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GENERALIZED NORMS INEQUALITIES FOR ABSOLUTE VALUE OPERATORS

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Abstract. In this article, we generalize some norms inequalities for sums, differences, and products of absolute value operators. Our results based on Minkowski type inequalities and generalized forms of the Cauchy-Schwarz inequality. Some other related inequalities are also discussed.

1. Introduction

In this article, notations are same as in [3], for reader convenience we recall that let H be a complex separable Hilbert space and $B(H)$ denote the C^*-algebra of all bounded linear operators on H. Let $|A|$ denote the absolute value of $A \in B(H)$, and is defined as $|A| = (A^*A)^{\frac{1}{2}}$, where A^* is the adjoint operator of A. If A is compact operator on complex separable Hilbert space H, then the singular values of A enumerated as $s_1(A) \geq s_2(A) \geq ...$ which are the eigenvalues of positive operator $|A|$. A norm $|||.|||$ stand for untarily invariant norm i.e., a norm with the property that $|||UAV||| = |||A|||$ for all A and for all unitary operators U, V in $B(H)$. Operator norm and Schatten p-norms are denoted as $||.||$ and $||.||_P$ respectively. Except the operator norm, which is defined on all of $B(H)$, each unitarily invariant norm is defined on an ideal in $B(H)$. When we use the symbol $|||A|||$ it is implicit understood that operator A is in this ideal.

For $0 < p < 1$, a norm $\|.\|_p$ defines a quasi-norm. For this norm it is well-known that

$$\|A + B\|_p \leq 2^{\frac{1}{p}-1} \left(\|A\|_p + \|B\|_p\right). \tag{1.1}$$

By the definition of the Schatten p-norm, we have

$$\| \mid A \mid^r \|_p \leq \|A\|_{rp}^p, \tag{1.2}$$

where r, p are real numbers. Also, since the singular values of $\mid A \mid^r$ and $\mid A^* \mid^r$ are same, so

$$||| \mid A \mid^r ||| = ||| \mid A^* \mid^r |||. \tag{1.3}$$

The unitarily invariant norms for differences of the absolute values of Hilbert space operators have attracted the attention of several mathematicians. It has been

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proved by K. Shebrawi and H. Albadawi in [3] that if A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and $0 < r \leq 1$. Then

$$\begin{aligned} & ||| \sum_{i=1}^{n} \mid A_i^* X_i B_i + B_i^* X_i A_i \mid^r ||| \\ \leq \quad & 2n^{1-\frac{r}{2}} \sum_{i=1}^{n} ||| \, (A_i^* \mid X_i \mid A_i)^r \, |||^{\frac{1}{2}} ||| \, (B_i^* \mid X_i \mid B_i)^r \, |||^{\frac{1}{2}}, \end{aligned} \tag{1.4}$$

which leads to the following inequality

$$||| \mid A \mid^{2r} - \mid B \mid^{2r} ||| \leq 2^{1-r} ||| \mid A + B \mid^{2r} |||^{\frac{1}{2}} ||| \mid A - B \mid^{2r} |||^{\frac{1}{2}}. \tag{1.5}$$

Inequality (1.5) generalize the result presented by Bhatia in [5] as follows:

$$||| \mid A \mid - \mid B \mid ||| \leq \sqrt{2} ||| \mid A + B \mid |||^{\frac{1}{2}} ||| \mid A - B \mid |||^{\frac{1}{2}}. \tag{1.6}$$

K. Shebrawi and H. Albadawi also proved in [3] that if A, B, X be operators in $B(H)$ such that X is self adjoint operator and $0 < r \leq \frac{1}{2}$, $1 \leq p \leq 2$, then

$$\begin{aligned} & \| \mid A^* X B + B^* X A \mid^r \|_p \\ \leq \quad & 2^{\frac{1}{p} - r + \frac{1}{2}} \| \, (A^* \mid X \mid A)^r \, \|_p^{\frac{1}{2}} \| \, (B^* \mid X \mid B)^r \, \|_p^{\frac{1}{2}}, \end{aligned} \tag{1.7}$$

this leads to the following inequality

$$\| \mid A \mid^{2r} - \mid B \mid^{2r} \|_p \leq 2^{\frac{1}{p} - 2r + \frac{1}{2}} \| \mid A + B \mid^{2r} \|_p^{\frac{1}{2}} \| \mid A - B \mid^{2r} \|_p^{\frac{1}{2}}. \tag{1.8}$$

Inequality (1.8) generalize the following result in [5]

$$\| \mid A \mid - \mid B \mid \|_p \leq 2^{\frac{1}{p} - \frac{1}{2}} \| \mid A + B \mid \|_p^{\frac{1}{2}} \| \mid A - B \mid \|_p^{\frac{1}{2}}, \tag{1.9}$$

where $1 \leq p \leq 2$.

This article we have organized as: In Section 2, we generalize the inequality (1.5) and also we discuss some other related results. In Section 3, we present some Schatten p-norms inequalities, one of which generalize the inequality (1.8).

2. Generalized unitarily invariant norms inequalities for absolute value operators

In this section, we generalize some unitarily invariant norms inequalities for absolute value operators. Our results based on several lemmas. First two lemmas contain norm inequalities of Minkowski type and generalized forms of the Cauchy-Schwarz inequality, see [4] and [2] respectively.

Lemma 2.1. Let $A_i, B_i \in B(H)$, $i = 1, 2, ..., n$. Then

$$n^{\frac{1}{2} - \frac{1}{r}} ||| \sum_{i=1}^{n} \mid A_i + B_i \mid^r |||^{\frac{1}{r}} \leq 2^{\frac{1}{r} - 1} \left(||| \sum_{i=1}^{n} \mid A_i \mid^r |||^{\frac{1}{r}} + ||| \sum_{i=1}^{n} \mid B_i \mid^r |||^{\frac{1}{r}} \right) \tag{2.1}$$

for $0 < r \leq 1$,

$$n^{-\left|\frac{1}{r} - \frac{1}{2}\right|} ||| \sum_{i=1}^{n} \mid A_i + B_i \mid^r |||^{\frac{1}{r}} \leq ||| \sum_{i=1}^{n} \mid A_i^r \mid |||^{\frac{1}{r}} + ||| \sum_{i=1}^{n} \mid B_i \mid^r |||^{\frac{1}{r}} \tag{2.2}$$

for $r \geq 1$, and

$$n^{-(1-\frac{1}{p})/r} ||| \sum_{i=1}^{n} | A_i + B_i |^r |||^{\frac{1}{r}}_{p} \leq ||| \sum_{i=1}^{n} | A_i^r | |||^{\frac{1}{r}}_{p} + ||| \sum_{i=1}^{n} | B_i |^r |||^{\frac{1}{r}}_{p} \tag{2.3}$$

for $1 \leq p, r < \infty$.

Lemma 2.2. For $A, B, X \in B(H)$, for all unitarily invariant norms and for all positive real numbers μ_1, μ_2 and r such that $\mu_1^{-1} + \mu_2^{-1} = 1$, we have

$$||| \, | A^*XB |^r \, ||| \leq ||| \, | AA^*X |^{\frac{\mu_1 r}{2}} \, |||^{\frac{1}{\mu_1}} ||| \, | XBB^* |^{\frac{\mu_2 r}{2}} \, |||^{\frac{1}{\mu_2}}, \tag{2.4}$$

and also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$||| \, | A^*XB |^r \, ||| \leq ||| \left(A^*f^2(| X^* |)A\right)^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| \left(B^*g^2(| X |)B\right)^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}. \tag{2.5}$$

For following two lemmas see [1] and [6, pp. 293, 294].

Lemma 2.3. Let A be a positive operator in $B(H)$. Then for every normalized unitarily invariant norm (i.e.,$|||diag(1, 0, 0, ..., 0)||| = 1$), we have

$$|||A|||^r \leq |||A^r||| \tag{2.6}$$

for $0 \leq r \leq 1$ and

$$|||A^r||| \leq |||A|||^r \tag{2.7}$$

for $r \geq 1$.

Lemma 2.4. Let A and B be a positive operator in $B(H)$. Then

$$|||A^r - B^r||| \leq ||| \, | A - B |^r \, ||| \tag{2.8}$$

for $0 \leq r \leq 1$ and

$$||| \, | A - B |^r \, ||| \leq |||A^r - B^r||| \tag{2.9}$$

for $r \geq 1$.

Last lemma is a consequence of the concavity (convexity) of the function $f(t) = t^r$, $0 \leq r \leq 1$ $(r \geq 1)$.

Lemma 2.5. Let a and b be two positive real numbers

$$(a + b)^r \leq a^r + b^r \tag{2.10}$$

for $0 \leq r \leq 1$ and

$$(a + b)^r \leq 2^{r-1}(a^r + b^r) \tag{2.11}$$

for $r \geq 1$.

Theorem 2.1. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers, such that $\mu_1^{-1}+\mu_2^{-1} = 1$

and $0 < r \leq 1$. Then

$$
\begin{aligned}
& ||| \sum_{i=1}^{n} | A_i^* X_i B_i + B_i^* X_i A_i |^r ||| \\
\leq \quad & 2n^{1-\frac{r}{2}} \sum_{i=1}^{n} ||| \, | A_i A_i^* X_i |^{\frac{\mu_1 r}{2}} \, |||^{\frac{1}{\mu_1}} ||| \, | X_i B_i B_i^* |^{\frac{\mu_2 r}{2}} \, |||^{\frac{1}{\mu_2}}, \qquad (2.12)
\end{aligned}
$$

also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$
\begin{aligned}
& ||| \sum_{i=1}^{n} | A_i^* X_i B_i + B_i^* X_i A_i |^r ||| \\
\leq \quad & 2n^{1-\frac{r}{2}} \sum_{i=1}^{n} ||| \left(A_i^* f^2(| X_i |) A_i \right)^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| \left(B_i^* g^2(| X_i |) B_i \right)^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}.
\end{aligned}
$$

(2.13)

Proof. By applying (2.1), the triangle inequality, (1.3) and (2.4), respectively, we obtain

$$
\begin{aligned}
& ||| \sum_{i=1}^{n} | A_i^* X_i B_i + B_i^* X_i A_i |^r |||^{\frac{1}{r}} \\
\leq \quad & 2^{\frac{1}{r}-1} n^{\frac{1}{r}-\frac{1}{2}} \left(||| \sum_{i=1}^{n} | A_i^* X_i B_i |^r |||^{\frac{1}{r}} + ||| \sum_{i=1}^{n} | B_i^* X_i A_i |^r |||^{\frac{1}{r}} \right) \\
\leq \quad & 2^{\frac{1}{r}-1} n^{\frac{1}{r}-\frac{1}{2}} \left(\left(\sum_{i=1}^{n} ||| \, | A_i^* X_i B_i |^r \, ||| \right)^{\frac{1}{r}} + \left(\sum_{i=1}^{n} ||| \, | B_i^* X_i A_i |^r \, ||| \right)^{\frac{1}{r}} \right) \\
\leq \quad & 2^{\frac{1}{r}} n^{\frac{1}{r}-\frac{1}{2}} \left(\sum_{i=1}^{n} ||| \, | A_i^* X_i B_i |^r \, ||| \right)^{\frac{1}{r}} \\
\leq \quad & 2^{\frac{1}{r}} n^{\frac{1}{r}-\frac{1}{2}} \left(\sum_{i=1}^{n} ||| \, | A_i A_i^* X_i |^{\frac{\mu_1 r}{2}} \, |||^{\frac{1}{\mu_1}} ||| \, | X_i B_i B_i^* |^{\frac{\mu_2 r}{2}} \, |||^{\frac{1}{\mu_2}} \right)^{\frac{1}{r}}.
\end{aligned}
$$

The proof is completed. By applying (2.5) and the proof of the first part of Theorem 2.1, we can obtain (2.13). □

Replace A_i, B_i by $A_i + B_i$, $A_i - B_i$ respectively and also take $f(t) = g(t) = t^{\frac{1}{2}}$ in Theorem 2.1, then, we obtain the following result.

Corollary 2.1. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers, such that $\mu_1^{-1} + \mu_2^{-1} = 1$ and $0 < r \leq 1$. Then

$$
\begin{aligned}
& 2^{r-1} n^{\frac{r}{2}-1} ||| \sum_{i=1}^{n} | A_i^* X_i A_i - B_i^* X_i B_i |^r ||| \\
\leq \quad & \sum_{i=1}^{n} ||| \, | (A_i + B_i)(A_i + B_i)^* X_i |^{\frac{\mu_1 r}{2}} \, |||^{\frac{1}{\mu_1}} ||| \, | X_i (A_i - B_i)(A_i - B_i)^* |^{\frac{\mu_2 r}{2}} \, |||^{\frac{1}{\mu_2}},
\end{aligned}
$$

and

$$2^{r-1}n^{\frac{r}{2}-1}||| \sum_{i=1}^{n} | A_i^* X_i A_i - B_i^* X_i B_i |^r |||$$

$$\leq \sum_{i=1}^{n} ||| ((A_i + B_i)^* | X_i | (A_i + B_i))^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| ((A_i - B_i)^* | X_i | (A_i - B_i))^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}.$$

By similar way applying to the proof of theorem 2.1, based on the inequality (2.2), we can obtain the following result.

Theorem 2.2. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers, such that $\mu_1^{-1}+\mu_2^{-1} = 1$ and $r \geq 1$. Then

$$||| \sum_{i=1}^{n} | A_i^* X_i B_i + B_i^* X_i A_i |^r |||$$

$$\leq 2^r n^{|1-\frac{r}{2}|} \sum_{i=1}^{n} ||| | A_i A_i^* X_i |^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| | X_i B_i B_i^* |^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}, \tag{2.14}$$

also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$||| \sum_{i=1}^{n} | A_i^* X_i B_i + B_i^* X_i A_i |^r |||$$

$$\leq 2^r n^{|1-\frac{r}{2}|} \sum_{i=1}^{n} ||| \left(A_i^* f^2(| X_i |) A_i\right)^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| \left(B_i^* g^2(| X_i |) B_i\right)^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}. \tag{2.15}$$

Corollary 2.2. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers, such that $\mu_1^{-1}+\mu_2^{-1} = 1$ and $r \geq 1$. Then

$$n^{-|1-\frac{r}{2}|} ||| \sum_{i=1}^{n} | A_i^* X_i A_i - B_i^* X_i B_i |^r |||$$

$$\leq \sum_{i=1}^{n} ||| | (A_i + B_i)(A_i + B_i)^* X_i |^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| | X_i (A_i - B_i)(A_i - B_i)^* |^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}},$$

and

$$n^{-|1-\frac{r}{2}|} ||| \sum_{i=1}^{n} | A_i^* X_i A_i - B_i^* X_i B_i |^r |||$$

$$\leq \sum_{i=1}^{n} ||| ((A_i + B_i)^* | X_i | (A_i + B_i))^{\frac{\mu_1 r}{2}} |||^{\frac{1}{\mu_1}} ||| ((A_i - B_i)^* | X_i | (A_i - B_i))^{\frac{\mu_2 r}{2}} |||^{\frac{1}{\mu_2}}.$$

Remark 2.1. If we take $f(t) = t^{\alpha}$ and $g(t) = t^{(1-\alpha)}$ for $\alpha \in [0, 1]$, then from the inequality (2.13) we can obtain important special case. Also, if we take $f(t) =$

$g(t) = t^{\frac{1}{2}}$ then from (2.13) we have

$$\begin{aligned} & ||| \sum_{i=1}^{n} \mid A_i^* X_i B_i + B_i^* X_i A_i \mid^r ||| \\ \leq \quad & 2n^{1-\frac{r}{2}} \sum_{i=1}^{n} |||\, (A_i^* \mid X_i \mid A_i)^{\frac{\mu_1 r}{2}} \,|||^{\frac{1}{\mu_1}} |||\, (B_i^* \mid X_i \mid B_i)^{\frac{\mu_2 r}{2}} \,|||^{\frac{1}{\mu_2}}, \end{aligned}$$

which is the more general form of the inequality (1.4). Similar remark we can give for the inequality (2.15), which is more general form of the inequality (2.19) in [3].

Our following result contains a promised generalization of (1.5).

Corollary 2.3. Let A and B be operators in $B(H)$ and μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$ then

$$||| \mid A \mid^{2r} - \mid B \mid^{2r} ||| \leq 2^{1-r} ||| \mid A + B \mid^{\mu_1 r} |||^{\frac{1}{\mu_1}} ||| \mid A - B \mid^{\mu_2 r} |||^{\frac{1}{\mu_2}} \tag{2.16}$$

for $0 < r \leq 1$, and

$$||| \mid A^* A - B^* B \mid^r ||| \leq ||| \mid A + B \mid^{\mu_1 r} |||^{\frac{1}{\mu_1}} ||| \mid A - B \mid^{\mu_2 r} |||^{\frac{1}{\mu_2}} \tag{2.17}$$

for $r \geq 1$.

Proof. By (2.8) and from second result of Corollary (2.1), we have

$$\begin{aligned} ||| \mid A \mid^{2r} - \mid B \mid^{2r} ||| &\leq ||| \mid\mid A \mid^2 - \mid B \mid^2 \mid^r ||| \\ &\leq 2^{1-r} ||| \mid A + B \mid^{\mu_1 r} |||^{\frac{1}{\mu_1}} ||| \mid A - B \mid^{\mu_2 r} |||^{\frac{1}{\mu_2}}, \end{aligned}$$

for $0 < r \leq 1$. and inequality (2.17) is a special case of the first result of Corollary (2.2).

Remark 2.2. Special cases of second results in Corollary 2.1 and 2.2 respectively are: Let A, B, X be operators in $B(H)$ such that X is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$\begin{aligned} & ||| \mid A^* X A - B^* X B \mid^r ||| \\ \leq \quad & 2^{1-r} |||\, ((A+B)^* \mid X \mid (A+B))^{\frac{\mu_1 r}{2}} \,|||^{\frac{1}{\mu_1}} |||\, ((A-B)^* \mid X \mid (A-B))^{\frac{\mu_2 r}{2}} \,|||^{\frac{1}{\mu_2}}. \end{aligned}$$

for $0 < r \leq 1$, and

$$\begin{aligned} & ||| \mid A^* X A - B^* X B \mid^r ||| \\ \leq \quad & |||\, ((A+B)^* \mid X \mid (A+B))^{\frac{\mu_1 r}{2}} \,|||^{\frac{1}{\mu_1}} |||\, ((A-B)^* \mid X \mid (A-B))^{\frac{\mu_2 r}{2}} \,|||^{\frac{1}{\mu_2}}. \end{aligned}$$

for $r \geq 1$.

Corollary 2.4. Let A be an operator in $B(H)$ and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$||| \mid A^* A - A A^* \mid^r ||| \leq 2^{1+r} ||| \mid ReA \mid^{\mu_1 r} |||^{\frac{1}{\mu_1}} ||| \mid ImA \mid^{\mu_2 r} |||^{\frac{1}{\mu_2}},$$

for $0 < r \leq 1$, and

$$||| \mid A^* A - A A^* \mid^r ||| \leq 2^{2r} ||| \mid ReA \mid^{\mu_1 r} |||^{\frac{1}{\mu_1}} ||| \mid ImA \mid^{\mu_2 r} |||^{\frac{1}{\mu_2}},$$

for $r \geq 1$.

3. Generalized norm inequalities for the Schatten p-norm

Schatten p-norm for absolute value operators are discussed in this section. Our these results refine some of the results in Section 2 and also, our first result leads to a generalization of (1.8).

Theorem 3.1. Let A, B, X be operators in $B(H)$ such that X is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$\begin{aligned} & \| \mid A^*XB + B^*XA \mid^r \|_p \\ \leq \; & 2^{\frac{1}{p}-r+\frac{1}{2}} \| \mid AA^*X \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid XBB^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned} \tag{3.1}$$

also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$\begin{aligned} & \| \mid A^*XB + B^*XA \mid^r \|_p \\ \leq \; & 2^{\frac{1}{p}-r+\frac{1}{2}} \| \left(A^* f^2(\mid X \mid) A\right)^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \left(B^* g^2(\mid X \mid) B\right)^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}. \end{aligned} \tag{3.2}$$

for $0 < r \leq \frac{1}{2}$ and $1 \leq p \leq 2$.

Proof. By applying (1.2), (1.1), (2.10), (1.3) and (2.4) respectively, we obtain

$$\begin{aligned} \| \mid A^*XB + B^*XA \mid^r \|_p &= \|A^*XB + B^*XA\|_{rp}^r \\ &\leq \left(2^{\frac{1}{rp}-1} \left(\|A^*XB\|_{rp} + \|B^*XA\|_{rp}\right)\right)^r \\ &\leq 2^{\frac{1}{p}-r} \left(\|A^*XB\|_{rp}^{2r} + \|B^*XA\|_{rp}^{2r}\right)^{\frac{1}{2}} \\ &\leq 2^{\frac{1}{p}-r} \left(\| \mid A^*XB \mid^r \|_p^2 + \| \mid B^*XA \mid^r \|_p^2\right)^{\frac{1}{2}} \\ &= 2^{\frac{1}{p}-r+\frac{1}{2}} \| \mid A^*XB \mid^r \|_p \\ &\leq 2^{\frac{1}{p}-r+\frac{1}{2}} \| \mid AA^*X \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid XBB^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}. \end{aligned}$$

The proof is completed. By applying (2.5) and the proof of the first part of Theorem 3.1, we can obtain (3.2). □

Corollary 3.1. Let A, B, X be operators in $B(H)$ such that X is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$\begin{aligned} & \| \mid A^*XA - B^*XB \mid^r \|_p \\ \leq \; & 2^{\frac{1}{p}-2r+\frac{1}{2}} \| \mid (A+B)(A+B)^*X \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid X(A-B)(A-B)^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned}$$

and

$$\begin{aligned} & \| \mid A^*XA - B^*XB \mid^r \|_p \\ \leq \; & 2^{\frac{1}{p}-2r+\frac{1}{2}} \| \left((A+B)^* \mid X \mid (A+B)\right)^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \left((A-B)^* \mid X \mid (A-B)\right)^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned}$$

for $0 < r \leq \frac{1}{2}$ and $1 \leq p \leq 2$.

Remark 3.1. By using (2.8) and from second inequality in Corollary (3.1), we can obtain

$$\begin{aligned} \| \mid A \mid^{2r} + \mid B \mid^{2r} \|_p &\leq \| \| A \mid^2 - \mid B \mid^2 \mid^r \|_p \\ &\leq 2^{\frac{1}{p}-2r+\frac{1}{2}} \| \mid A+B \mid^{\mu_1 r} \| \|_p^{\frac{1}{\mu_1}} \| \mid A-B \mid^{\mu_2 r} \|_p^{\frac{1}{\mu_2}}, \end{aligned}$$

which is the generalized form of the inequality (1.8).

Similarly to the proof of Theorem 3.1, based on (2.11), we can obtain the following result.

Theorem 3.2. Let A, B, X be operators in $B(H)$ such that X is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$\begin{aligned} & \| \mid A^*XB + B^*XA \mid^r \|_p \\ \le\ & 2^r \| \mid AA^*X \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid XBB^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned} \tag{3.3}$$

also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$\begin{aligned} & \| \mid A^*XB + B^*XA \mid^r \|_p \\ \le\ & 2^r \| \left(A^* f^2(\mid X \mid)A\right)^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \left(B^* g^2(\mid X \mid)B\right)^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned} \tag{3.4}$$

for $r \ge \frac{1}{2}$ and $p \ge 2$.

Corollary 3.2. Let A, B, X be operators in $B(H)$ such that X is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, then

$$\begin{aligned} & \| \mid A^*XA - B^*XB \mid^r \|_p \\ \le\ & \| \mid (A+B)(A+B)^*X \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid X(A-B)(A-B)^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned}$$

and

$$\begin{aligned} & \| \mid A^*XA - B^*XB \mid^r \|_p \\ \le\ & \| \left((A+B)^* \mid X \mid (A+B)\right)^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \left((A-B)^* \mid X \mid (A-B)\right)^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned}$$

for $r \ge \frac{1}{2}$ and $p \ge 2$.

Similarly to the proof of Theorem 3.1, based on (2.3), we can also obtain the following result.

Theorem 3.3. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1} + \mu_2^{-1} = 1$, Then

$$\begin{aligned} & \| \sum_{i=1}^{n} \mid A_i^* X_i B_i + B_i^* X_i A_i \mid^r \|_p \\ \le\ & 2^r n^{1-\frac{1}{p}} \sum_{i=1}^{n} \| \mid A_i A_i^* X_i \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid X_i B_i B_i^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned} \tag{3.5}$$

also, if f and g are nonnegative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$, for all $t \in [0, \infty)$, then we have

$$\begin{aligned} & \| \sum_{i=1}^{n} \mid A_i^* X_i B_i + B_i^* X_i A_i \mid^r \|_p \\ \le\ & 2^r n^{1-\frac{1}{p}} \sum_{i=1}^{n} \| \left(A_i^* f^2(\mid X_i \mid)A_i\right)^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \left(B_i^* g^2(\mid X_i \mid)B_i\right)^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}}, \end{aligned} \tag{3.6}$$

for $r, p \geq 1$.

Corollary 3.3. Let A_i, B_i, X_i $(i = 1, 2, ..., n)$ be operators in $B(H)$ such that X_i is self adjoint operator and if μ_1, μ_2 are positive real numbers such that $\mu_1^{-1}+\mu_2^{-1} = 1$, then

$$n^{\frac{1}{p}-1} \| \sum_{i=1}^{n} \mid A_i^* X_i A_i - B_i^* X_i B_i \mid^r \|_p$$

$$\leq \sum_{i=1}^{n} \||\mid (A_i + B_i)(A_i + B_i)^* X_i \mid^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| \mid X_i (A_i - B_i)(A_i - B_i)^* \mid^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}},$$

and

$$n^{\frac{1}{p}-1} \| \sum_{i=1}^{n} \mid A_i^* X_i A_i - B_i^* X_i B_i \mid^r \|_p$$

$$\leq \sum_{i=1}^{n} \| ((A_i + B_i)^* \mid X_i \mid (A_i + B_i))^{\frac{\mu_1 r}{2}} \|_p^{\frac{1}{\mu_1}} \| ((A_i - B_i)^* \mid X_i \mid (A_i - B_i))^{\frac{\mu_2 r}{2}} \|_p^{\frac{1}{\mu_2}},$$

for $r, p \geq 1$.

Remark 3.2. For $r \leq 2$ and $p \leq \frac{2}{r}$ or $r \geq 2$ and $p(4 - r) \leq 2$, the results in Corollary 3.3 refine the results in Corollary 2.2 respectively.

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