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Direct Solution of Black-Scholes-Merton European Put Option Model on Dividend Yield With Modified-Log Payoff Function

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Abstract. This paper proposes a framework based on the celebrated transform of Mellin type (MT) for the direct solution of the Black-Scholes-Merton European Put Option Model (BSMEPOM) on Dividend Yield (DY) with Modified-Log Payoff Function (MLPF) under the geometric Brownian motion. The focal goal of this paper is to use MT to obtain a valuation formula for the European Put Option (EPO) which pays a DY with MLPF. By means of the MT and its inversion formula, the price of EPO on DY was expressed in terms of integral equation. The valuation formula of EPO was obtained with the help of the convolution property of MT and final time condition. MT was tested on an illustrative example in order to measure its performance, effectiveness and suitability. The MLPF was compared with other existing payoff functions. Hence, the effect of DY on the pricing of EPO with MLPF was also investigated.

1. Introduction

Option valuation has become extremely popular in computational finance. This popularity has been displayed as one of the key major areas in derivative security. In other words, option valuation has contributed greatly to the financial markets. There is a massive growth in trading activities on derivatives globally from the inception of the Black-Scholes pricing formula [1, 2]. It is noteworthy to say that the Black-Scholes models for linear payoff function has been used by many researchers and as well as become one of the utmost areas in financial markets over the last few decades. Immediately

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after the huge success recorded by the Black-Scholes model for vanilla option flavours, several other valuation formula were developed for options pricing with different payoff functions such as Mellin transform, binomial model, finite difference method, Monte Carlo method, e.t.c; see [3] – [6]. For mathematical framework, some implementations of transform methods of different types in financial markets; see [7]– [15]. Ghevariya [16] solved the classical Black-Scholes European put option model for Modified-Log payoff function with the help of the MT. In this paper, a direct solution of BSMEPOM via the celebrated transform of Mellin type is proposed in the sense of DY and MLPF. The remaining part of the paper is listed as follows, Section 2 captures the brief concepts of MT. A new result that captures the governing model for EPO on DY with MLPF and the solution of BSMEPOM on DY with MLPF is stated and proved in Section 3. An illustrative example on the application of MT to EPO is captured by Section 4. Section 5 is the concluding part of the paper.

2. Mellin transform

This section captures some definitions of terms based on the framework of the Mellin transform.

2.1. **Definitions of Terms.**

Definition 2.1. *Let $f(x)$ be a locally Lebesgue integrable function. The Mellin transform of $f(x)$ is defined as*

$$\mathcal{M}[f(x),\omega] := \tilde{f}(\omega) = \int_0^\infty f(x)x^{\omega-1}dx \tag{2.1}$$

The Mellin transform variable ω is a complex number, $\omega = \Re(.) + i\Im(.)$, where $\Re(.)$ is the real part, i is the imaginary unit and $\Im(.)$ is the imaginary part.

Definition 2.2. *If $f(x)$ is an integrable function with fundamental strips (a, b), then if c is such that $a < c < b$ and $\{\tilde{f}(\omega) : \omega = c + it\}$ is integrable, the inverse Mellin transform is defined as*

$$\mathcal{M}^{-1}[\tilde{f}(\omega)] = f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{f}(\omega)x^{-\omega}d\omega \tag{2.2}$$

Remark 2.1. *For more details on the condition that ensures the existence of MT; see [17].*

Remark 2.2. *The fundamental operational properties of the Mellin transforms such as scaling, shifting, derivatives, integrals, convolution, multiplicative convolution and Parseval's formula are well detailed in [9, 12, 17, 18].*

3. The Solution of BSMEPOM on DY with MLPF

Ghevariya derived BSM formula on non-dividend yield for ML-payoff function [16]. In this section, Black-Scholes-Merton formula on dividend yield with MLPF is derived via the MT in the following result.

Theorem 3.1. *Consider the BSMEPOM on DY with MLPF of the form*

$$\frac{\partial P_E(S_t,t)}{\partial t}+(r-q)S_t\frac{\partial P_E(S_t,t)}{\partial S_t}+\frac{(\sigma S_t)^2}{2}\frac{\partial^2 P_E(S_t,t)}{\partial (S_t)^2}-rP_E(S_t,t)=0 \tag{3.1}$$

subject to the boundary conditions

$$\lim_{S_t\to\infty} P_E(S_t,t)=0 \qquad \text{on } [0,T) \tag{3.2}$$

$$\lim_{S_t\to 0} P_E(S_t,t)=\frac{K}{e^{r(T-t)}} \qquad \text{on } [0,T) \tag{3.3}$$

and MLPF

$$P_E(S_T,T)=\left[S_T\ln\left(\frac{K}{S_T}\right)\right]^+ \qquad \text{on } [0,\infty) \tag{3.4}$$

where $P_E(S_t,t)$, t, T, S_t, K, σ, r and q are the price of EPO, current time, time to expiry, underlying asset price, strike price, volatility, risk-free interest rate and DY, respectively, then the valuation formula for BSMEPOM on DY with MLPF is given by

$$P_E(S_t,t)=S_te^{-q\tau}\left[\sigma\sqrt{\tau}\eta(d)-(D_1+D_2\tau)\mathcal{N}(-d)\right] \tag{3.5}$$

with

$$D_1=\ln\left(\frac{S_t}{K}\right), \tag{3.6}$$

$$D_2=\left(r-q+\frac{\sigma^2}{2}\right), \tag{3.7}$$

$$d=\frac{D_1+D_2\tau}{\sigma\sqrt{\tau}}, \tag{3.8}$$

$$\tau=T-t,\eta(\kappa)=\frac{1}{\sqrt{2\pi}}e^{-\frac{\kappa^2}{2}},\mathcal{N}(\kappa)=\int_{-\infty}^{\kappa}\eta(\kappa)d\kappa. \tag{3.9}$$

Proof. Taking the MT of (3.1) and using its linearity, independence of time derivatives and shifting properties and rearranging terms, one obtains

$$\frac{\partial \tilde{P}_E(\omega,t)}{\partial t}=-\frac{\sigma^2}{2}(\omega^2+\omega(1-\alpha_1)-\alpha_2)\tilde{P}_E(\omega,t) \tag{3.10}$$

where

$$\alpha_1=\frac{2(r-q)}{\sigma^2},\alpha_2=\frac{2r}{\sigma^2}$$

Solving (3.1), yields

$$\tilde{P}_E(\omega,t)=m(\omega)e^{-\frac{1}{2}\sigma^2(\omega^2+\omega(1-\alpha_1)-\alpha_2)t} \tag{3.11}$$

But

$$m(\omega)=\mathcal{M}(P_E(S_T,T),\omega)e^{\frac{1}{2}\sigma^2(\omega^2+(1-\alpha_1)\omega-\alpha_2)T} \tag{3.12}$$

which is equivalent to

$$m(\omega)=\tilde{g}(\omega)e^{\frac{1}{2}\sigma^2(\omega^2+(1-\alpha_1)\omega-\alpha_2)T} \tag{3.13}$$

Substituting (3.13) into (3.11), yields

$$\tilde{P}_E(\omega, t) = \tilde{g}(\omega)e^{\frac{1}{2}\sigma^2(\omega^2+(1-\alpha_1)\omega-\alpha_2)\tau} \tag{3.14}$$

with $\tau = T - t$. By means of (2.2), (3.14) yields

$$P_E(S_t, t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \tilde{g}(\omega)e^{\frac{1}{2}\sigma^2(\omega^2+(1-\alpha_1)\omega-\alpha_2)\tau}S_t^{-\omega}d\omega \tag{3.15}$$

which is the integral equation for governing equation (3.1). Let

$$\xi(S_t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{\sigma^2}{2}(\omega^2+\omega(1-\alpha_1)-\alpha_2)}S_t^{-\omega}d\omega \tag{3.16}$$

Using the fact that

$$e^{\frac{1}{2}\sigma^2(\omega^2+(1-\alpha_1)\omega-\alpha_2)\tau} = e^{-\beta_1(\beta_1^2+\alpha_2)+\beta_1(\omega+\beta_2)^2} \tag{3.17}$$

where

$$\beta_1 = \frac{\sigma^2\tau}{2}, \beta_2 = \frac{1-\alpha_1}{2} \tag{3.18}$$

Thus

$$\xi(S_t) = \frac{e^{-\beta_1(\beta_1^2+\alpha_2)}}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\beta_1(\omega+\beta_2)^2}S_t^{-\omega}d\omega \tag{3.19}$$

Using the transformation given by [19].

$$e^{\phi\omega^2} = \frac{1}{2\sqrt{\pi}}\int_0^\infty \frac{1}{\sqrt{\phi}}\exp\left(\frac{-(\ln S_t)^2}{4\phi}\right)(S_t)^{\omega-1}dS_t, \ \ \Re(\phi) \geq 0 \tag{3.20}$$

yields

$$\xi(S_t) = e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{S^{\beta_2}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{1}{2}\left(\frac{\ln(S)}{\sigma\sqrt{\tau}}\right)^2} \tag{3.21}$$

Similarly,

$$\xi\left(\frac{S_t}{v}\right) = e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{(\frac{S_t}{v})^{\beta_2}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2} \tag{3.22}$$

Using the terminal condition (3.4), then

$$g(S_t) = \mathcal{M}^{-1}(\tilde{g}(\omega)) = \left[S_T\ln\left(\frac{K}{S_T}\right)\right]^+ \tag{3.23}$$

Also

$$g(v) = \left[v\ln\left(\frac{K}{v}\right)\right]^+ \tag{3.24}$$

With the help of the convolution property of MT, (3.15) becomes

$$P_E(S_t, t) = \int_0^\infty g(v)\xi\left(\frac{S_t}{v}\right)\frac{1}{v}dv \tag{3.25}$$

Substituting (3.22) and (3.24) into (3.25), one gets

$$P_E(S_t, t) = \int_0^\infty \left[v\ln\left(\frac{K}{v}\right)\right]^+ e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{(\frac{S_t}{v})^{\beta_2}}{\sigma\sqrt{2\pi\tau}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}\frac{1}{v}dv \tag{3.26}$$

$$P_E(S_t,t)=e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{S_t^{\beta_2}}{\sigma\sqrt{2\pi\tau}}\int_0^K\left[v\ln\left(\frac{K}{v}\right)\right]e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}\frac{1}{v}dv \tag{3.27}$$

Simplifying further, yields

$$\begin{aligned}P_E(S_t,t)&=e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{S_t^{\beta_2}}{\sigma\sqrt{2\pi\tau}}\int_0^K\ln(K)\frac{1}{v^{\beta_2}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}dv\\&-e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{S_t^{\beta_2}}{\sigma\sqrt{2\pi\tau}}\int_0^K\ln(v)\frac{1}{v^{\beta_2}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}dv\end{aligned} \tag{3.28}$$

$$P_E(S_t,t)=e^{-\beta_1(\beta_1^2+\alpha_2)}\frac{S_t^{\beta_2}}{\sigma\sqrt{\tau}}\left[\ln(K)G_1-G_2\right] \tag{3.29}$$

where

$$G_1=\frac{1}{\sqrt{2\pi}}\int_0^K\frac{1}{v^{\beta_2}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}dv \tag{3.30}$$

$$G_2=\frac{1}{\sqrt{2\pi}}\int_0^K\frac{\ln(v)}{v^{\beta_2}}e^{-\frac{1}{2}\left(\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}\right)^2}dv \tag{3.31}$$

Let

$$y=\frac{\ln\left(\frac{S}{v}\right)}{\sigma\sqrt{\tau}} \tag{3.32}$$

Thus

$$G_2=\sigma\sqrt{\tau}S_t^{-\beta_2+1}e^{\beta_1(\beta_2-1)^2}[\sigma\sqrt{\tau}J_1-\ln(S_t)J_2] \tag{3.33}$$

where

$$J_1=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}}ye^{-\frac{1}{2}(y-\sigma\sqrt{\tau}(\beta_2-1))^2}dy \tag{3.34}$$

$$J_2=\frac{1}{\sqrt{2\pi}}\int_{\infty}^{\frac{\ln\left(\frac{S_t}{v}\right)}{\sigma\sqrt{\tau}}}e^{-\frac{1}{2}(y-\sigma\sqrt{\tau}(\beta_2-1))^2}dy \tag{3.35}$$

Taking

$$t=y-\sigma\sqrt{\tau}(\beta_2-1),\, d=\frac{\ln\left(\frac{S_t}{K}\right)-\sigma^2\tau(\beta_2-1)}{\sigma\sqrt{\tau}}=\frac{\ln\left(\frac{S_t}{K}\right)+\left(r-q+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \tag{3.36}$$

Equations (3.34) and (3.35) become

$$J_1=-[\eta(d)+\sigma\sqrt{\tau}(\beta_2-1)\mathcal{N}(-d)] \tag{3.37}$$

and

$$J_2=-\mathcal{N}(-d) \tag{3.38}$$

respectively. Substituting (3.37) and (3.38) into (3.33), yields

$$G_2 = -\sigma\sqrt{\tau}S_t^{-\beta_2+1}e^{\beta_1(\beta_2-1)^2}[\sigma\sqrt{\tau}\eta(d) + (\sigma^2\tau(\beta_2-1) - \ln(S_t))\mathcal{N}(-d)] \tag{3.39}$$

Similarly,

$$G_1 = \sigma\sqrt{\tau}S_t^{-\beta_2+1}e^{\beta_1(\beta_2-1)^2}[\mathcal{N}(-d)] \tag{3.40}$$

Using (3.39), (3.40), the values of α_2, β_1, β_2 and (3.29), the result follows. Hence, this completes the proof □

4. Numerical Example

Consider the valuation of the EPO on a DY with MLPF via the MT using the following parameters S, K, r, σ, q, T in Table 1.

Table 1. The parameters

Parameters	**Values**
S in dollars	100
K in dollars	100, 110, 120, 130, 140, 150
r	8%
σ	0.5
q	$0, 5\%, 20\%, 60\%, 100\%$
T in months	6

The results obtained are displayed in Tables 2 and 3.

Table 2. The comparative study of MLPF, Log Payoff [20] and Linear Payoff [1] with q =0

K	**MLPF**	**Log Payoff [20]**	**Linear Payoff [1]**
100	0	0	0
110	9.5310	0.0953	10
120	18.2322	0.1823	20
130	26.2364	0.2624	30
140	33.6472	0.3365	40
150	40.5465	0.4055	50

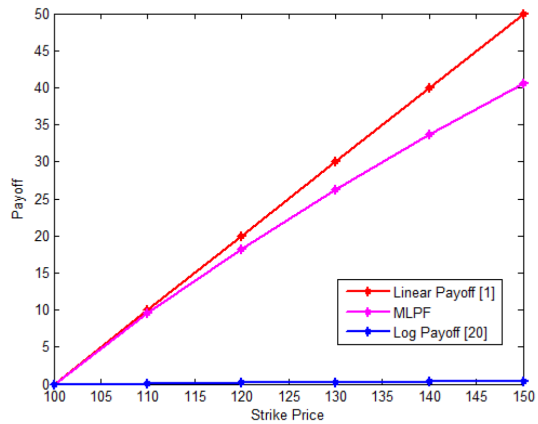

Figure 1. The Plots of Table 2.

Table 3. The Effect of DY on the price of EPO with MLPF

K/q	0	0.05	0.2	0.6	1
100	9.5684	10.3063	12.6497	19.3538	25.5118
110	13.7482	14.6425	17.3986	24.6869	30.6874
120	18.4538	19.4684	22.5140	30.0284	35.6212
130	23.5158	24.6103	27.8197	35.2583	40.2814
140	28.7839	29.9193	33.1776	40.3074	44.6658
150	34.1365	35.2780	38.4899	45.1414	48.7877

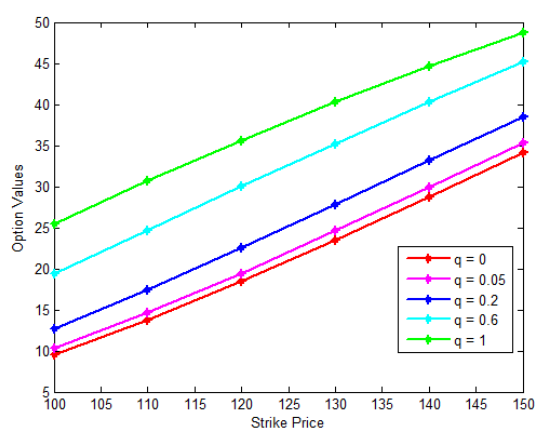

Figure 2. Physical Interpretation of the Effect of Dividend Yield on the Price of EPO using Table 3.

5. Conclusion

A direct solution of BSMEPOM via the celebrated transform of Mellin type in the sense of DY and MLPF has been proposed in this paper. The MT has the ability of handling complex functions by means of its fundamental properties and it is closely related to other well-known transforms such as Laplace and Fourier types. The integral equation for the representation of the price of EPO with DY was obtained. The closed form approximation formula for EPO was also obtained via MT with the help of its convolution property and final time condition. Moreover, the MT was tested on some parameters to show its performance, effectiveness, and suitability. From Table 2, it is clearly seen that the MLPF used in this present paper performed better than the log payoff function used in [20] and also was found to be very close to the linear payoff function of plain vanilla [1]. It is observed from Table 3, that the holder is more beneficial to enter into a European put option. In other words, however, the benefits of these cash flows are given to the holder of a put option. Table 3 shows that increase in DY leads to increase in the prices of the EPO with MLPF. The effect of DY is captured in Figure 2. Hence, from the results displayed in Figures 1 and 2, it can be concluded that MT is suitable for the valuation of EPO on MLPF with DY due to its capacity power of solving BSMEPOM directly in terms of market price.

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