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On the Uphill Zagreb Indices of Graphs

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Abstract. One of the tools, to research and investigation the structural dependence of various properties and some activities of chemical structures and networks is the topological indices of graphs. In this research work, we introduce novel indices of graphs which they based on the uphill degree of the vertices termed as uphill Zagreb topological indices. Exact formulae of these new indices for some important and famous families of graphs are established.

1. Introduction

In this research, by graphs, we mean undirected finite simple graph. We denote $G = (V, E)$ for a graph, where V is the set of vertices and E is the set of edges. For a vertex $v \in V(G)$ the degree of v, $d(v)$ is the number of edges incident with v. Any terminology or notation which, we did not mention its definition, we refer the reader to [3].

Topological indices have a widespread position specifically in pharmacology, chemistry, networks and many others. (see [8, 9, 13–15, 18, 24, 25]). Almost of the indices of contemporary interesting in mathematical chemistry are introduced based on vertex degrees of the chemical graph. The two well-known topological indices of graphs are the Zagreb indices that have been introduced by Gutman and Trinajstic by their work in [16], and described as $M_1(G) = \sum_{u\in V(G)} (d(u))^2$ and $M_2(G) = \sum_{uv\in E(G)} d(u)\, d(v)$, respectively. The forgotten topological index was introduced by Furtula and Gutman [10] as $F(G) = \sum_{u\in V(G)} (d(u))^3$.

Zagreb indices were studied considerably due to their numerous applications inside the area of present chemical methods which want extra time and more charges. Many new reformulated and prolonged versions of the Zagreb indices have been delivered for several similar reasons (cf. [1,4,12,19,22,27–29]).

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The uphill domination and some related concepts are introduced and studied in ([7,23]). In this paper motivated by the large applications of topological indices and the concept of uphill domination, we introduce novel indices of graphs based on a new degree (uphill degree) of the vertices termed as uphill Zagreb topological indices. Some properties and exact formulae of these new topological indices for some standard and famous families of graphs are established.

2. Some Results on the Uphill Zagreb Indices of Graphs

Definition 2.1. *[7] For any graph $G = (V, E)$. A path $u - v$ is a sequence of vertices in G, initialing with u and terminal at v, such that sequential vertices are adjacent, and no vertex is repeated. A path $P = v_1, v_2, ...v_{k+1}$ in G is an uphill path if for every i,$1 \leq i \leq k$, $deg(v_i) \leq deg(v_{i+1})$.*
For any vertices u and v in G, if there is an uphill path from u to v we say that u is uphill adjacent to v.

Definition 2.2. *A vertex v is uphill dominates a vertex u in a graph G if v uphill adjacent to u. An uphill neighborhood of the vertex v is denoted by $N_{up}(v)$ and described as: $N_{up}(v) = \{u : v\, uphill\, adjacent\, to\, u\ \}$. The uphill degree of the vertex v, denoted by $d_{up}(v)$, is the number of vertices which v uphill adjacent them, that means $d_{up}(v) = |N_{up}(v)|$.*
The uphill closed neighborhood, $N_{up}[v]$, of the vertex v is the uphill open neighborhood of v together with the vertex v.
The maximum and minimum uphill degrees in the graph G denoted by $\Delta_{up}(G)$ and $\delta_{up}(G)$, respectively. The vertex with uphill degree equal to zero is called uphill isolated vertex.

In this paper by $E_{x,y}$, we mean that $E_{x,y} = \{uv \in E(G) : d_{up}(u) = x \text{ and } d_{up}(v) = y\}$.

Definition 2.3. *For any graph $G = (V, E)$ the first uphill Zagreb, second uphill Zagreb, forgotten uphill Zagreb index and modified first uphill Zagreb are defined as:*

$$UPM_1(G) = \sum_{v \in V(G)} (d_{up}(v))^2,$$

$$UPM_2(G) = \sum_{vu \in E(G)} d_{up}(v) d_{up}(u),$$

$$UPM_1^*(G) = \sum_{vu \in E(G)} (d_{up}(v) + d_{up}(u)),$$

and

$$UPF(G) = \sum_{v \in V(G)} (d_{up}(v))^3.$$

Lemma 2.1. *Suppose G be a graph, for any two vertices u and v, where u is uphill adjacent to v, if $d(u) < d(v)$. Then $d_{up}(u) > d_{up}(v)$.*

Proof. If G is a graph, where u and v be any two vertices in G, where u is uphill adjacent to v. Then there are two cases:

Case 1. When u is adjacent with v. Without loss of generality, suppose that $N_{up}(v) = \{v_1, v_2, ..., v_r\}$, where $r = d_{up}(v)$. In this case,

$$N_{up}[v] \subseteq N_{up}(u).$$

Hence,

$$d_{up}(u) > d_{up}(v).$$

Case 2. When u not adjacent with v. Then, there is an uphill path Γ from u to v. Clearly, $d_{up}(u) \geq d_{up}(v) + k + 1$, where k is the number of internal vertices in Γ.

Hence,

$$d_{up}(u) > d_{up}(v).$$

□

Definition 2.4. *A graph G is called k-uphill regular graph if $\Delta_{up}(G) = \delta_{up}(G) = k$.*

Theorem 2.1. *Let G be any graph. Then, the graph G is regular if and only if G is uphill regular.*

Proof. If G is regular graph of n vertices. Then it is straightforward, that G is $n - 1$ uphill regular graph.

To prove the other direction of the theorem, we will prove that if G is not regular then G is not uphill regular. Suppose that G is not regular. Then, there exist at least two vertices say u and v such that $d(u) < d(v)$. We have two cases:

Case 1. If u is uphill adjacent to v. Then, By Lemma 2.1, the graph G is not uphill regular.

Case 2. If u is not uphill adjacent to v. Then, there exist at least two adjacent vertices in some paths from u to v say w and w_1, where $d(w) > d(w_1)$. By Lemma 2.1, we get $d_{up}(w_1) > d_{up}(w)$. Hence, the graph G is not uphill regular. □

Proposition 2.1. *For any graph G, $UPM_2(G) = 0$ if and only if for each edge $e = uv$, either u or v is uphill isolated vertex.*

Proposition 2.2. *For any graph G, $\sum_{v \in V(G)} d_{up}(v) \leq \tau(G)$, where $\tau(G)$ is the number of uphill paths in G. Furthermore, the equality holds for the graph without cycle (acylic).*

Proof. Let G be any graph and let $\tau(G)$ be the number of uphill paths in G, for any vertex v in G it is obviously from the definition of uphill degree of the vertex that, $d_{up}(v)$ is less than or equal the number of paths in G which originated from v. Also if we denote by Γ_v as the number of uphill paths in G originated from v, then $\tau(G) = \sum_{v\in V(G)} \Gamma_v$. Therefore,

$$\sum_{v\in V(G)} d_{up}(v) \leq \sum_{v\in V(G)} \Gamma_v = \tau(G).$$

Furthermore, it is easy to check that if G is acyclic, then $d_{up}(v) = \Gamma_v$. Hence the equality holds. $\square$

Proposition 2.3. *Let $G \cong P_n$, be a path of $n \geq 3$ vertices. Then,*

i. $UPM_1(G) = n^3 - 6n^2 + 13n - 10$,
ii. $UPM_2(G) = n^3 - 7n^2 + 17n - 15$,
iii. $UPM_1^*(G) = 2n^2 - 8n + 8$,
iv. $UPF(G) = n^4 - 9n^3 + 33n^2 - 57n + 38$.

Proof. Suppose $G \cong P_n$ be a path of $n \geq 3$ vertices. Then there are two vertices with uphill degree $n-2$ and $n-2$ vertices of uphill degree $n-3$. So by using the definition of first uphill index we get,

$$\begin{aligned} UPM_1(G) &= 2(n-2)^2 + (n-2)(n-3)^2 \\ &= n^3 - 6n^2 + 13n - 10. \end{aligned}$$

Similarly,

$$\begin{aligned} UPF(G) &= 2(n-2)^3 + (n-2)(n-3)^3 \\ &= n^4 - 9n^3 + 33n^2 - 57n + 38. \end{aligned}$$

There are two edges of the type $E_{n-2,n-3}$ and $n-3$ edges of the type $E_{n-3,n-3}$. Then,

$$\begin{aligned} UPM_2(G) &= 2[(n-2)(n-3)] + (n-3)^3 \\ &= n^3 - 7n^2 + 17n - 15. \end{aligned}$$

In the same way we get

$$\begin{aligned} UPM_1^*(G) &= 2(2n-5) + (n-3)(2n-6) \\ &= 2n^2 - 8n + 8. \end{aligned}$$

$\square$

Proposition 2.4. *Let $G = (V, E)$ be a regular graph of degree k and has n vertices. Then,*

i. $UPM_1(G) = n(n-1)^2$,
ii. $UPM_2(G) = \frac{nk(n-1)^2}{2}$,
iii. $UPM_1^*(G) = nk(n-1)$,
iv. $UPF(G) = n(n-1)^3$.

Proof. Let $G = (V, E)$ be $k-$ regular graph with n vertices. Then it isn't difficult to see that between any two vertices of G there exists an uphill path, so for any $v \in V(G)$, we have $d_{up}(v) = n-1$. Hence, $UPM_1(G) = n(n-1)^2$, $UPM_2(G) = \frac{nk(n-1)^2}{2}$, $UPM_1^*(G) = nk(n-1)$ and $UPF(G) = n(n-1)^3$. $\square$

Corollary 2.1. *For any complete graph K_n, we have*

i. $UPM_1(K_n) = UPM_1^*(G) = n(n-1)^2$,
ii. $UPM_2(K_n) = \frac{n(n-1)^3}{2}$,
iii. $UPF(K_n) = n(n-1)^3$.

Corollary 2.2. *For any cycle C_n, where $n \geq 3$,*

i. $UPM_1(C_n) = UPM_2(C_n) = n(n-1)^2$,
ii. $UPM_1^*(C_n) = 2n(n-1)$,
iii. $UPF(C_n) = n(n-1)^3$.

By a graph W_n, we mean a wheel graph of $n+1$ vertices.

Proposition 2.5. *Let $G \cong W_n$, where $n \geq 3$ be a wheel graph. Then,*

i. $UPM_1(G) = UPM_2(G) = n^3$,
ii. $UPM_1^*(G) = 2n^2$,
iii. $UPF(G) = n^4$.

Proof. Let $G \cong W_n$, where $n \geq 3$ be a wheel graph. The graph G has one vertex of uphill degree zero and n vertices of uphill degree n. Then,

$$\begin{aligned} UPM_1(G) &= n(n)^2 + 0 \\ &= n^3. \end{aligned}$$

In the same way, we get the forgotten uphill index

$$\begin{aligned} UPF(G) &= n(n)^3 + 0 \\ &= n^4. \end{aligned}$$

There are n edges of the type $E_{n,n}$ and n edges of the type $E_{n,0}$. Then,

$$UPM_1^*(G) = 2n^2$$

and

$$\begin{aligned} UPM_2(G) &= n(n)^2 \\ &= n^3. \end{aligned}$$

□

A tadpole graph $T_{m,n}$ is constructed by joining between C_m and P_n by a bridge [17].

Proposition 2.6. *Let $G \cong T_{m,n}$, where $m, n \geq 3$ be a tadpole graph of $m+n$ vertices. Then,*

i. $UPM_1(G) = m^3 - 3m^2 + 3m - 2 + n^3 - 2n^2 + 3n$,
ii. $UPM_2(G) = m^3 - 4m^2 + 5m - 4 + n^3 - 3n^2 + 4n$,

iii. $UPM_1^*(G) = 2m^2 - 4m + 2n^2 - 5n + 5$,

iv. $UPF(G) = m^4 - 4m^3 + 6m^2 - 4m + 2 + n^4 - 3n^3 + 6n^2 - 4n$.

Proof. Let $G \cong T_{m,n}$, where $m, n \geq 3$ be a tadpole graph of $m+n$ vertices. There are $m-1$ vertices of uphill degree $m-1$, one vertex of uphill degree zero, $n-1$ vertices of uphill degree $n-1$ and one vertex of uphill degree n. Then, we get

$$UPM_1(G) = (m-1)^3 + (n-1)^3 + (n)^2$$
$$= m^3 - 3m^2 + 3m - 2 + n^3 - 2n^2 + 3n.$$

Similarly,

$$UPF(G) = (m-1)^4 + (n-1)^4 + (n)^3$$
$$= m^4 - 4m^3 + 6m^2 - 4m + 2 + n^4 - 3n^3 + 6n^2 - 4n.$$

Type	Number of edges
$E_{m-1,0}$	2
$E_{m-1,m-1}$	$m-2$
$E_{0,n-1}$	1
$E_{n-1,n-1}$	$n-2$
$E_{n-1,n}$	1

Table 1. Edge partition of tadpole graph based on uphill degree of end vertices.

Now, by using the partition given in Table1, we get

$$UPM_2(G) = (m-1)^2(m-2) + (n-1)^2(n-2) + n(n-1)$$
$$= m^3 - 4m^2 + 5m - 4 + n^3 - 3n^2 + 4n.$$

Also,

$$UPM_1^*(G) = 2(m-1) + (m-2)(2m-2) + (n-1) + (2n-2)(n-2)$$
$$= 2m^2 - 4m + 2n^2 - 5n + 5.$$

□

The graph which obtained from a wheel graph with extra vertex between each pair of adjacent vertices of the outer cycle is called gear graph G_n [17].

Proposition 2.7. *Let $G \cong G_n$, where $n \geq 4$ be a gear graph. Then,*

i. $UPM_1(G) = 10n$,

ii. $UPM_2(G) = 6n$,
iii. $UPM_1^*(G) = 9n$,
iv. $UPF(G) = 28n$.

Proof. Let $G \cong G_n$, where $n \geq 4$ be a gear graph. Then, there are n vertices of uphill degree one, one vertex of uphill degree zero and n vertices of uphill degree three. Clearly, we get

$$UPM_1(G) = n + 9n$$
$$= 10n.$$

In the same way, we get

$$UPF(G) = n + 27n$$
$$= 28n.$$

There are n edges of the type $E_{1,0}$ and $2n$ edges of the type $E_{1,3}$. Then,

$$UPM_1^*(G) = 9n,$$
$$UPM_2(G) = 6n.$$

□

The windmill graph $Wd(s,k)$, where $s,k \geq 2$, is a graph of k copies of complete graph K_s at a shared common vertex [17].

Proposition 2.8. *Let $G \cong Wd(s,k)$, where $s \geq 3$ and $k \geq 2$ be a windmill graph of $k(s-1)+1$ vertices. Then,*
i. $UPM_1(G) = k(s-1)^3$,
ii. $UPM_2(G) = k(s-1)^3(\frac{s-2}{2})$,
iii. $UPM_1^*(G) = k\,(s-1)^3$,
iv. $UPF(G) = k(s-1)^4$.

Proof. Let $G \cong Wd(s,k)$, where $s \geq 3$ and $k \geq 2$ be a windmill graph of $k(s-1)+1$ vertices. The graph G has one vertex of uphill degree zero and $k(s-1)$ vertices of uphill degree $s-1$. So,

$$UPM_1(G) = k(s-1)(s-1)^2$$
$$= k(s-1)^3.$$

Similarly,

$$UPF(G) = k(s-1)(s-1)^3$$
$$= k(s-1)^4.$$

There are $\frac{sk(s-1)}{2} - k(s-1)$ edges of the type $E_{s-1,s-1}$ and $k(s-1)$ edges of the type $E_{s-1,0}$. Then,

$$UPM_2(G) = (s-1)^2[\frac{sk(s-1)}{2} - k(s-1)]$$

$$= k(s-1)^3[\frac{s-2}{2}].$$

Also,

$$UPM_1^*(G) = \frac{k\,(s-1)\,(s-2)}{2}\,(2s-2) + k\,(s-1)\,(s-1)$$
$$= (s-1)\,(k\,(s-1)\,(s-2) + k\,(s-1))$$
$$= k\,(s-1)^3\,.$$

□

The graph which is obtained from W_n by adding an end edge to each outer vertex of W_n, is called helm graph and denoted by H_n [17].

Proposition 2.9. *Let $G \cong H_n$, where $n \geq 3$ be a helm graph of $2n+1$ vertices. Then,*

i. $UPM_1(G) = 2n^3 + 2n^2 + n$,
ii. $UPM_2(G) = 2n^3 + n^2$,
iii. $UPM_2^*(G) = 5n^2 + n$,
iv. $UPF(G) = 2n^4 + 3n^3 + 3n^2 + n$.

Proof. Let $G \cong H_n$, where $n \geq 3$ be a helm graph. The graph G has only one vertex of uphill degree zero, n vertices of uphill degree $n+1$ and n vertices of uphill degree n. Then,

$$UPM_1(G) = n(n+1)^2 + n(n)^2$$
$$= 2n^3 + 2n^2 + n.$$

Similarly,

$$UPF(G) = n(n+1)^3 + n(n)^3$$
$$= 2n^4 + 3n^3 + 3n^2 + n.$$

There are n edges of the type $E_{n+1,n}$, n edges of the type $E_{n,n}$ and n edges of the type $E_{n,0}$. Then,

$$UPM_2(G) = n^2(n+1) + n^3$$
$$= 2n^3 + n^2.$$

Similarly,

$$UPM_1^*(G) = n(2n+1) + 2n(n) + n^2$$
$$= 5n^2 + n.$$

□

The double star graph $S_{r,t}$ which obtained from complete graph of two vertices by joining r pendent edges to one vertex and t pendent edges to the other vertex of the complete graph K_2 [17].

Theorem 2.2. *Let $G \cong S_{r,t}$, where $r, t \geq 2$ be a double star graph. Then,* *i.* $UPM_1(G) = \begin{cases} 8r+2 & if \quad r=t; \\ 4r+t+1 & if \quad r<t. \end{cases}$

ii. $UPM_2(G) = \begin{cases} 4r+1 & if \quad r=t; \\ 2r & if \quad r<t. \end{cases}$

iii. $UPM_1^*(G) = \begin{cases} 6r+2 & if \quad r=t; \\ 3r+t+1 & if \quad r<t. \end{cases}$

iv. $UPF(G) = \begin{cases} 16r+2 & if \quad r=t; \\ 8r+t+1 & if \quad r<t. \end{cases}$

Proof. Let $G \cong S_{r,t}$, where $r, t \geq 2$ be a double star with $r+t+2$ vertices. Then,

i. There are two cases:

Case 1. If $r = t$, it has two vertices of uphill degree one and there are $2r$ vertices of uphill degree 2, then $UPM_1(G) = 8r + 2$.

Case 2. If $r < t$, there are r vertices of uphill degree 2 and $s + 1$ vertices of uphill degree one, also one uphill isolated vertex, then $UPM_1(G) = 4r + t + 1$.

ii. We have two cases:

Case 1. If $r = t$, it has one edge of the type $E_{1,1}$ and there are $2r$ edges in G, where each edge of the type $E_{1,2}$,then $UPM_2(G) = 4r + 1$.

Case 2. When $r < t$, there are $t + 1$ edges where each edge of the type $E_{1,0}$. Also, there are r edges where each edge of the type $E_{2,1}$,then $UPM_2(G) = 2r$.

iii. Similarly as in part ii, if $r = t$, then $UPM_1^*(G) = 6r + 2$. If $r < t$, then $UPM_1^*(G) = 3r + t + 1$.

iv. The proof is similarly to part i.

□

The subdivision graph $S(G)$ of the graph G is a graph obtained from G by replacing each of its edge by a path of length 2. By simple calculation, we get the uphill Zagreb indices for the subdivision graphs of path, cycle and complete graph.

Proposition 2.10. *Let $G \cong S(P_n)$, where $n \geq 3$. Then,*

i. $UPM_1(G) = 8n^3 - 36n^2 + 56n - 30$,

ii. $UPM_2(G) = 8n^3 - 40n^2 + 68n - 40$,

iii. $UPM_1^*(G) = 8n^2 - 24n + 18$,

iv. $UPF(G) = 16n^4 - 104n^3 + 264n^2 - 308n + 138$.

Proposition 2.11. *Let $G \cong S(C_n)$, where $n \geq 3$. Then,*

i. $UPM_1(G) = UPM_2(G) = 2n(2n-1)^2$,

ii. $UPM_1^*(G) = 2n(4n-2)$,

iii. $UPF(G) = 2n(2n-1)^3$.

Proposition 2.12. *Let $G \cong S(K_n)$, where $n \geq 4$. Then,*

i. $UPM_1(G) = 2n(n-1)$,

ii. $UPM_2(G) = 0$,

iii. $UPM_1^*(G) = 2n(n-1)$,

iv. $UPF(G) = 4n(n-1)$.

For each $p \geq 0$, the p-sun tree, denoted by Su_p, is the tree of order $n = 2p+1$ formed by taking the star on $p+1$ vertices and subdividing each edge. For $p, q \geq 0$, the (p, q)-double sun, denoted by $DSu_{p,q}$, is the tree of order $n = 2(p+q+1)$ obtained by connecting the centers of DSu_p and DSu_q with an edge [11].

Proposition 2.13. *Let $G \cong Su_p$, where $p \geq 3$ be a sun graph of $2p+1$ vertices. Then,*

i. $UPM_1(G) = 5p$,

ii. $UPM_2(G) = 2p$,

iii. $UPM_1^*(G) = 4p$,

iv. $UPF(G) = 9p$.

Proposition 2.14. *Let $G \cong Su_{p,q}$, where $p, q \geq 2$ be a double sun graph of $2(p+q+1)$ vertices. Then,*

i. $UPM_1(G) = \begin{cases} 26p+2 & if \quad p = q; \\ 13p+5q+1 & if \quad p < q. \end{cases}$

ii. $UPM_2(G) = \begin{cases} 16p+1 & if \quad p = q; \\ 8p+3q+1 & if \quad p < q. \end{cases}$

iii. $UPM_1^*(G) = \begin{cases} 16p+2 & if \quad p = q; \\ 8p+3q+1 & if \quad p < q. \end{cases}$

iv. $UPF(G) = \begin{cases} 70p+2 & if \quad p = q; \\ 35p+9q+1 & if \quad p < q. \end{cases}$

The central graph of a graph G is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G and denoted by $C(G)$ [2].

Proposition 2.15. *Let $G \cong C(P_n)$, where $n \geq 4$ be a central graph of a path with $2n-1$ vertices. Then,*

i. $UPM_1(G) = n(n-1)(2n-1)$,

ii. $UPM_2(G) = \frac{n^4-n^3+n^2-3n+2}{2}$,

iii. $UPM_1^*(G) = n^3 - n$,

iv. $UPF(G) = n(n-1)((n-1)^2 + n^2)$.

Proof. From the definition of the central graph of a graph obviously in $C(P_n)$ there are n vertices of uphill degree $n-1$ and $n-1$ vertices of uphill degree n. So,

$$UPM_1(G) = n(n-1)(2n-1).$$

Then clearly,

$$UPF(G) = n(n-1)((n-1)^2 + n^2).$$

Type	Number of edges
$E_{n-1,n}$	$2(n-1)$
$E_{n-1,n-1}$	$\frac{2(n-2)+(n-2)(n-3)}{2}$

Table 2. Edge partition of $C(P_n)$ graph based on uphill degree of end vertices.

Now, by using the partition in Table 2, we get

$$\begin{aligned} UPM_2(G) &= 2n\,(n-1)^2 + \frac{(n-1)^3\,(n-2)}{2} \\ &= \frac{n^4 - n^3 + n^2 - 3n + 2}{2}. \end{aligned}$$

By using the same partition in Table 2, we get

$$\begin{aligned} UPM_1^*(G) &= 2\,(2n-1)\,(n-1) + (2n-2)\,\frac{2\,(n-2) + (n-2)\,(n-3)}{2} \\ &= n^3 - n. \end{aligned}$$

□

Proposition 2.16. *Let $G \cong C(C_n)$, where $n \geq 5$ be a central graph of a cycle with $2n$ vertices. Then,*

i. $UPM_1(G) = n(n-1)^2 + n^3$,

ii. $UPM_2(G) = 2n^2(n-1) + \frac{n(n-3)(n-1)^2}{2}$,

iii. $UPM_1^*(G) = n^3 + n$,

iv. $UPF(G) = n(n-1)^3 + n^4$.

Proposition 2.17. *For any graph $G \cong C(K_n)$,*

i. $UPM_1(G) = UPM_1^*(G) = \frac{UPF(G)}{2}$, *ii.* $UPM_2(G) = 0$.

Proof. Clearly, there are $\frac{n(n-1)}{2}$ vertices of uphill degree 2 and all other vertices of uphill degree zero. Also there are $n(n-1)$ edges of type $E_{2,0}$. Then we have

$$UPM_1 = 2n(n-1),$$

$$UPF(G) = 4n(n-1),$$

$$UPM_1^*(G) = 2n(n-1),$$

$$UPM_2(G) = 0.$$

□

An (r, s) banana tree denoted by $B_{r,s}$, is a graph obtained by connecting one leaf of each of r copies of an star graph of s vertices with a single root vertex that is distinct from all the stars [6].

Theorem 2.3. *Let $G \cong B_{r,s}$, where $s \geq 4$ be a banana tree graph with $rs + 1$ vertices. Then,*

i. $UPM_1(G) = \begin{cases} 2s + 44 & if \quad r = 2; \\ r(s+2) & if \quad r \geq 3. \end{cases}$

ii. $UPM_2(G) = \begin{cases} 32 & if \quad r = 2; \\ 0 & if \quad r \geq 3. \end{cases}$

iii. $UPM_1^*(G) = \begin{cases} rs - 2r + 24 & if \quad r = 2; \\ rs + 2r & if \quad r \geq 3. \end{cases}$

iv. $UPF(G) = \begin{cases} 2s + 188 & if \quad r = 2; \\ 8r + s - 2 & if \quad r \geq 3. \end{cases}$

Proof. Let $G \cong B_{r,s}$, where $s \geq 4$ be a banana tree graph with $rs + 1$ vertices. Then,

i. We have two cases:

Case 1. If $r = 2$, the graph G has two vertices of uphill degree zero, three vertices of uphill degree four and $2(s-2)$ vertices of uphill degree one, then $UPM_1(G) = 2s + 44$.

Case 2. If $r \geq 3$, there are $r + 1$ vertices of uphill degree zero, r vertices of uphill degree two and $r(s-2)$ vertices of uphill degree one, then $UPM_1(G) = r(s+2)$.

ii. We have two cases:

Case 1. If $r = 2$, it has two edges of the type $E_{4,4}$, two edges of the type $E_{0,4}$ and there are $r(s-2)$ edges in G, where each edge of the type $E_{1,0}$, then $UPM_2(G) = 32$.

Case 2. If $r \geq 3$, there are $2r$ edges where each edge of the type $E_{0,2}$. Also, there are $r(s-2)$ edges where each edge of the type $E_{1,0}$, then $UPM_2(G) = 0$.

iii. As part ii, we get $UPM_1^*(G) = rs + 2r$ if $r \geq 3$ and if $r = 2$, $UPM_1^*(G) = rs - 2r + 24$.

iv. In the same way as part i.

□

A firecracker graph $F_{r,s}$ is a graph obtained by the concatenation of n stars, each consists of s vertices, by linking one leaf from each star [6].

Theorem 2.4. *Let $G \cong F_{r,s}$, where $s \geq 5$ be a firecracker graph with rs vertices. Then,*

i. $UPM_1(G) = \begin{cases} 2S + 14 & if \quad r = 2; \\ 2(2r-3)^2 + (r-2)(2r-5)^2 + r(s-2) & if \quad r \geq 3. \end{cases}$

ii. $UPM_2(G) = \begin{cases} 9 & if \quad r = 2; \\ 2(2r-3)(2r-5) + (r-3)(2r-5)^2 & if \quad r \geq 3. \end{cases}$

iii. $UPM_1^*(G) = \begin{cases} 2S + 8 & if \quad r = 2; \\ 6r^2 + rs - 21r + 18 & if \quad r \geq 3. \end{cases}$

iv. $UPF(G) = \begin{cases} 50 + 2s & if \quad r = 2; \\ 2(2r-3)^3 + (r-2)(2r-5)^3 + r(s-2) & if \quad r \geq 3. \end{cases}$

Proof. Let $G \cong F_{r,s}$, where $s \geq 5$ be a firecracker graph with rs vertices. Then,

i. There are two cases:

Case 1. If $r = 2$, the graph G has two vertices of uphill degree zero, two vertices of uphill degree three and $2(s-2)$ vertices of uphill degree one, then $UPM_1(G) = 2s + 14$.

Case 2. If $r \geq 3$, there are two vertices of uphill degree $2r-3$, $r-2$ vertices of uphill degree $2r-5$,r vertices of uphill degree zero and $r(s-2)$ vertices of uphill degree one, then $UPM_1(G) = 2(2r-3)^2 + (r-2)(2r-5)^2 + r(s-2)$.

ii. There are two cases:

Case 1. If $r = 2$, the graph G has $2(s-2)$ edges of type $E_{1,0}$, two edges of the type $E_{0,3}$ and one edge of type $E_{3,3}$. So, $UPM_2(G) = 9$.

Case 2. If $r \geq 3$, in this case, there are $r(s-2)$ of type $E_{1,0}$, $r-2$ edges of the type $E_{0,2r-5}$, two edges of the type $E_{0,2r-3}$, two edges of the type $E_{2r-3,2r-5}$ and $r-3$ edges of type $E_{2r-5,2r-5}$, then $UPM_2(G) = 2(2r-3)(2r-5) + (r-3)(2r-5)^2$.

iii. As the method in ii.

iv. As the method in i.

□

Book graph is a Cartesian product of a star and single edge, denoted by B_r. The r-book graph is defined as the graph Cartesian product $S_{r+1} \times P_2$, where S_{r+1} is a star graph and P_2 is the path graph. The stacked book graph of order (r, t) is defined as the graph Cartesian product $S_{r+1} \times P_t$, where S_r is a star graph and P_t is the path graph on t nodes, and it is denoted by $B_{r,t}$ [17].

Proposition 2.18. *Let $G \cong B_r$, where $r \geq 2$ be a book graph of $2(r + 1)$ vertices. Then,*

i. $UPM_1(G) = 2(9r+1)$,
ii. $UPM_2(G) = 15r+1$,
iii. $UPM_1^*(G) = 14r+2$,
iv. $UPF(G) = 2(27r+1)$.

Proof. Let $G \cong B_r$, where $r \geq 2$ be a book graph of $2(r+1)$ vertices. There are two vertices of uphill degree one and $2r$ vertices of uphill degree three. Then,

$$UPM_1(G) = 2(9r+1),$$

In the same way,

$$UPF(G) = 2(27r+1).$$

The graph G has three kinds of edges, one edge of the type $E_{1,1}$, $2r$ edges of the type $E_{3,1}$ and r edges of the type $E_{3,3}$. Then,

$$UPM_2(G) = 15r+1.$$

Also,

$$UPM_1^*(G) = 14r+2.$$

□

Theorem 2.5. *Let $G \cong B_{r,t}$, where $r \geq 2$ and $t \geq 3$ be a stacked book graph with $t(r+1)$ vertices. Then,*

i. $UPM_1(G) = 2r(2t-3)^2 + (t-2)[(t-3)^2 + 2(t-2) + r(2t-5)^2]$,
ii. $UPM_2(G) = 2r(2t-3)(3t-7) + (t-3)[2(t-2) + (t-3)^2 + r(2t-5)^2 + r(t-2)(2t-5)]$,
iii. $UPM_1^*(G) = -22rt + 20r + 2t^2 - 8t + 7rt^2 + 8$,
iv. $UPF(G) = 2r(2t-3)^3 + (t-2)[(t-3)^3 + 2(t-2) + r(2t-5)^3]$.

Proof. Let $G \cong B_{r,t}$, where $r \geq 2$ and $t \geq 3$, be a stacked book graph with $t(r+1)$ vertices. In Figure 1, we can see the graph G has $2r$ vertices are labeled by $(v_{1,1}, v_{2,1}, ..., v_{r,1})$ and $(v_{1,t}, v_{2,t}, ..., v_{r,t})$ of uphill degree $2t-3$, $r(t-2)$ vertices are labeled by $(v_{1,2}, v_{1,3}, ..., v_{1,t-1}), (v_{2,2}, v_{2,3}, ..., v_{2,t-1}), ..., (v_{r,2}, v_{r,3}, ..., v_{r,t-1})$ of uphill degree $2t-5$, two vertices are labeled by $(v_{0,1})$ and $(v_{0,t})$ of uphill degree $t-2$, $t-2$ vertices are labeled by $(v_{0,2}, v_{0,3}, ..., v_{0,t-1})$ of uphill degree $t-3$. Then,

$$UPM_1(G) = 2r(2t-3)^2 + (t-2)[(t-3)^2 + 2(t-2) + r(2t-5)^2].$$

Similarly,

$$UPF(G) = 2r(2t-3)^3 + (t-2)[(t-3)^3 + 2(t-2) + r(2t-5)^3].$$

There are 6 types of edges.

Type	Number of edges
$E_{2t-3,2t-5}$	$2r$
$E_{2t-3,t-2}$	$2r$
$E_{t-2,t-3}$	2
$E_{t-3,t-3}$	$t-3$
$E_{2t-5,2t-5}$	$r(t-3)$
$E_{t-3,2t-5}$	$r(t-2)$

Table 3. Edge partition of $B_{r,t}$ graph based on uphill degree of end vertices.

In Figure 1, the types of edges, $E_{2t-3,2t-5}$, $E_{2t-3,t-2}$, $E_{t-2,t-3}$, $E_{t-3,t-3}$, $E_{2t-5,2t-5}$ and $E_{t-3,2t-5}$ are colored by green, purple, red, yellow, blue and black, respectively.

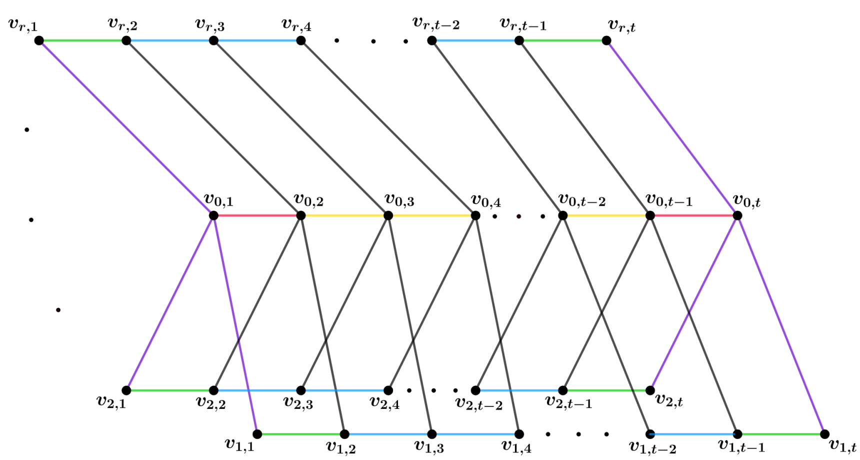

Figure 1. Stacked book graph $B_{r,t}$

Now, by using the partition in Table 3, we get

$$UPM_2(G) = 2r(2t-3)(3t-7) + (t-3)[2(t-2) + (t-3)^2 + r(2t-5)^2 + r(t-2)(2t-5)].$$

Also,

$$UPM_1^*(G) = -22rt + 20r + 2t^2 - 8t + 7rt^2 + 8.$$

□

A firefly graph $F_{a,b,c}$ is a graph of $n = 2a + 2b + c + 1$ vertices that consists of c pendant edges, a triangles, and b pendant paths of length 2, all of them sharing a common vertex [5].

Proposition 2.19. *Let $G \cong F_{a,b,c}$ be the firefly graph with $2a + 2b + c + 1$ vertices. Then,*

i. $UPM_1(G) = 8a + 5b + c$,
ii. $UPM_2(G) = 2(2a + b)$,
iii. $UPM_1^*(G) = 8a + 4b + c$,
iv. $UPF(G) = 16a + 9b + c$.

Proof. Let $G \cong F_{a,b,c}$ be the firefly graph with $2a + 2b + c + 1$ vertices. In Figure 2, we can see the graph G has one uphill isolated vertex, $2a$ vertices are labeled by $(v_1, v_2, ..., v_{2a})$ of uphill degree two, b vertices are labeled by $(u_1, u_2, ...u_b)$ of uphill degree one, b vertices are labeled by $(u'_1, u'_2, ..., u'_b)$ of uphill degree two and c vertices are labeled by $(w_1, w_2, ...w_c)$ of uphill degree one. Then,

$$UPM_1(G) = 8a + 5b + c.$$

Similarly,

$$UPF(G) = 16a + 9b + c.$$

There are 4 types of edges.

Type	Number of edges
$E_{2,2}$	a
$E_{2,0}$	$2a$
$E_{1,0}$	$b + c$
$E_{2,1}$	b

Table 4. Edge partition of $F_{a,b,c}$ graph based on uphill degree of end vertices.

In Figure 2, the types of edges, $E_{2,2}, E_{2,0}, E_{1,0}$ and $E_{2,1}$ are colored by blue, red, black blue and green, respectively.

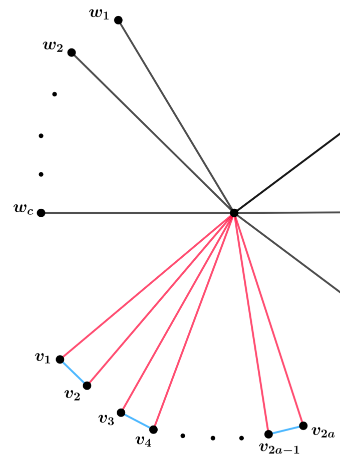

Figure 2. Firefly graph $F_{a,b,c}$

Now, by using the edge partition in Table 4, we get

$$UPM_2(G) = 2(2a + b),$$

and

$$UPM_1^*(G) = 8a + 4b + c.$$

$\square$

Corollary 2.3. *Let $G \cong BF$, be the butterfly graph with $2a + c + 1$ vertices. Then,*

i. $UPM_1(G) = UPM_1^*(G) = 8a + c$,

ii. $UPM_2(G) = 4a$,

iii. $UPF(G) = 16a + c$.

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