end{pmatrix}.$$

Similarly, if we (parallel) project the hypersurface $\mathbf{P}(s,t,r)$ into the $x_1 = 0$ subspace, and setting $r = 1$ we get

$$M : \mathbf{P}_{x_1}(s,t,1) = \left(\frac{1}{2}\sin s, \frac{1}{2}s, \frac{1}{\sqrt{2}}s + \frac{s}{\sqrt{6}}\,(t-1)\right),$$

where $0 \le t \le 1$, and $0 \le s \le 2\pi$, in 3-space drawn in Figure 3-Type (c).

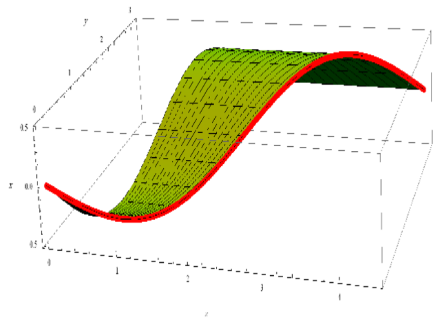

Figure 3. Projection of a member of the hypersurface family and its geodesic.

4. Conclusion

In this study, we have considered a mathematical framework, for constructing a surface family whose members all share a given geodesic curve as an isoparametric curve in $\mathbb{E}^4$. Given a regular spatial curve, we answer question about the necessary and sufficient condition for the given curve to be a geodesic. Lastly, as an application of our approach one example for each type of marching-scale functions is given. Hopefully these results will lead to a wider usage of surfaces in geometric modeling, garment-manufacture industry, and the manufacturing of products.

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