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Generalized Ulam-Hyers Stability Results of a Quadratic Functional Equation in Felbin's Type Fuzzy Normed Linear Spaces

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Abstract. This paper presents the generalized Ulam-Hyers stability of the following quadratic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right)=\frac{3}{4}\left(f(z-x)+f(z-y)+f(x-y)\right)$$

in Felbin's type fuzzy normed linear spaces (f-NLS) using direct and fixed point methods.

1. Introduction

In 1940, S.M. Ulam [37] posed the stability problem for approximate homomorphisms. In 1941, D.H. Hyers [12] provided a partial solution to Ulam's problem for mappings between Banach spaces. In 1950, T. Aoki [2] generalized Hyers' Theorem for additive mappings. In 1978, Th.M.Rassias [28] proved a further generalzation of Hyers' Theorem by introducing the concept of the unbounded Cauchy difference for the sum of powers of two $p-$norms. During the last three decades the stability theorem of Th.M. Rassias [28] provided a lot of influence for the development of stability theory of a large variety of functional equations. This new concept is known today with the term Hyers-Ulam-Rassias

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stability for functional equations. Following the innovative approach of Th.M. Rassias similar theorems were formulated and proved by a number of mathematicians. For example four years later in 1982, J.M. Rassias [27] proved a similar theorem for the case "the unbounded Cauchy difference" is the "product of two $p-$norms".

In 1994, the above stability results were further extended by P. Gavruta [9] who considered a more general control function in the real variables x, y for the unbounded Cauchy difference in the spirit of Th.M. Rassias' stability approach. In 2008, a special case of Gavruta's theorem for the unbounded Cauchy difference was obtained in [31] by considereing the summation of both the sum and product of two $p-$norms.

The functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y) \tag{1.1}$$

is said to be quadratic functional equation because the quadratic function $f(x)=ax^2$ is a solution of the functional equation (1.1).

This paper presents the generalized Ulam-Hyers stability of the following quadratic functional equation

$$f\left(\frac{x+y}{2}-z\right)+f\left(\frac{y+z}{2}-x\right)+f\left(\frac{z+x}{2}-y\right) = \frac{3}{4}\left(f(z-x)+f(z-y)+f(x-y)\right) \tag{1.2}$$

in Felbin's fuzzy normed linear spaces (f-NLS) using direct and fixed point methods

2. Some preliminaries on fuzzy real number

This section, some preliminaries in the theory of fuzzy real numbers are given. Furthermore, we give some definition which help to investigate the stability in Felbin's type normed linear spaces.

In [10] Grantner takes the fuzzy real number as a decreasing mapping from the real line to the unit interval or lattice in general. Lowen [23] applies the fuzzy real numbers as non-decreasing, left continuous mapping from the real line to the unit interval so that its supremum over $\mathbb{R}$ is 1. Also fuzzy arithmetic operations on $L-$fuzzy real line were studied by Rodabaugh [34], where he showed that the binary addition is the only extension of addition to $\mathbb{R}((L))$.

Hoehle [13] especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut. In this paper a fuzzy real number is taken as a fuzzy normal and convex mapping from the real line to the unit interval. The concept of the fuzzy metric space has been studied by Kaleva [19, 20] by using fuzzy number as a fuzzy set on the real axis. Kaleva also has recently showed that a fuzzy metric space can be embedded in a complete fuzzy metric space [21].

In [8], Felbin introduced the concept of fuzzy normed linear space (f-NLS); Xiao and Zhu [38] studied its linear topological structures and some basic properties of a fuzzy normed linear space. It is known that theories of classical normed space and Menger probabilistic normed spaces are special cases of fuzzy normed linear spaces.

Let η be a fuzzy subset on $\mathbb{R}$, i.e., a mapping $\eta : \mathbb{R} \to [0,1]$ associating with each real number t its grade of membership ηt.

Definition 2.1. *[8] A fuzzy subset η on $\mathbb{R}$ is called a fuzzy real number, whose $\alpha-$level set is denoted by $[\eta]_\alpha$*

$$i.e., [\eta]_\alpha = \{t : \eta(t) \geq \alpha\},$$

if it satisfies two axioms:

- *$(N1)$ There exists $t_0 \in \mathbb{R}$ such that $\eta(t_0) = 1$.*
- *$(N2)$ For each $\alpha \in (0,1]$, $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$ where $-\infty < \eta^-_\alpha \leq \eta^+_\alpha < +\infty$.*

The set of all fuzzy real number is denoted by $F(\mathbb{R})$. If $\eta \in F(\mathbb{R})$ and $\eta(t) = 0$ whenever $t < 0$, then η is called a non-negative fuzzy real number and $F^*(\mathbb{R})$ denotes the set of all non-negative fuzzy real numbers.

The number $\bar{0}$ stands for the fuzzy real number as:

$$\bar{0} = \begin{cases} t, & t = 0, \\ 0, & t \neq 0. \end{cases}$$

Clearly, $\bar{0} \in F^*(\mathbb{R})$. Also the set of all real numbers can be embedded in $F(\mathbb{R})$ because if $r \in (-\infty, \infty)$, then $\bar{r} \in F(\mathbb{R})$ satisfies $\bar{r}(t) = \bar{0}(t - r)$.

Definition 2.2. *[8] Fuzzy arithmetic operations $\oplus, \ominus, \otimes, \oslash$ on $F(\mathbb{R}) \times F(\mathbb{R})$ can be defined as:*

1. $(\eta \oplus \delta)(t) = \sup\limits_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(t - s)\}, t \in \mathbb{R}$,
2. $(\eta \ominus \delta)(t) = \sup\limits_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(s - t)\}, t \in \mathbb{R}$,
3. $(\eta \otimes \delta)(t) = \sup\limits_{s\in\mathbb{R}}\{\eta(s) \wedge \delta(t/s)\}, t \in \mathbb{R}$,
4. $(\eta \oslash \delta)(t) = \sup\limits_{s\in\mathbb{R}}\{\eta(st) \wedge \delta(s)\}, t \in \mathbb{R}$.

The additive and multiplicative identities in $F(\mathbb{R})$ are $\bar{0}$ and $\bar{1}$, respectively. Let $\ominus\eta$ be defined as $\bar{0} \ominus \eta$. It is clear that $\eta \ominus \delta = \eta \oplus (\ominus\delta)$.

Definition 2.3. *[8] For $k, 0 \in \mathbb{R}$, fuzzy scalar multiplication $k \odot \eta$ is defined as $(k \odot \eta)(t) = \eta(t/k)$ and $0 \odot \eta$ is defined to be 0.*

Lemma 2.1. *Let η, δ be fuzzy real numbers. Then*

$$\forall\ t \in \mathbb{R},\ \eta(t) = \delta(t)\ \Leftrightarrow\ \forall\ \alpha \in (0,1], [\eta]_\alpha = [\delta]_\alpha.$$

Lemma 2.2. *Let $\eta, \delta \in F(\mathbb{R})$ and $[\eta]_\alpha = [\eta^-_\alpha, \eta^+_\alpha]$, $[\delta]_\alpha = [\delta^-_\alpha, \delta^+_\alpha]$. Then*

- *(i) $[\eta \oplus \delta]_\alpha = [\eta^-_\alpha + \delta^-_\alpha, \eta^+_\alpha + \delta^+_\alpha]$,*

(ii) $[\eta \ominus \delta]_\alpha = [\eta^-_\alpha - \delta^-_\alpha, \eta^+_\alpha - \delta^+_\alpha]$,

(iii) $[\eta \otimes \delta]_\alpha = [\eta^-_\alpha \delta^-_\alpha, \eta^+_\alpha \delta^+_\alpha], \eta, \delta \in F^*(\mathbb{R})$,

(iv) $[\bar{1} \oslash \delta]_\alpha = [1/\delta^+_\alpha, 1/\delta^-_\alpha], \delta^-_\alpha > 0$.

Definition 2.4. *[8] Define a partial ordering $\preceq$ in $F(\mathbb{R})$ by $\eta \preceq \delta$ if and only if $\eta^-_\alpha \leq \delta^-_\alpha$ and $\eta^+_\alpha \leq \delta^+_\alpha$ for all $\alpha \in (0, 1]$. The strict inequality in $F(\mathbb{R})$ is defined by $\eta \prec \delta$ if and only if $\eta^-_\alpha < \delta^-_\alpha$ and $\eta^+_\alpha < \delta^+_\alpha$ for all $\alpha \in (0, 1]$.*

Definition 2.5. *[38] Let X be a real linear space, L and $\mathbb{R}$ (respectively, left norm and right norm) be symmetric and non-decreasing mappings from $[0, 1] \times [0, 1] \to [0, 1]$ satisfying $L(0, 0) = 0, \mathbb{R}(1, 1) = 1$. Then $||\cdot||$ is called a fuzzy norm and $(X, ||\cdot||, L, \mathbb{R})$ is a fuzzy normed linear space (abbreviated to f-NLS) if the mapping $||\cdot|| : X \to F^*(R)$ satisfies the following axioms, where $[||x||]_\alpha = [||x||^-_\alpha, ||x||^+_\alpha]$ for $x \in X$ and $\alpha \in (0, 1]$:*

(A1) $||x|| = 0$ *if and only if* $x = 0$,

(A2) $||rx|| = |r| \odot ||x||$ *for all* $x \in X$ *and* $r \in (-\infty, \infty)$,

(A3) *For all* $x, y \in X$,

(A3L) *if* $s \leq ||x||^-_1$, $t \leq ||y||^-_1$ *and* $s + t \leq ||x + y||^-_1$, *then* $||x + y||(s + t) \geq L(||x||(s), ||y||(t))$,

(A3R) *if* $s \geq ||x||^-_1$, $t \geq ||y||^-_1$ *and* $s + t \geq ||x + y||^-_1$, *then* $||x + y||(s + t) \leq L(||x||(s), ||y||(t))$,

Lemma 2.3. *[39] Let $(X, ||\cdot||, L, R)$ be an f-NLS, and suppose that*

(R1) $R(a, b) \leq \max(a, b)$,

(R2) $\forall\ \alpha \in (0, 1]$, $\exists \beta \in (0, \alpha]$ *such that* $R(\beta, y) \leq \alpha$ *for all* $y \in (0, \alpha)$,

(R3) $\lim_{a \to 0^+} R(a, a) = 0$.

Then $(R1) \Rightarrow (R2) \Rightarrow (R3)$ *but not conversely.*

Lemma 2.4. *[39] Let $(X, ||\cdot||, L, R)$ be an f-NLS. Then we have the following:*

(A) *If* $R(a, b) \leq \max(a, b)$, *then* $\forall\ \alpha \in (0, 1], ||x + y||^+_\alpha \leq ||x||^+_\alpha + ||y||^+_\alpha$ *for all* $x, y \in X$.

(B) *If* $(R2)$ *then for each* $\alpha \in (0, 1]$ *there is* $\beta \in (0, \alpha]$ *such that* $||x + y||^+_\alpha \leq ||x||^+_\beta + ||y||^+_\alpha$ *for all* $x, y \in X$.

(C) *If* $\lim_{a \to 0^+} R(a, a) = 0$, *then for each* $\alpha \in (0, 1]$ *there is* $\beta \in (0, \alpha]$ *such that* $||x + y||^+_\alpha \leq ||x||^+_\beta + ||y||^+_\beta$ *for all* $x, y \in X$.

Lemma 2.5. *[39] Let $(X, ||\cdot||, L, R)$ be an f-NLS, and suppose that*

(L1) $L(a, b) \geq \min(a, b)$,

(L2) $\forall\ \alpha \in (0, 1]$, $\exists \beta \in [\alpha, 1]$ *such that* $L(\beta, \gamma) \geq \alpha$ *for all* $\gamma \in [\alpha, 1)$,

(L3) $\lim_{a \to 1^-} L(a, a) = 1$.

Then $(L1) \Rightarrow (L2) \Rightarrow (L3)$.

Lemma 2.6. *[39] Let $(X, ||\cdot||, L, R)$ be an f-NLS. Then we have the following:*

(A) *If* $L(a,b) \geq \min(a,b)$*, then* $\forall\ \alpha \in (0,1], ||x+y||_\alpha^- \leq ||x||_\alpha^- + ||y||_\alpha^-$ *for all* $x, y \in X$.

(B) *If* $(L2)$ *then for each* $\alpha \in (0,1]$ *there is* $\beta \in [\alpha, 1]$ *such that* $||x+y||_\alpha^- \leq ||x||_\beta^- + ||y||_\alpha^-$ *for all* $x, y \in X$.

(C) *If* $\lim\limits_{a \to 1^-} R(a,a) = 1$*, then for each* $\alpha \in (0,1]$ *there is* $\beta \in [\alpha, 1]$ *such that* $||x+y||_\alpha^- \leq ||x||_\beta^- + ||y||_\beta^-$ *for all* $x, y \in X$.

Lemma 2.7. *[39] Let* $(X, ||\cdot||, L, R)$ *be an f-NLS.*

(a) *If* $R(a,b) \geq \max(a,b)$ *and* $\forall \alpha \in (0,1]$, $||x+y||_\alpha^+ \leq ||x||_\alpha^+ + ||y||_\alpha^+$ *for all* $x, y \in X$ *then* $(A3R)$.

(b) *If* $L(a,b) \leq \min(a,b)$ *and* $\forall \alpha \in (0,1]$ $||x+y||_\alpha^- \leq ||x||_\alpha^- + ||y||_\alpha^-$ *for all* $x, y \in X$ *then* $(A3L)$.

Theorem 2.1. *[35] Let* $(X, ||\cdot||, L, R)$ *be an f-NLS and* $\lim\limits_{a \to 0^+} R(a,a) = 0$*. Then* $(X, ||\cdot||, L, R)$ *is a Hausdorff topological vector space, whose neighborhood base of origin is* $\{N(\epsilon, \alpha) : \epsilon > 0, \alpha \in (0,1]\}$*, where* $N(\epsilon, \alpha) = \{x : ||x||_\alpha^+ \leq \epsilon\}$.

Definition 2.6. *Let* $(X, ||\cdot||, L, R)$ *be an f-NLS. A sequence* $\{x_n\}_{n=1}^\infty \subseteq X$ *converges to* $x \in X$*, if* $\lim\limits_{n \to \infty} \|x_n - x\|_\alpha^+$*, for every* $\alpha \in (0,1]$ *denoted by* $\lim\limits_{n \to \infty} x_n = x$.

Definition 2.7. *Let* $(X, ||\cdot||, L, R)$ *be an f-NLS. A sequence* $\{x_n\}_{n=1}^\infty \subseteq X$ *is called a Cauchy sequence if* $\lim\limits_{m,n \to \infty} \|x_m - x_n\|_\alpha^+ = 0$ *for every* $\alpha \in (0,1]$.

Definition 2.8. *Let* $(X, ||\cdot||, L, R)$ *be an f-NLS. A subset* $A \subseteq X$ *is said to be complete if every Cauchy sequence in A, converges in A. The fuzzy normed space* $(X, ||\cdot||, L, R)$ *is said to be a fuzzy Banach space (f-BS) if it is complete.*

3. Felbin's stability results: Direct method

The generalized Ulam-Hyers stability of the quadratic functional equation is discussed in this section.

Through out this section, let U be a normed space and V be a Banach space respectively. Define a mapping $F : U^3 \to V$ by

$$\begin{aligned} F(x,y,z) =& f\left(\frac{x+y}{2} - z\right) + f\left(\frac{y+z}{2} - x\right) + f\left(\frac{z+x}{2} - y\right) \\ & - \frac{3}{4}\left(f(z-x) + f(z-y) + f(x-y)\right) \end{aligned}$$

for all $x, y, z \in U$.

Theorem 3.1. *Let* $j \in \{-1, 1\}$*. Let* $\xi : U^3 \to F^*(R)$ *be a function such that*

$$\sum_{n=0}^{\infty} \frac{\xi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)_\alpha^+}{2^{2nj}} \ converges\ and \lim_{n \to \infty} \frac{\xi\left(2^{nj}x, 2^{nj}y, 2^{nj}z\right)_\alpha^+}{2^{2nj}} < \infty \tag{3.1}$$

for all $x, y, z \in U$ *and let* $F : U^3 \to V$ *be an even function satisfying the inequality*

$$\|F(x,y,z)\|_\alpha^+ \preceq \xi(x,y,z)_\alpha^+ \tag{3.2}$$

for all $x, y, z \in U$. Then there exists a unique quadratic function $Q : U^3 \to V$ such that

$$\|F(x) - Q(x)\|_{\alpha}^{+} \preceq \frac{1}{2^2} \odot \sum_{i=\frac{1-j}{2}}^{\infty} \frac{\xi\left(2^{ij}x, 2^{ij}x, -2^{ij}x\right)_{\alpha}^{+}}{2^{2ij}} \tag{3.3}$$

for all $x \in U$. The mapping $Q(x)$ is defined by

$$Q(x) = \lim_{n\to\infty} \frac{f(2^{nj}x)}{2^{2nj}} \tag{3.4}$$

for all $x \in U$.

Proof. Assume $j = 1$.

Replacing (x, y, z) by $(x, x, -x)$ and dividing by 2^2 in (1.2), we get

$$\left\|\frac{f(2x)}{2^2} - f(x)\right\|_{\alpha}^{+} \preceq \frac{1}{2^2} \odot \xi\,(x, x, -x)_{\alpha}^{+} \tag{3.5}$$

for all $x \in U$. Replacing x by $2x$ in (3.5) and divided by 2^2, we get

$$\left\|\frac{f(2^2x)}{2^4} - \frac{f(2x)}{2^2}\right\|_{\alpha}^{+} \preceq \frac{1}{2^4} \odot \xi\,(2x, 2x, -2x)_{\alpha}^{+} \tag{3.6}$$

for all $x \in U$. Combining (3.5) and (3.6), we obtain

$$\left\|\frac{f(2^2x)}{2^4} - f(x)\right\|_{\alpha}^{+} \preceq \frac{1}{2^2} \odot \left(\xi\,(x, x, -x)_{\alpha}^{+} + \frac{1}{2^2} \odot \xi\,(2x, 2x, -2x)_{\alpha}^{+}\right) \tag{3.7}$$

for all $x \in U$. Using induction on a positive integer n, we obtain that

$$\begin{aligned}\left\|\frac{f(2^nx)}{2^{2n}} - f(x)\right\|_{\alpha}^{+} &\preceq \frac{1}{2^2} \odot \sum_{i=0}^{n-1} \frac{\xi\left(2^ix, 2^ix, -2^ix\right)_{\alpha}^{+}}{2^{2i}} \\ &\preceq \frac{1}{2^2} \odot \sum_{i=0}^{\infty} \frac{\xi\left(2^ix, 2^ix, -2^ix\right)_{\alpha}^{+}}{2^{2i}}\end{aligned} \tag{3.8}$$

for all $x \in U$. In order to prove the convergence of the sequence $\left\{\frac{f(2^nx)}{2^{2n}}\right\}$, replace x by 2^mx and divided by 2^{2m} in (3.8), for any $m, n > 0$, we arrive

$$\begin{aligned}\left\|\frac{f(2^n2^mx)}{2^{2n+2m}} - \frac{f(2^mx)}{2^{2m}}\right\|_{\alpha}^{+} &= \frac{1}{2^{2m}} \odot \left\|\frac{f(2^n2^mx)}{2^{2nk}} - f(2^mx)\right\|_{\alpha}^{+} \\ &\preceq \frac{1}{2^2} \odot \sum_{i=0}^{n-1} \frac{\xi\left(2^{i+m}x, 2^{i+m}x, -2^{i+m}x\right)_{\alpha}^{+}}{2^{2(i+m)}} \\ &\preceq \frac{1}{2^2} \odot \sum_{i=0}^{\infty} \frac{\xi\left(2^{i+m}x, 2^{i+m}x, -2^{i+m}x\right)_{\alpha}^{+}}{2^{2(i+m)}}\end{aligned} \tag{3.9}$$

for all $x \in U$. Since the right hand side of the inequality (3.9) tends to 0 as $m \to \infty$, the sequence $\left\{\frac{f(2^nx)}{2^{2n}}\right\}$ is Cauchy sequence. Since Y is complete, there exists a mapping $Q : U \to Y$ such that

$$Q(x) = \lim_{n\to\infty} \frac{f(2^nx)}{2^{2n}}, \quad \forall\ x \in U.$$

Letting $n \to \infty$ in (3.8), we see that (3.3) holds for all $x \in U$. Now we need to prove Q satisfies (1.2), replacing (x, y, z) by $(2^n x, 2^n y, 2^n z)$ and divided by 2^{2n} in (3.2), we arrive

$$\frac{1}{2^{2n}} \odot \|Df(2^n x, 2^n y, 2^n z)\|_\alpha^+ \preceq \frac{1}{2^{2n}} \odot \xi(2^n x, 2^n y, 2^n z)_\alpha^+$$

for all $x, y, z \in U$. Hence we arrive

$$\|Q(2^n x, 2^n y, 2^n z)\|_\alpha^+ = 0.$$

Hence Q satisfies (1.2) for all $x, y, z \in U$. In order to prove Q is unique, let $Q'(x)$ be another quadratic mapping satisfying (3.3) and (1.2). Then

$$\begin{aligned}
\left\|Q(x) - Q'(x)\right\|_\alpha^+ &= \frac{1}{2^{2n}} \odot \left\|Q(2^n x) - Q'(2^n x)\right\|_\alpha^+ \\
&\preceq \frac{1}{2^{2n}} \odot \left\{\|Q(2^n x) - f(2^n x)\|_\alpha^+ + \left\|f(2^n x) - Q'(2^n x)\right\|\right\} \\
&\preceq \frac{2}{2^2} \odot \sum_{i=0}^{\infty} \frac{\xi(2^{n+i}x, 2^{n+i}x, -2^{n+i}x)_\alpha^+}{2^{2(n+i)}} \\
&\to 0 \quad as \quad n \to \infty
\end{aligned}$$

for all $x \in U$. Hence Q is unique.

For $j = -1$, we can prove the similar stability result. Hence it completes the proof. □

The following corollary is an immediate consequence of Theorem 3.1 concerning the Ulam-Hyers, Hyers-Ulam-Rassias, and Rassias stabilities of (1.2).

Corollary 3.1. *Let λ and s be nonnegative real numbers. If an even function $f : U \to Y$ satisfies the inequality*

$$\|F(x, y, z)\|_\alpha^+ \preceq \begin{cases} \lambda, & \\ \lambda \otimes (||x||^s \oplus ||y||^s \oplus ||z||^s), & s \neq 2; \\ \lambda \otimes (||x||^s \otimes ||y||^s \otimes ||z||^s), & s \neq \frac{2}{3}; \\ \lambda \otimes \left\{(||x||^s \otimes ||y||^s \otimes ||z||^s) \oplus ||x||^{3s} \oplus ||y||^{3s} \oplus ||z||^{3s}\right\}, & s \neq \frac{2}{3}; \end{cases} \tag{3.10}$$

for all $x, y, z \in U$, then there exists a unique quadratic function $Q : U \to Y$ such that

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \begin{cases} \dfrac{\lambda}{|3|}, \\ \dfrac{3 \odot \lambda \otimes (||x||^s)_\alpha^+}{|2^2 - 2^s|}, \\ \dfrac{\lambda \otimes \left(||x||^{3s}\right)_\alpha^+}{|2^2 - 2^{3s}|}, \\ \dfrac{4 \odot \lambda \otimes \left(||x||^{3s}\right)_\alpha^+}{|2^2 - 2^{3s}|} \end{cases} \tag{3.11}$$

for all $x \in U$.

4. Felbin's stability results: Fixed point method

This section deals with the generalized Ulam - Hyers stability of the functional equation (1.2) in Banach space using fixed point method.

Now we will recall the fundamental results in fixed point theory.

Theorem 4.1. *(Banach's contraction principle) Let (X, d) be a complete metric space and consider a mapping $T : X \to X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \le Ld(x, y)$ *for some (Lipschitz constant)* $L < 1$. *Then,*

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;

(ii)The fixed point for each given element x^* is globally attractive, that is

(A2) $lim_{n\to\infty} T^n x = x^*$, *for any starting point* $x \in X$;

(iii) One has the following estimation inequalities:

(A3) $d(T^n x, x^*) \le \frac{1}{1-L}\ d(T^n x, T^{n+1}x), \forall\ n \ge 0, \forall\ x \in X$;

(A4) $d(x, x^*) \le \frac{1}{1-L}\ d(x, x^*), \forall\ x \in X$.

Theorem 4.2. *[24](The alternative of fixed point) Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \to X$ with Lipschitz constant L. Then, for each given element $x \in X$, either*

$(B1)$ $$d(T^n x, T^{n+1}x) = \infty \quad \forall\ n \ge 0,$$

or

$(B2)$ *there exists a natural number n_0 such that:*

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all* $n \ge n_0$;

(ii)*The sequence $(T^n x)$ is convergent to a fixed point y^* of T*

(iii) *y^* is the unique fixed point of T in the set $Y = \{y \in X : d(T^{n_0}x, y) < \infty\}$;*

(iv) $d(y^*, y) \le \frac{1}{1-L}\ d(y, Ty)$ *for all* $y \in Y$.

Theorem 4.3. *Let $F : U^3 \to V$ be a mapping for which there exist a function $\varphi : U^3 \to [0, \infty)$ with the condition*

$$\lim_{n\to\infty} \frac{1}{\mu_i^{2n}} \varphi\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)_\alpha^+ = 0 \tag{4.1}$$

where $\mu_i = 2$ if $i = 0$ and $\mu_i = \frac{1}{2}$ if $i = 1$ such that the functional inequality with

$$\|F(x, y, z)\|_\alpha^+ \preceq \varphi(x, y, z)_\alpha^+ \tag{4.2}$$

for all $x, y, z \in U$. If there exists $L = L(i) < 1$ such that the function

$$x \to \gamma(x)_\alpha^+ = \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}\right)_\alpha^+, \tag{4.3}$$

has the property

$$\gamma(x)_\alpha^+ \preceq L\frac{1}{\mu_i^2} \odot \gamma\left(\mu_i x\right)_\alpha^+ \tag{4.4}$$

for all $x \in U$, then there exists a unique quadratic mapping $Q : U \to Y$ satisfying the functional equation (1.2) and

$$\| f(x) - Q(x) \|_\alpha^+ \preceq \frac{L^{1-i}}{1-L}\gamma(x)_\alpha^+ \tag{4.5}$$

for all $x \in U$.

Proof. Consider the set

$$\Omega = \{p/p : U \to Y,\ p(0) = 0\}$$

and introduce the generalized metric on Ω,

$$d(p, q) = d_\gamma(p, q) = \inf\{K \in (0, \infty) :\| p(x) - q(x) \|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U\}.$$

It is easy to see that (Ω, d) is complete.

Define $T : \Omega \to \Omega$ by

$$Tp(x) = \frac{1}{\mu_i^2} \odot p(\mu_i x)_\alpha^+,$$

for all $x \in U$. Now $p, q \in \Omega$, we have

$$\begin{aligned}
& d(p, q) \leq K \\
& \Rightarrow \| p(x) - q(x) \|_\alpha^+ \preceq K\gamma(x)_\alpha^+, x \in U. \\
& \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i x) - \frac{1}{\mu_i^2} q(\mu_i x) \right\|_\alpha^+ \preceq \frac{1}{\mu_i^2} \odot K\gamma(\mu_i x)_\alpha^+, x \in U, \\
& \Rightarrow \left\| \frac{1}{\mu_i^2} p(\mu_i x) - \frac{1}{\mu_i^2} q(\mu_i x) \right\|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\
& \Rightarrow \| Tp(x) - Tq(x) \|_\alpha^+ \preceq LK\gamma(x)_\alpha^+, x \in U, \\
& \Rightarrow d_\gamma(p, q) \preceq LK.
\end{aligned}$$

This implies $d(Tp, Tq) \preceq Ld(p, q)$, for all $p, q \in \Omega$. i.e., T is a strictly contractive mapping on Ω with Lipschitz constant L.

Replacing (x, y, z) by $(x, x, -x)$ in (1.2), we get,

$$\left\| 2^2 f(x) - f(2x) \right\|_\alpha^+ \preceq \varphi\left(x, x, -x\right)_\alpha^+ \tag{4.6}$$

Hence from the above inequality, we have

$$\left\| f(x) - \frac{f(2x)}{2^2} \right\|_\alpha^+ \preceq \frac{1}{2^2} \odot \varphi\left(x, x, -x\right)_\alpha^+ \tag{4.7}$$

for all $x \in U$. Using (4.3) and (4.4) for the case $i = 0$, it reduces to

$$\left\| f(x) - \frac{f(2x)}{2^2} \right\|_\alpha^+ \preceq \frac{1}{2^2} \odot \gamma(x)_\alpha^+$$

for all $x \in U$.

$$\text{i.e.,}\quad d_{\varphi}(f,Tf) \preceq L \Rightarrow d(f,Tf) \preceq L \preceq L^{1} < \infty.$$

Again replacing $x = \frac{x}{2}$ in (4.6), we get,

$$\left\|2^{2} f\left(\frac{x}{2}\right) - f(x)\right\|_{\alpha}^{+} \preceq \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}\right)_{\alpha}^{+} \tag{4.8}$$

for all $x \in U$. Using (4.3) and (4.4) for the case $i = 1$ it reduces to

$$\left\|f(x) - 2^{2} f\left(\frac{x}{2}\right)\right\|_{\alpha}^{+} \preceq \gamma(x)_{\alpha}^{+}$$

for all $x \in U$,

$$\text{i.e.,}\quad d_{\varphi}(f,Tf) \leq 1 \Rightarrow d(f,Tf) \preceq 1 \preceq L^{0} < \infty.$$

In both cases, we arrive

$$d(f,Tf) \preceq L^{1-i}.$$

Therefore $(A1)$ holds.

By $(A2)$, it follows that there exists a fixed point Q of T in Ω such that

$$Q(x) = \lim_{n\to\infty} \frac{1}{\mu_i^{2n}} \left(f(\mu_i^{n} x)\right) \tag{4.9}$$

for all $x \in U$.

To prove $Q : U \to Y$ is quadratic. Replacing (x,y,z) by $\left(\mu_i^n x, \mu_i^n y, \mu_i^n z\right)$ in (4.2) and dividing by μ_i^{2n}, it follows from (4.1) that

$$\begin{aligned}\|Q(x,y,z)\|_{\alpha}^{+} &= \lim_{n\to\infty} \frac{\left\|D\ f(\mu_i^n x, \mu_i^n y, \mu_i^n z)\right\|_{\alpha}^{+}}{\mu_i^{2n}} \\ &\preceq \lim_{n\to\infty} \frac{\varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z)_{\alpha}^{+}}{\mu_i^{2n}} = 0\end{aligned}$$

for all $x, y, z \in U$. That is, Q satisfies the functional equation (1.2).

By $(A3)$, Q is the unique fixed point of T in the set $\Delta = \{Q \in \Omega : d(f,Q) < \infty\}$, Q is the unique function such that

$$\|f(x) - Q(x)\|_{\alpha}^{+} \preceq K\gamma(x)_{\alpha}^{+}$$

for all $x \in U$ and $K > 0$. Finally by $(A4)$, we obtain

$$d(f,Q) \preceq \frac{1}{1-L} d(f,Tf)$$

this implies

$$d(f,Q) \preceq \frac{L^{1-i}}{1-L}$$

which yields

$$\|\ f(x) - Q(x)\ \|_{\alpha}^{+} \preceq \frac{L^{1-i}}{1-L}\gamma(x)_{\alpha}^{+}.$$

This completes the proof of the theorem. □

The following corollary is an immediate consequence of Theorem 4.3 concerning the stability of (1.2).

Corollary 4.1. *Let $F : U \to Y$ be a mapping and there exit real numbers λ and s such that*

$$\|F(x,y,z)\|_{\alpha}^{+} \preceq \begin{cases} \lambda, & \\ \lambda \otimes (||x||^s \oplus ||y||^s \oplus ||z||^s), & s \neq 2; \\ \lambda \otimes (||x||^s \otimes ||y||^s \otimes ||z||^s), & s \neq \frac{2}{3}; \\ \lambda \otimes \left\{(||x||^s \otimes ||y||^s \otimes ||z||^s) \oplus ||x||^{3s} \oplus ||y||^{3s} \oplus ||z||^{3s}\right\}, & s \neq \frac{2}{3}; \end{cases} \tag{4.10}$$

for all $x, y, z \in U$, then there exists a unique quadratic function $Q : U \to U$ such that

$$\|f(x) - Q(x)\|_{\alpha}^{+} \preceq \begin{cases} \dfrac{\lambda}{|3|}, \\ \dfrac{2 \odot \lambda_{\alpha}^{+} \otimes (||x||^s)_{\alpha}^{+}}{|2^2 - 2^s|}, \\ \dfrac{\lambda_{\alpha}^{+} \otimes \left(||x||^{3s}\right)_{\alpha}^{+}}{|2^2 - 2^{3s}|}, \\ \dfrac{4 \odot \lambda_{\alpha}^{+} \otimes \left(||x||^{3s}\right)_{\alpha}^{+}}{|2^2 - 2^{3s}|}. \end{cases} \tag{4.11}$$

for all $x \in U$.

Proof. Set

$$\varphi(x,y,z) = \begin{cases} \lambda, \\ \lambda \otimes (||x||^s \oplus ||y||^s \oplus ||z||^s), \\ \lambda \otimes (||x||^s \otimes ||y||^s \otimes ||z||^s), \\ \lambda \otimes \left\{||x||^s \otimes ||y||^s \otimes ||z||^s \oplus ||x||^{3s} \oplus ||y||^{3s} \oplus ||z||^{3s}\right\} \end{cases}$$

for all $x, y, z \in U$. Now,

$$\begin{aligned} \frac{\varphi(\mu_i^n x, \mu_i^n y, \mu_i^n z)}{\mu_i^{2n}} &= \begin{cases} \dfrac{\lambda}{\mu_i^{2n}}, \\ \dfrac{\lambda}{\mu_i^{2n}} \otimes (||\mu_i^n x||^s \oplus ||\mu_i^n y||^s \oplus ||\mu_i^n z||^s), \\ \dfrac{\lambda}{\mu_i^{2n}} \otimes (||\mu_i^n x||^s \otimes ||\mu_i^n y||^s \otimes ||\mu_i^n z||^s) \\ \dfrac{\lambda}{\mu_i^{2n}} \otimes \{(||\mu_i^n x||^s \otimes ||\mu_i^n y||^s \otimes ||\mu_i^n z||^s) \\ \qquad \oplus ||\mu_i^n x||^{3s} \oplus ||\mu_i^n y||^{3s} \oplus ||\mu_i^n z||^{3s}\} \end{cases} \\ &= \begin{cases} \lambda \otimes \mu_i^{-2n}, \\ \lambda \otimes \mu_i^{(s-2)n} \otimes (||x||^s \oplus ||y||^s \oplus ||z||^s), \\ \lambda \otimes \mu_i^{(3s-2)n} \otimes (||x||^s \otimes ||y||^s \otimes ||z||^s) \\ \lambda \otimes \mu_i^{(3s-2)n} \otimes \{(||x||^s \otimes ||y||^s \otimes ||z||^s) \\ \qquad \oplus ||x||^{3s} \oplus ||y||^{3s} \oplus ||z||^{3s}\} \end{cases} \\ &= \begin{cases} \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty, \\ \to 0 \text{ as } n \to \infty. \end{cases} \end{aligned}$$

Thus, (4.1) holds.

But we have $\gamma(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}\right)$ has the property $\gamma(x) \preceq L \cdot \frac{1}{\mu_i^2}\,\gamma\left(\mu_i x\right)$ for all $x \in U$. Hence

$$\gamma(x) = \varphi\left(\frac{x}{2}, \frac{x}{2}, \frac{-x}{2}\right) = \begin{cases} \lambda, \\ \lambda \otimes \frac{3}{2^s} \otimes ||x||^s, \\ \lambda \otimes \frac{1}{2^{3s}} \otimes ||x||^{3s}, \\ \lambda \otimes \frac{4}{2^{3s}} \otimes ||x||^{3s}. \end{cases}$$

Now,

$$\frac{1}{\mu_i^2}\gamma(\mu_i x) = \begin{cases} \frac{\lambda}{\mu_i^2}, \\ \frac{\lambda}{\mu_i^2} \odot \frac{3}{2^s} \otimes (||\mu_i x||^s), \\ \frac{\lambda}{\mu_i^2} \odot \frac{1}{2^{3s}} \otimes (||\mu_i x||^{ns}), \\ \frac{\lambda}{\mu_i^2} \odot \frac{4}{2^{3s}} \otimes \left(||\mu_i x||^{3s}\right) \end{cases} = \begin{cases} \mu_i^{-2}\gamma(x)_\alpha^+, \\ \mu_i^{s-2}\gamma(x)_\alpha^+, \\ \mu_i^{3s-2}\gamma(x)_\alpha^+, \\ \mu_i^{3s-2}\gamma(x)_\alpha^+. \end{cases}$$

Now from (4.5),

Case:1 $L = 2^{-2}$ for $s = 0$ if $i = 0$,

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \frac{\left(2^{-2}\right)^{1-0}}{1 - 2^{(-2)}}\gamma(x)_\alpha^+ \preceq \lambda_\alpha^+ \odot \left(\frac{2^{-2}}{1 - 2^{-2}}\right) \preceq \frac{\lambda_\alpha^+}{3}.$$

Case:2 $L = 2^2$ for $s = 0$ if $i = 1$,

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \frac{\left(2^{2}\right)^{1-1}}{1 - 2^{2}}\gamma(x)_\alpha^+ \preceq \lambda_\alpha^+ \odot \left(\frac{1}{1 - 2^{2}}\right) \preceq \frac{\lambda_\alpha^+}{-3}.$$

Case:3 $L = 2^{s-2}$ for $s > 2$ if $i = 0$,

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \frac{\left(2^{s-2}\right)^{1-0}}{1 - 2^{s-2}}\gamma(x)_\alpha^+ \preceq \frac{3 \odot \lambda_\alpha^+}{2^2 - 2^s} \otimes (||x||^s)_\alpha^+ .$$

Case:4 $L = 2^{2-s}$ for $s < 2$ if $i = 1$,

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \frac{\left(2^{2-s}\right)^{1-1}}{1 - 2^{2-s}}\gamma(x)_\alpha^+ \preceq \frac{3 \odot \lambda_\alpha^+}{2^s - 2^2} \otimes (||x||^s)_\alpha^+ .$$

Case:5 $L = 2^{3s-2}$ for $s > \frac{2}{3}$ if $i = 0$,

$$\|f(x) - Q(x)\| \preceq \left(\frac{\left(2^{(2s-2)}\right)^{1-0}}{1 - 2^{(2s-2)}}\right)\gamma(x)_\alpha^+ \preceq \frac{\lambda_\alpha^+}{2^2 - 2^{3s}} \otimes \left(||x||^{3s}\right)_\alpha^+ .$$

Case:6 $L = 2^{2-3s}$ for $s < \frac{2}{3}$ if $i = 1$,

$$\|f(x) - Q(x)\|_\alpha^+ \preceq \left(\frac{\left(2^{(2-3s)}\right)^{1-1}}{1 - 2^{(2-3s)}}\right)\gamma(x)_\alpha^+ \preceq \frac{\lambda_\alpha^+}{2^{3s} - 2^2} \otimes \left(||x||^{3s}\right)_\alpha^+ .$$

Hence it completes the proof. □

5. Conclusion

This article has proved the stability results of the quadratic functional equation in Felbin's fuzzy normed linear spaces (f-NLS) using both the direct and fixed point methods.

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References

[1] J. Aczel, J. Dhombres, Functional Equations in Several Variables, Cambridge University Press, Cambridge, 1989.
[2] T. Aoki, On the Stability of the linear Transformation in Banach Spaces, J. Math. Soc. Japan. 2 (1950), 64-66. https://doi.org/10.2969/jmsj/00210064.
[3] M. Arunkumar, S. Karthikeyan, Solution and Stability of n-Dimensional Additive Functional Equation, Int. J. Appl. Math. 25 (2012), 163-174.
[4] I.S. Chang, E.H. Lee, H.M. Kim, On the Hyers-Ulam-Rassias Stability of a Quadratic Functional Equations, Math. Ineq. Appl. 6 (2003), 87-95.
[5] S. Czerwik, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.
[6] S. Czerwik, On the Stability of the Quadratic Mapping in Normed Spaces, Abh. Math. Semin. Univ. Hambg. 62 (1992), 59–64. https://doi.org/10.1007/BF02941618.
[7] D.K. Singha, S. Grover, On the Stability of a Sum Form Functional Equation Related to Entropies of Type (α, β), J. Nonlinear Sci. Appl. 14 (2021), 168-180. https://doi.org/10.22436/jnsa.014.03.06.
[8] C. Felbin, Finite Dimensional Fuzzy Normed Linear Space, Fuzzy Sets Syst. 48 (1992), 239–248. https://doi.org/10.1016/0165-0114(92)90338-5.
[9] P. Gavruta, A Generalization of the Hyers-Ulam-Rassias Stability of Approximately Additive Mappings, J. Math. Anal. Appl. 184 (1994), 431–436. https://doi.org/10.1006/jmaa.1994.1211.
[10] T.E. Gantner, R.C. Steinlage, R.H. Warren, Compactness in Fuzzy Topological Spaces, J. Math. Anal. Appl. 62 (1978), 547–562. https://doi.org/10.1016/0022-247X(78)90148-8.
[11] E.S. El-Hady, On Hyperstability of Cauchy Functional Equation in $(2, \gamma)$-Banach Spaces, J. Math. Computer Sci. 23 (2021), 354-363. https://doi.org/10.22436/jmcs.023.04.08.
[12] D.H. Hyers, On the Stability of the Linear Functional Equation, Proc. Nat. Acad. Sci. 27 (1941), 222–224. https://doi.org/10.1073/pnas.27.4.222.
[13] U. Höhle, Fuzzy Real Numbers as Dedekind Cuts With Respect to a Multiple-Valued Logic, Fuzzy Sets Syst. 24 (1987), 263–278. https://doi.org/10.1016/0165-0114(87)90027-3.
[14] D.H. Hyers, G. Isac, T.M. Rassias, Stability of Functional Equations in Several Variables, Birkhauser, Basel, 1998.
[15] K.-W. Jun, H.-M. Kim, On the Stability of an N-Dimensional Quadratic and Additive Functional Equation, Math. Inequal. Appl. 9 (2006), 153–165. https://doi.org/10.7153/mia-09-16.
[16] S.M. Jung, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.
[17] Pl. Kannappan, Functional Equations and Inequalities with Applications, Springer, New York, 2009.
[18] S. Karthikeyan, C. Park, P. Palani, T.R.K. Kumar, Stability of an Additive-Quartic Functional Equation in Modular Spaces, J. Math. Computer Sci. 26 (2021), 22–40. https://doi.org/10.22436/jmcs.026.01.04.
[19] O. Kaleva, S. Seikkala, On Fuzzy Metric Spaces, Fuzzy Sets Syst. 12 (1984), 215–229. https://doi.org/10.1016/0165-0114(84)90069-1.

[20] O. Kaleva, The Completion of Fuzzy Metric Spaces, J. Math. Anal. Appl. 109 (1985), 194-198.

[21] O. Kaleva, A Comment on the Completion of Fuzzy Metric Spaces, Fuzzy Sets Syst. 159 (2008), 2190–2192. https://doi.org/10.1016/j.fss.2008.03.011.

[22] H.-M. Kim, On the Stability Problem for a Mixed Type of Quartic and Quadratic Functional Equation, J. Math. Anal. Appl. 324 (2006), 358–372. https://doi.org/10.1016/j.jmaa.2005.11.053.

[23] R. Lowen, Fuzzy Set Theory, (Ch. 5: Fuzzy real numbers), Kluwer, Dordrecht, 1996.

[24] J.B. Diaz, B. Margolis, A Fixed Point Theorem of the Alternative, for Contractions on a Generalized Complete Metric Space, Bull. Amer. Math. Soc. 74 (1968), 305-309.

[25] F. Moradlou, S. Rezaee, I. Sadeqi, Stability of Cauchy Functional Equation in Felbin's Type Spaces: A Fixed Point Approach, Submitted.

[26] A.K. Mirmostafaee, M.S. Moslehian, Fuzzy Versions of Hyers–ulam–rassias Theorem, Fuzzy Sets Syst. 159 (2008), 720–729. https://doi.org/10.1016/j.fss.2007.09.016.

[27] J.M. Rassias, On Approximation of Approximately Linear Mappings by Linear Mappings, J. Funct. Anal. 46 (1982), 126–130. https://doi.org/10.1016/0022-1236(82)90048-9.

[28] T.M. Rassias, On the Stability of the Linear Mapping in Banach Spaces, Proc. Amer. Math. Soc. 72 (1978), 297–300. https://doi.org/10.1090/S0002-9939-1978-0507327-1.

[29] T.M. Rassias, On the Stability of Functional Equations and a Problem of Ulam, Acta Appl. Math. 62 (2000), 23–130. https://doi.org/10.1023/A:1006499223572.

[30] T.M. Rassias, Functional Equations, Inequalities and Applications, Kluwer Acedamic Publishers, Dordrecht, 2003.

[31] K. Ravi, M. Arunkumar, J.M. Rassias, Ulam Stability for the Orthogonally General Euler-Lagrange Type Functional Equation, Int. J. Math. Stat. 3 (2008), 36-46.

[32] J.M. Rassias, R. Saadati, G. Sadeghi, J. Vahidi, On Nonlinear Stability in Various Random Normed Spaces, J. Inequal. Appl. 2011 (2011), 62. https://doi.org/10.1186/1029-242X-2011-62.

[33] J.M. Rassias, N. Pasupathi, R. Saadati, M. de la Sen, Approximation of Mixed Euler-Lagrange σ-Cubic-Quartic Functional Equation in Felbin's Type f-NLS, J. Funct. Spaces. 2021 (2021), 8068673. https://doi.org/10.1155/2021/8068673.

[34] S.E. Rodabaugh, Fuzzy Addition in the L-Fuzzy Real Line, Fuzzy Sets Syst. 8 (1982), 39–52. https://doi.org/10.1016/0165-0114(82)90028-8.

[35] I. Sadeqi, F. Moradlou, M. Salehi, On Approximate Cauchy Equation in Felbin's Type Fuzzy Normed Linear Spaces, Iran. J. Fuzzy Syst. 10 (2013), 51-63. https://doi.org/10.22111/ijfs.2013.862.

[36] I. Sadeqi, M. Salehi, Fuzzy Compact Operators and Topological Degree Theory, Fuzzy Sets Syst. 160 (2009), 1277–1285. https://doi.org/10.1016/j.fss.2008.08.014.

[37] S.M. Ulam, Problems in Modern Mathematics, Science Editions, Wiley, New York, 1964.

[38] J. Xiao, X. Zhu, On Linearly Topological Structure and Property of Fuzzy Normed Linear Space, Fuzzy Sets Syst. 125 (2002), 153–161. https://doi.org/10.1016/S0165-0114(00)00136-6.

[39] J. Xiao, X. Zhu, Topological Degree Theory and Fixed Point Theorems in Fuzzy Normed Space, Fuzzy Sets Syst. 147 (2004), 437–452. https://doi.org/10.1016/j.fss.2004.01.003.