Int. J. Anal. Appl. (2026), 24:155

International Journal of Analysis and Applications

Semihypergroups in Which the Radical of Every Hyperideal Is a Subsemihypergroup

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Abstract. The concept of a semihypergroup is a notable generalization of the concept of a semigroup. For nonempty subset A of a semihypergroup H, the radical of A denoted by $\sqrt{A}$, is $\sqrt{A} = \{a \in H \mid a^n \subseteq A$ for some positive integer $n\}$. A characterization when the radical of every hyperideal of H is a subsemihypergroup of H is investigated and given in this paper. Indeed, we characterize when the radical of every hyperideal of H is a bi-hyperideal of H; when the radical of every bi-hyperideal of H is a left hyperideal of H; and when the radical of every subsemihypergroup of H is a hyperideal of H.

1. Introduction

In 1934, the concept of algebraic hyperstructures has been defined by Marty [11]. Hyperstructures in various algebraic structures for example, rings, semirings, semigroups, ordered semigroups and ternary semigroups have been studied by many authors (for example, see [1,4,7–10,12]). The concept of a semihypergroup is a generalization of the concept of a semigroup and many classical notions such as ideals, quasi-ideals and bi-ideals defined for semigroups and regular semigroups have been generalized to semihypergroups.

A commutative semigroup is a semilattice of Archimedean semigroups was shown by T. Tamura and N. Kimura [15]. A semigroup which is a semilattice of Archimedean semigroups is completely was described by M.S. Putcha [13]. A nonempty subset A of a semigroup S, $\sqrt{A}$ denotes the *radical* of A, i.e., $\sqrt{A} = \{a \in S \mid a^n \in A$ for some positive integer $n\}$. S is called *Archimedean* if and only if for all $a, b \in S$ and $k \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $b^n \in Sa^kS$ (cf. [3, 14]). M. Ćirić and S. Bogdanović [3] proved that a semigroup S is a semilattice of Archimedean semigroups if and only if the radical of every ideal of S is an ideal of S. Moreover, in [3], the same authors characterized

Received: Oct. 14, 2022.

2020 *Mathematics Subject Classification.* 20N10, 06F05.

Key words and phrases. semihypergroup; hyperideal; bi-hyperideal; subsemihypergroup.

https://doi.org/10.28924/2291-8639-24-2026-155
ISSN: 2291-8639

a semigroup S in which the radical of every ideal of S is a subsemigroup. Recently, J. Sanborisoot and T. Changphas [14] characterized a semigroup in which the radical of every quasi-ideal Q of S is a subsemigroup. According to the fact that the concept of semihypergroups is a generalization of the concept of semigroups, several classical notions such as ideals and bi-ideals introduced in semigroups and regular semigroups have been generalized to semihypergroups (see [5,6]).

In this paper, we do in the line of Ćirić and Bogdanović in [3]. We characterize semihypergroups in which the radical of every hyperideal (or subsemihypergroup or left hyperideal or bi-hyperideal) is a subsemihypergroup (or left hyperideal or hyperideal or bi-hyperideal).

2. Preliminaries

The concept of a semihypergroup is a notable generalization of the concept of a semigroup. In a semigroup the product of two elements is an element, while in a semihypergroup, the composition of two elements is a nonempty set. Let H be a nonempty set and $\mathcal{P}^*(H)$ be the set of all nonempty subsets of H. A *hyperoperation* on H is a map $\circ : H \times H \to \mathcal{P}^*(H)$ and the system $(H, \circ)$ is called a *hypergroupoid*. Let A and B be nonempty subsets of a hypergroupoid $(H, \circ)$. For $a \in H$, we denote that

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \; a \circ A = \{a\} \circ A, \text{ and } A \circ a = A \circ \{a\}.$$

A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if

$$(a \circ b) \circ c = a \circ (b \circ c)$$

for all $a, b, c \in H$.

Let H be a semihypergroup. A nonempty subset A of H is called a *subsemihypergroup* if for any $a, b \in A$, $a \circ b \subseteq A$. For any nonempty sets $A, B \subseteq H$, the subsemihypergroup of H generated by $\{A, B\}$ denoted by $\langle A, B \rangle$. A nonempty subset A of H is called a *left hyperideal* (*right hyperideal*) of H if $H \circ A \subseteq A$ $(A \circ H \subseteq A)$. A nonempty subset A of H is called a *hyperideal* of H if A is both a left hyperideal and a right hyperideal of H. A subsemihypergroup B of H is called a *bi-hyperideal* of H if $B \circ H \circ B \subseteq B$ (cf. [5,6,9]).

Example 2.1. [12] Let $H = \{a, b, c, d\}$. Define the hyperoperation $\circ$ as follow:

$\circ$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$
d	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b, c\}$

Then the hyperstructure $(H, \circ)$ is a semihypergroup. It is easy to see that $\{a\}, \{a, b\}$, $\{a, b, c\}$ and H are hyperideals of H. And, the set $\{a, c\}$ is a bi-hyperideal of H.

Throughout this paper, let $\mathbb{N} = \{1, 2, 3, \ldots\}$ be the set of all positive integers. Also, we shall write a^n for $a \circ a \circ \cdots \circ a$ (n-copies of a) for any $n \in \mathbb{N}$. The *radical* of subset A of a semihypergroup H, we mean the set $\sqrt{A}$ defined by

$$\sqrt{A} = \{a \in H \mid \ a^n \subseteq A \text{ for some } n \in \mathbb{N}\}.$$

3. Main Results

In [3], the authors characterized when the radical of every ideal of a semigroup S is a subsemigroup of S. The section begins the following theorem.

Theorem 3.1. *Let H be a semihypergroup. Then the radical of every hyperideal of H is a subsemihypergroup of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ \{a^k, b^l\} \circ H.$$

Proof. Assume that the radical of every hyperideal of H is a subsemihypergroup of H. Let $a, b, r \in H$ be such that $r \in a \circ b$, and let $k, l \in \mathbb{N}$. Define

$$A = H \circ \{a^k, b^l\} \circ H.$$

Then A is a hyperideal of H. By assumption, $\sqrt{A}$ is a subsemihypergroup of H. Since $a^{k+2}, b^{l+2} \subseteq A$, we have $a, b \in \sqrt{A}$. Hence $a \circ b \subseteq \sqrt{A} \circ \sqrt{A} \subseteq \sqrt{A}$. As $r \in a \circ b$, there exists $n \in \mathbb{N}$ such that

$$r^n \subseteq H \circ \{a^k, b^l\} \circ H.$$

Conversely, assume that for any $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ \{a^k, b^l\} \circ H.$$

Let A be a hyperideal of H. We want show that $\sqrt{A}$ is a subsemihypergroup of H. Now, let $r \in \sqrt{A} \circ \sqrt{A}$; then there exist $a, b \in \sqrt{A}$ such that $r \in a \circ b$. Thus, there exist $k, l \in \mathbb{N}$ such that $a^k, b^l \subseteq A$. By assumption, there exists $n \in \mathbb{N}$ where $r^n \subseteq H \circ \{a^k, b^l\} \circ H$. From

$$H \circ \{a^k, b^l\} \circ H \subseteq H \circ A \circ H \subseteq A,$$

it follows that $r \in \sqrt{A}$. Therefore $\sqrt{A}$ is a subsemihypergroup of H. □

Theorem 3.2. *Let H be a semihypergroup. Then the radical of every left hyperideal of H is a subsemihypergroup of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ \{a^k, b^l\}.$$

Proof. Assume that the radical of every left hyperideal of H is a subsemihypergroup of H. Let $a, b, r \in H$ such that $r \in a \circ b$. For every $k, l \in \mathbb{N}$, we define

$$L = H \circ \{a^k, b^l\};$$

then L is a left hyperideal of H and $a, b \in \sqrt{L}$. Using assumption, we have $\sqrt{L}$ is a subsemihypergroup of H. Therefore $a \circ b \in \sqrt{L} \circ \sqrt{L} \subseteq \sqrt{L}$. As $r \in a \circ b$, we get $r^n \subseteq L$ for some $n \in \mathbb{N}$.

Conversely, suppose that for each $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ \{a^k, b^l\}.$$

Let L be a left ideal of H. Let $r \in H$ and $a, b \in \sqrt{L}$ be such that $r \in a \circ b$. Then there exist $k, l \in \mathbb{N}$ such that $a^k, b^l \subseteq L$. We obtain by assumption that $r^n \subseteq H \circ \{a^k, b^l\}$. Since

$$H \circ \{a^k, b^l\} \subseteq H \circ L \subseteq L,$$

it follows that $r \in \sqrt{L}$, i.e. $\sqrt{L}$ is a subsemihypergroup of H. □

The following example shows that the radical of a left hyperideal of a semihypergroup H need not be a subsemihypergroup of H.

Example 3.1. Let $H = \{a, b, c, d, e\}$. Define the hyperoperation $\circ$ on H as follows:

$\circ$	a	b	c	d	e
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,d\}$	$\{a\}$
c	$\{a\}$	$\{a,e\}$	$\{a,c\}$	$\{a,c\}$	$\{a,e\}$
d	$\{a\}$	$\{a,b\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b\}$
e	$\{a\}$	$\{a,e\}$	$\{a\}$	$\{a,c\}$	$\{a\}$

Then H is a semihypergroup. We have $\{a, c, d\}$ is a left hyperideal of H. But $\sqrt{\{a, c, d\}} = \{a, c, d, e\}$ which is not a subsemihypergroup of H.

As Theorem 3.2, we obtain the following theorem.

Theorem 3.3. *Let H be a semihypergroup. Then the radical of every right hyperideal of H is a subsemihypergroup of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \{a^k, b^l\} \circ H.$$

Theorem 3.4. *Let H be a semihypergroup. Then the radical of every subsemihypergroup of H is a bi-hyperideal of H if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists m \in \mathbb{N})\ r^m \subseteq \langle a^k, b^l\rangle$;
(2) $(\forall a, b, c, s \in H)\ s \in a \circ c \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ s^n \subseteq \langle a^k, b^l\rangle$.

Proof. Suppose that (1) and (2) hold and let A be a subsemihypergroup of H. To show that $\sqrt{A}$ is a bi-hyperideal of H. Now, let $r \in \sqrt{A} \circ \sqrt{A}$. Then $r \in a \circ b$ for some $a, b \in \sqrt{A}$; hence, there exist $k, l \in \mathbb{N}$ such that $a^k, b^l \subseteq A$. Using (1), $r^m \subseteq \langle a^k, b^l\rangle \subseteq A$ for some $m \in \mathbb{N}$ and so $r \in \sqrt{A}$. It follows that $\sqrt{A}$ is a subsemihypergroup of H. Let $s \in \sqrt{A} \circ H \circ \sqrt{A}$; then there exist $u, w \in \sqrt{A}$ and $v \in H$ such that $s \in u \circ v \circ w$. Now, we have $u^i, w^j \subseteq A$ for some $i, j \in \mathbb{N}$. Using (2), $s^n \subseteq \langle u^i, w^j\rangle \subseteq A$. Consequently, $s \in \sqrt{A}$.

Conversely, assume that the radical of every subsemihypergroup of H is a bi-hyperideal of H. Let $a, b, c, r, s \in H$ be such that $r \in a \circ b$ and $s \in a \circ c \circ b$. For every $k, l \in \mathbb{N}$, we let $A = \langle a^k, b^l\rangle$. Then A is a subsemihypergroup; we have by assumption that $\sqrt{A}$ is a bi-hyperideal of H. Because $a^k, b^l \subseteq A$, we get $a, b \in \sqrt{A}$. Consider

$$r \in a \circ b \subseteq \sqrt{A} \circ \sqrt{A} \subseteq \sqrt{A} \text{ and } s \in a \circ c \circ b \subseteq \sqrt{A} \circ H \circ \sqrt{A} \subseteq \sqrt{A}.$$

This shows that $r^m, s^n \subseteq A$ for some $m, n \in \mathbb{N}$, whence (1) and (2) hold. □

Theorem 3.5. *Let H be a semihypergroup. Then the radical of every left hyperideal of H is a bi-hyperideal of H if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists m \in \mathbb{N})\ r^m \subseteq H \circ \{a^k, b^l\}$;
(2) $(\forall a, b, c, s \in H)\ s \in a \circ c \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ s^n \subseteq H \circ \{a^k, b^l\}$.

Proof. Suppose first that the radical of every left hyperideal of H is a bi-hyperideal of H. Let $a, b, c, r, s \in H$ be such that $r \in a \circ b$ and $s \in a \circ c \circ b$, and let $k, l \in \mathbb{N}$. Set

$$L = H \circ \{a^k, b^l\}.$$

Then L is a left hyperideal of H. We have by assumption that $\sqrt{L}$ is a bi-hyperideal of H. From $a^{k+1}, b^{l+1} \subseteq L$, we get that $a, b \in \sqrt{L}$. Since $\sqrt{L}$ is a bi-hyperideal of H, it follows that $\sqrt{L}$ is a subsemihypergroup of H. Whence $a \circ b \subseteq \sqrt{L} \circ \sqrt{L} \subseteq \sqrt{L}$. We obtain that $r^m \subseteq L$ for some $m \in \mathbb{N}$, because $r \in a \circ b$. Thus (1) holds. Since

$$s \in a \circ c \circ b \subseteq \sqrt{L} \circ H \circ \sqrt{L} \subseteq \sqrt{L},$$

then $s^n \subseteq L$ for some $n \in \mathbb{N}$. Accordingly, (2) holds.

Conversely, assume that (1) and (2) hold. Let L be a left hyperideal of H. We have to show that $\sqrt{L}$ is a bi-hyperideal of H. Firstly, we shall show that $\sqrt{L}$ is a subsemihypergroup. Let $r \in \sqrt{L} \circ \sqrt{L}$. Then $r \in a \circ b$ for some $a, b \in \sqrt{L}$. It follows that there are $k, l \in \mathbb{N}$ where $a^k, b^l \subseteq L$. Using (1), we have $r^m \subseteq H \circ \{a^k, b^l\}$ for some $m \in \mathbb{N}$. From

$$H \circ \{a^k, b^l\} \subseteq H \circ L \subseteq L,$$

we conclude that $r \in \sqrt{L}$. Suppose that $s \in \sqrt{L} \circ H \circ \sqrt{L}$. Then $s \in u \circ v \circ w$ for some $u, w \in \sqrt{L}$ and $v \in H$. For all $u^i, w^j \subseteq L$ for some $i, j \in \mathbb{N}$ we have by (2) that

$$s^n \subseteq H \circ \{u^i, w^j\} \subseteq H \circ L \subseteq L.$$

it follows that $s \in \sqrt{L}$; this completes the proof. □

The following theorem is giving sufficient and necessary conditions in order that the radical of every hyperideal of a semihypergroup H is a bi-hyperideal of H.

Theorem 3.6. *Let H be a semihypergroup. Then the radical of every hyperideal of H is a bi-hyperideal of H if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists m \in \mathbb{N})\ r^m \subseteq H \circ \{a^k, b^l\} \circ H$;
(2) $(\forall a, b, c, s \in H)\ s \in a \circ c \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ s^n \subseteq H \circ \{a^k, b^l\} \circ H$.

Proof. Suppose that the radical of every hyperideal of H is a bi-hyperideal of H. Let $a, b, c, r, s \in H$ where $r \in a \circ b$ and $s \in a \circ c \circ b$. For any $k, l \in \mathbb{N}$, we let

$$A = H \circ \{a^k, b^l\} \circ H.$$

Since A is a hyperideal of H, we obtain by hypothesis that $\sqrt{A}$ is a bi-hyperideal of H. Clearly, $a, b \in \sqrt{A}$ and $\sqrt{A}$ is a subsemihypergroup. Then $a \circ b \subseteq \sqrt{A}$, i.e., $r^m \subseteq A$ for some $m \in \mathbb{N}$. Thus (1) follows. Because

$$s \in a \circ c \circ b \subseteq \sqrt{A} \circ H \circ \sqrt{A} \subseteq \sqrt{A},$$

we have $s^n \subseteq A$ for some $n \in \mathbb{N}$. Consequently, (2) follows.

Conversely, suppose (1) and (2) hold and let A be a hyperideal of H. Let $r \in a \circ b$ and $s \in a \circ c \circ b$ be such that $a, b \in \sqrt{A}$ and $c, r, s \in H$. Then there exist $k, l \in \mathbb{N}$ such that $a^k, b^l \subseteq A$. Using (1) and (2), there exist $m, n \in \mathbb{N}$ where

$$r^m, s^n \subseteq H \circ \{a^k, b^l\} \circ H \subseteq H \circ A \circ H \subseteq A.$$

That is $r, s \in \sqrt{A}$. Hence $\sqrt{A}$ is a bi-hyperideal of H. □

Theorem 3.7. *Let H be a semihypergroup. Then the radical of every bi-hyperideal of H is a bi-hyperideal of H if and only if* (1) *and* (2) *hold:*

(1) $(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \{a^k, b^l\} \circ H \circ \{a^k, b^l\}$;

(2) $(\forall a, b, c, s \in H)\ s \in a \circ c \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists m \in \mathbb{N})\ s^m \subseteq \{a^k, b^l\} \circ H \circ \{a^k, b^l\}$.

Proof. Suppose that the radical of every bi-hyperideal of H is a bi-hyperideal of H. Now let $a, b, c, r, s \in H$ where $r \in a \circ b$ and $s \in a \circ c \circ b$. For any $k, l \in \mathbb{N}$, we put

$$B = \{a^k, b^l\} \circ H \circ \{a^k, b^l\}.$$

Then $a, b \in \sqrt{B}$ and B is a bi-hyperideal of H. By hypothesis, we obtain that $\sqrt{B}$ is a bi-hyperideal of H. Now we have

$$a \circ b \subseteq \sqrt{B} \circ \sqrt{B} \subseteq \sqrt{B} \text{ and } a \circ c \circ b \subseteq \sqrt{B} \circ H \circ \sqrt{B} \subseteq \sqrt{B}.$$

So $r^m, s^n \subseteq B$ for some $m, n \in \mathbb{N}$. Hence (1) and (2) hold.

For the converse, assume (1) and (2) hold. Let B be a bi-hyperideal of H. Let $c, r, s \in H$ and $a, b \in \sqrt{B}$ be such that $r \in a \circ b$ and $s \in a \circ c \circ b$. Then there are $k, l \in \mathbb{N}$ such that $a^k, b^l \subseteq B$. Using (1) and (2), $r^m, s^m \subseteq \{a^k, b^l\} \circ H \circ \{a^k, b^l\}$ for some $m, n \in \mathbb{N}$. We obtain that $r, s \in \sqrt{B}$, since

$$\{a^k, b^l\} \circ H \circ \{a^k, b^l\} \subseteq B \circ H \circ B \subseteq B.$$

This proves that $\sqrt{B}$ is a bi-hyperideal of H. □

Theorem 3.8. *Let H be a semihypergroup. Then the radical of every bi-hyperideal of H is a hyperideal of H if and only if for every $a, b, c, d, r, s \in H$,*

$$r \in a \circ b \wedge s \in c \circ d \Rightarrow (\forall k, l \in \mathbb{N})(\exists m, n \in \mathbb{N})\ r^m, s^n \subseteq \{b^k, c^l\} \circ H \circ \{b^k, c^l\}.$$

Proof. Assume that the radical of every bi-hyperideal of H is a hyperideal of H and let $a, b, c, d, r, s \in H$ be such that $r \in a \circ b$ and $s \in c \circ d$. Now, let $k, l \in \mathbb{N}$. We have $B = \{b^k, c^l\} \circ H \circ \{b^k, c^l\}$ is a bi-hyperideal of H. Accordingly, $\sqrt{B}$ is a hyperideal of H by assumption. It is clear that $b, c \in \sqrt{B}$;

hence, $a \circ b \subseteq H \circ \sqrt{B} \subseteq \sqrt{B}$ and $c \circ d \subseteq \sqrt{B} \circ H \subseteq \sqrt{B}$. Hence $r^m, s^n \subseteq B$ for some $m, n \in \mathbb{N}$, because $r \in a \circ b$ and $s \in c \circ d$.

Conversely, assume that for all $a, b, c, d, r, s \in H$,

$$r \in a \circ b \wedge s \in c \circ d \Rightarrow (\forall k, l \in \mathbb{N})(\exists m, n \in \mathbb{N})\ r^m, s^n \subseteq \{b^k, c^l\} \circ H \circ \{b^k, c^l\}.$$

Let B be a bi-hyperideal of H. We need show that $\sqrt{B}$ is a hyperideal of H. Now, let $r \in H \circ \sqrt{B}$ and $s \in \sqrt{B} \circ H$. Then $r \in a \circ b$ and $s \in c \circ d$ for some $a, d \in H$ and $b, c \in \sqrt{B}$. For every $k, l \in \mathbb{N}$ such that $b^k, c^l \subseteq B$ we have by (1) and (2) that

$$r^m, s^n \subseteq \{b^k, c^l\} \circ H \circ \{b^k, c^l\} \subseteq B \circ H \circ B \subseteq B$$

for some $m, n \in \mathbb{N}$. That is $r, s \in \sqrt{B}$ so that $\sqrt{B}$ is a hyperideal of H. □

The following theorem can be proved similarly to the proof of Theorem 3.1.

Theorem 3.9. *Let H be a semihypergroup. Then the radical of every bi-hyperideal of H is a subsemihypergroup of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \{a^k, b^l\} \circ H \circ \{a^k, b^l\}.$$

Theorem 3.10. *Let H be a semihypergroup. Then the radical of every bi-hyperideal of H is a left hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq b^k \circ H \circ b^k.$$

Proof. Assume that the radical of every bi-hyperideal of H is a left hyperideal of H. Let $a, b, r \in H$ and $k \in \mathbb{N}$. Suppose that $r \in a \circ b$. Consider $B = b^k \circ H \circ b^k$; then B is a bi-hyperideal of H and $b \in \sqrt{B}$. According to assumption, $\sqrt{B}$ is a left hyperideal of H. Consequently, $a \circ b \subseteq H \circ \sqrt{B} \subseteq \sqrt{B}$; hence $r^n \subseteq B$ for some $n \in \mathbb{N}$.

For the converse, suppose that for any $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq b^k \circ H \circ b^k.$$

Let B be a bi-hyperideal of H. To show that $\sqrt{B}$ is a left hyperideal of H. Let $r \in H \circ \sqrt{B}$. Then $r \in a \circ b$ for some $a \in H$ and $b \in \sqrt{B}$; thus, there is $k \in \mathbb{N}$ where $b^k \subseteq B$. Using assumption, there is $n \in \mathbb{N}$ such that $r^n \subseteq b^k \circ H \circ b^k \subseteq B \circ H \circ B \subseteq B$. It follows that $r \in \sqrt{B}$. □

The following theorem can be proved similarly to the proof of Theorem 3.10.

Theorem 3.11. *Let H be a semihypergroup. Then the radical of every bi-hyperideal of H is a right hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq a^k \circ H \circ a^k.$$

Theorem 3.12. *Let H be a semihypergroup. Then the radical of every subsemihypergroup of H is a hyperideal of H if and only if*

$$(\forall a, b, c, d, r, s \in H)\ r \in a \circ b \wedge s \in c \circ d \Rightarrow (\forall k, l \in \mathbb{N})(\exists m, n \in \mathbb{N})\ r^m, s^n \subseteq \langle b^k, c^l \rangle.$$

Proof. Assume that the radical of every subsemihypergroup of H is a hyperideal of H. Let $a, b, c, d, r, s \in H$ be such that $r \in a \circ b$ and $s \in c \circ d$, and let $k, l \in \mathbb{N}$. Because $\langle b^k, c^l \rangle$ is a subsemihypergroup of H where $b^k, c^l \subseteq \sqrt{\langle b^k, c^l \rangle}$, we have by assumption that $\sqrt{\langle b^k, c^l \rangle}$ is a hyperideal of H containing b, c. Hence

$$a \circ b \subseteq H \circ \sqrt{\langle b^k, c^l \rangle} \subseteq \sqrt{\langle b^k, c^l \rangle} \text{ and } c \circ d \subseteq \sqrt{\langle b^k, c^l \rangle} \circ H \subseteq \sqrt{\langle b^k, c^l \rangle}.$$

Now, we have $r^m, s^n \subseteq \langle b^k, c^l \rangle$ for some $m, n \in \mathbb{N}$, since $r \in a \circ b$ and $s \in c \circ d$.

Conversely, assume that for any $a, b, c, d, r, s \in H$,

$$r \in a \circ b \wedge s \in c \circ d \Rightarrow (\forall k, l \in \mathbb{N})(\exists m, n \in \mathbb{N})\ r^m, s^n \subseteq \langle b^k, c^l \rangle.$$

Let A be a subsemihypergroup of H. We need show that $\sqrt{A}$ is a hyperideal of H. Now, let $r \in H \circ \sqrt{A}$ and $s \in \sqrt{A} \circ H$. Then there exist $b, c \in \sqrt{A}$ and $a, d \in H$ such that $r \in a \circ b$ and $s \in c \circ d$. Since $b, c \in \sqrt{A}$, then there exist $k, l \in \mathbb{N}$ such that $b^k, c^l \subseteq A$. Using hypothesis, there are $m, n \in \mathbb{N}$ such that

$$r^m, s^n \subseteq \langle b^k, c^l \rangle \subseteq A.$$

Accordingly, $r, s \in \sqrt{A}$. □

Theorem 3.13. *Let H be a semihypergroup. Then the radical of every subsemihypergroup of H is a right hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \langle a^k \rangle.$$

Proof. Suppose that the radical of every subsemihypergroup of H is a right hyperideal of H. Let $a, b, r \in H$ where $r \in a \circ b$, and let $k \in \mathbb{N}$. By assumption, $\sqrt{\langle a^k \rangle}$ is a right hyperideal of H containing a, so we have

$$a \circ b \subseteq \sqrt{\langle a^k \rangle} \circ H \subseteq \sqrt{\langle a^k \rangle}.$$

This shows that $r \in \sqrt{\langle a^k \rangle}$; therefore, there exists $n \in \mathbb{N}$ such that $r^n \subseteq \langle a^k \rangle$.

On the other hand, assume that for any $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \langle a^k \rangle.$$

Let A be a subsemihypergroup of H. To show that $\sqrt{A}$ is a right hyperideal of H, we let $a \in \sqrt{A}$ and $b, r \in H$ where $r \in a \circ b$. Then there is $k \in \mathbb{N}$ where $a^k \subseteq A$, by hypothesis, $r^n \subseteq \langle a^k \rangle \subseteq A$ for some $n \in \mathbb{N}$. As a result, $r \in \sqrt{A}$ as required. □

The following theorem can be proved similarly to the proof of Theorem 3.13.

Theorem 3.14. *Let H be a semihypergroup. Then the radical of every subsemihypergroup of H is a left hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \langle b^k \rangle.$$

Theorem 3.15. *Let H be a semihypergroup. Then the radical of every subsemihypergroup of H is a subsemihypergroup of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \langle a^k, b^l \rangle.$$

Proof. Suppose that the radical of every subsemihypergroup of H is a subsemihypergroup of H. Let $a, b, r \in H$ and $k, l \in \mathbb{N}$. Assume that $r \in a \circ b$. Since $A = \langle a^k, b^l \rangle$ is a subsemihypergroup of H, we have by assumption that $\sqrt{A}$ is a subsemihypergroup of H. As $a, b \in \sqrt{A}$, $a \circ b \subseteq \sqrt{A} \circ \sqrt{A} \subseteq \sqrt{A}$. Hence $r^n \subseteq A$ for some $n \in \mathbb{N}$.

Conversely, assume that for any $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k, l \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq \langle a^k, b^l \rangle.$$

Let A be a subsemihypergroup of H. Now, let $r \in \sqrt{A}^2$. Then $r \in a \circ b$ for some $a, b \in \sqrt{A}$. We have $a^k, b^l \subseteq A$ for some $k, l \in \mathbb{N}$. According to assumption, $r^n \subseteq \langle a^k, b^l \rangle \subseteq A$. It follows that $r \in \sqrt{A}$, i.e. $\sqrt{A}$ is a subsemihypergroup of H. □

Theorem 3.16. *Let H be a semihypergroup. Then the radical of every left hyperideal of H is a left hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ b^k.$$

Proof. Suppose that the radical of every left hyperideal of H is a left hyperideal of H. Let $r \in a \circ b$ be such that $a, b, r \in H$. Put $L = H \circ b^k$. Since L is a left ideal of H, $\sqrt{L}$ is a left hyperideal of H by hypothesis. From $b \in \sqrt{L}$ we have $a \circ b \in H \circ \sqrt{L} \subseteq \sqrt{L}$, and so $r^n \subseteq L$ for some $n \in \mathbb{N}$.

Conversely, assume that for all $a, b, r \in H$,

$$r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq H \circ b^k.$$

Let L be a left hyperideal of H and $r \in H \circ \sqrt{L}$. Then there exist $a \in H$ and $b \in \sqrt{L}$ such that $r \in a \circ b$. For every $k \in \mathbb{N}$ such that $b^k \subseteq L$ we have by assumption that $r^n \subseteq H \circ b^k$ for some $n \in \mathbb{N}$. But since $H \circ b^k \subseteq H \circ L \subseteq L$, then $r \in \sqrt{L}$. This proves that $\sqrt{L}$ is a left hyperideal of H. □

As Theorem 3.16, we obtain the following.

Theorem 3.17. *Let H be a semihypergroup. Then the radical of every right hyperideal of H is a left hyperideal of H if and only if*

$$(\forall a, b, r \in H)\ r \in a \circ b \Rightarrow (\forall k \in \mathbb{N})(\exists n \in \mathbb{N})\ r^n \subseteq b^k \circ H.$$

Acknowledgment: This research project was financially supported by Mahasarakham University.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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