International Journal of Analysis and Applications
Volume 19, Number 1 (2021), 123-137
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-19-2021-123

ON SUBORDINATION RESULTS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF RECIPROCAL ORDER

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Abstract. In this paper, we introduce the subclass $S_{\beta}(\alpha,\lambda)$ of analytic functions and obtain coefficient inequality for functions belong to this class. Furthermore, we give sufficient conditions for starlikeness of reciprocal order of analytic functions. In the last part, we obtain the subordination results of a new subclass of analytic functions of reciprocal order, which are defined here by means of a Hadamard product of analytic functions. The results presented in this work improve or generalize the recent works of other authors and also give rise to several new results.

1. Introduction and Preliminaries

Let $U = \{z : |z| < 1\}$. We denote by$A$ the class of analytic functions on the unit disc U having the following Taylor series representation:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Let S denote the subclass of A consisting of all analytic functions $f(z)$ which are also univalent in U.

A function $f \in A$ is said to be starlike of order α if it satisfies

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \ \ (0 \le \alpha < 1,\ z \in U). \tag{1.2}$$

Received October 12$^{\text{th}}$, 2020; accepted November 18$^{\text{th}}$, 2020; published December 17$^{\text{th}}$, 2020.

2010 *Mathematics Subject Classification.* 30C45, 30C80.

Key words and phrases. analytic function; starlike function; convex function; analytic function of reciprocal order; subordinating factor sequence; hadamard product.

We denote by $S^{*}(\alpha)$ the subclass of A consisting of functions which are starlike of order α in U.

A function $f \in A$ is said to be starlike of reciprocal order α in U if

$$\operatorname{Re}\left(\frac{f(z)}{zf'(z)}\right) > \alpha \ \ (z \in U) \tag{1.3}$$

for some $\alpha\,(0 \leq \alpha < 1)$. We denote the class such functions by $S^{-1_{*}}(\alpha)$ ([1], [4], [10]).

A function $f \in A$ is said to be convex of order α in U if it satisfies the condition

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha\,(z \in U) \tag{1.4}$$

for some $0 \leq \alpha < 1$.We denote by $K(\alpha)$ the subclass of A consisting of functions which are convex of orderα in U. Furthermore, a function $f \in A$ is said to be convex of reciprocal order α in U if

$$\operatorname{Re}\left(\frac{1}{1 + \frac{zf''(z)}{f'(z)}}\right) > \alpha \ \ (0 \leq \alpha < 1, \ z \in U) \tag{1.5}$$

We denote the class such functions by $K^{-1}(\alpha)$ [10]. Clearly, we have $S^{*}(\alpha) \subseteq S^{*}(0) = S^{*}$,$K(\alpha) \subseteq K(0) = K$ and $f(z) \in K(\alpha)$ if and only if $zf^{'}(z) \in S^{*}(\alpha)$ for $0 \leq \alpha < 1$.

For $|\beta| < \frac{\pi}{2}$ and $0 \leq \alpha < 1$, a function $f \in A$ is said to be $\beta-$spirallike of order α in U if it satisfies

$$\operatorname{Re}\left(e^{i\beta}\frac{zf^{'}(z)}{f(z)}\right) > \alpha\cos\beta. \tag{1.6}$$

The class of all such functions is denote by $S_{\beta}(\alpha)$ ([8], [10]).

Definition 1.1. *Let $H(z) = \frac{zf^{'}(z)}{f(z)}$ for $f(z) \in S$. A function $f(z) \in S$ is said to be in the class denote by $S_{\beta}(\alpha)$ if it satisfies the inequality*

$$\left|\frac{1}{e^{i\beta}H(z)} - \frac{1}{2\alpha}\right| < \frac{1}{2\alpha} \tag{1.7}$$

for some real β and $0 < \alpha < 1$.

Owa et al. [9] gave the following coefficient inequality for the function class $S_{\beta}(\alpha)$.

Theorem 1.1. *[9] If $f(z) \in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{n + \left|n - 2\alpha e^{-i\beta}\right|\right\}|a_n| \leq 1 - \left|1 - 2\alpha e^{-i\beta}\right| \tag{1.8}$$

for some real $|\beta| < \frac{\pi}{2}$ and $0 < \alpha < \cos\beta$, then $f(z) \in S_{\beta}(\alpha)$.

For $f \in A$, Salagean [2] has introduced the following operator called the Salagean operator:

$$\begin{aligned}
&D^{0}f(z) = f(z)\\
&D^{1}f(z) = Df(z) = zf^{'}(z) = z + \textstyle\sum_{n=2}^{\infty} na_n z^n\\
&\vdots\\
&D^{\Omega}f(z) = D\left(D^{\Omega-1}f(z)\right) = z + \textstyle\sum_{n=2}^{\infty} n^{\Omega}a_n z^n
\end{aligned}$$

where $\Omega \in \mathbb{N}_0 = \{0, 1, 2, ...\}$.

Definition 1.2. *[5] A function $f(z) \in A$ is said to be in the class $M(\alpha, \lambda, \Omega)$ if it satisfies the inequality*

$$\left| \frac{(1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)}{(1-\lambda) z \left(D^{\Omega} f(z)\right)' + \lambda z \left(D^{\Omega+1} f(z)\right)'} - \frac{1+\lambda}{2\alpha} \right| < \frac{1+\lambda}{2\alpha} \tag{1.9}$$

for $0 < \alpha < 1$, $0 \leq \lambda < 1$, $\Omega \in \mathbb{N}_0$ and $z \in U$.

M.Kamali [5] gave the following coefficients inequality for the function class $M(\alpha, \lambda, \Omega)$.

Theorem 1.2. *[5] Let $0 < \alpha < 1$ and $0 \leq \lambda < 1$. If $f(z) \in A$ satisfies the following coefficient inequality:*

$$\begin{aligned} & \sum_{n=2}^{\infty} n^{\Omega} (\lambda n + 1 - \lambda) \{|2\alpha - (1+\lambda) n| + (1+\lambda) n\} |a_n| \\ \leq\ & (1+\lambda) - |2\alpha - (1+\lambda)| \\ =\ & \begin{cases} 2\alpha; & 0 < \alpha \leq \frac{1+\lambda}{2} \\ 2(1+\lambda-\alpha); & \frac{1+\lambda}{2} \leq \alpha < 1+\lambda \end{cases} \end{aligned} \tag{1.10}$$

then $f(z) \in M(\alpha, \lambda, \Omega)$.

2. Some Results and Coefficient Inequality for Functions in the Class $S_\beta(\alpha, \lambda)$

Definition 2.1. *Let $G(z) = \frac{(1-\lambda) z \left(D^{\Omega} f(z)\right)' + \lambda z \left(D^{\Omega+1} f(z)\right)'}{(1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)}$ for $f(z) \in S$. A function $f(z) \in S$ is said to be in the class denote by $S_\beta(\alpha, \lambda)$ if it satisfies the inequality*

$$\left| \frac{1}{e^{i\beta} G(z)} - \frac{1+\lambda}{2\alpha} \right| < \frac{1+\lambda}{2\alpha} \tag{2.1}$$

for some real β and $0 < \alpha < 1$, $0 \leq \lambda < 1$, $\Omega \in \mathbb{N}_0$, $z \in U$.

Theorem 2.1. *If $f(z) \in S_\beta(\alpha, \lambda)$ iff*

$$\operatorname{Re}\left\{ e^{i\beta} \frac{(1-\lambda) z \left(D^{\Omega} f(z)\right)' + \lambda z \left(D^{\Omega+1} f(z)\right)'}{(1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)} \right\} > \frac{\alpha}{1+\lambda}. \tag{2.2}$$

Proof. Let $G(z) = \frac{(1-\lambda) z \left(D^{\Omega} f(z)\right)' + \lambda z \left(D^{\Omega+1} f(z)\right)'}{(1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)}$ for$f(z) \in S$. If $f(z) \in S_\beta(\alpha, \lambda)$,we can write

$$\left| \frac{1}{e^{i\beta} G(z)} - \frac{1+\lambda}{2\alpha} \right| < \frac{1+\lambda}{2\alpha}.$$

Then, we can obtain

$$\left| \tfrac{1}{e^{i\beta} G(z)} - \tfrac{1+\lambda}{2\alpha} \right| < \tfrac{1+\lambda}{2\alpha} \Leftrightarrow \left| \tfrac{2\alpha - (1+\lambda) e^{i\beta} G(z)}{2\alpha e^{i\beta} G(z)} \right|^2 < \left(\tfrac{1+\lambda}{2\alpha}\right)^2$$
$$\Leftrightarrow \left[2\alpha - (1+\lambda) e^{i\beta} G(z)\right] \left[\overline{2\alpha - (1+\lambda) e^{i\beta} G(z)}\right] < (1+\lambda)^2 \left[e^{i\beta} G(z)\right] \left[\overline{e^{i\beta} G(z)}\right]$$

$$\Leftrightarrow 2\alpha - 2(1+\lambda)\, Re\left[e^{i\beta} G(z)\right] < 0$$
$$\Leftrightarrow \operatorname{Re}\left[e^{i\beta} G(z)\right] > \tfrac{\alpha}{1+\lambda}$$
$$\Leftrightarrow \operatorname{Re}\left\{ e^{i\beta} \tfrac{(1-\lambda)\left(D^{\Omega} f(z)\right)' + \lambda\left(D^{\Omega+1} f(z)\right)'}{(1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)} \right\} > \tfrac{\alpha}{1+\lambda}.$$

□

Theorem 2.2. *If $f(z) \in A$ satisfies*

$$\begin{aligned} &\sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)n+\left|(1+\lambda)n-2\alpha e^{-i\beta}\right|\right\}|a_n| \\ (2.3) \qquad \leq\ & (1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right| \end{aligned}$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{1+\lambda}<\cos\beta$, then $f(z)\in S_\beta(\alpha,\lambda)$.

Proof. It suffices to show that

$$\left|\frac{2\alpha e^{-i\beta}-(1+\lambda)G(z)}{(1+\lambda)G(z)}\right|<1$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{1+\lambda}<\cos\beta$, where $G(z)=\frac{(1-\lambda)z\left(D^{\Omega}f(z)\right)'+\lambda z\left(D^{\Omega+1}f(z)\right)'}{(1-\lambda)(D^{\Omega}f(z))+\lambda(D^{\Omega+1}f(z))}$.

Note that

$$\begin{aligned} &\left|\frac{2\alpha e^{-i\beta}-(1+\lambda)G(z)}{(1+\lambda)G(z)}\right| \\ &=\left|\frac{2\alpha e^{-i\beta}\left\{z+\sum_{n=2}^{\infty}n^{\Omega}(1-\lambda+\lambda n)a_nz^n\right\}-(1+\lambda)\left\{z+\sum_{n=2}^{\infty}n^{\Omega+1}(1-\lambda+\lambda n)a_nz^n\right\}}{(1+\lambda)\left\{z+\sum_{n=2}^{\infty}n^{\Omega+1}(1-\lambda+\lambda n)a_nz^n\right\}}\right| \\ (2.4) \qquad &\leq\frac{\left|(1+\lambda)-2\alpha e^{-i\beta}\right|+\sum_{n=2}^{\infty}n^{\Omega}(1-\lambda+\lambda n)\left|(1+\lambda)n-2\alpha e^{-i\beta}\right||a_n||z|^{n-1}}{(1+\lambda)\left\{1-\sum_{n=2}^{\infty}n^{\Omega+1}(1-\lambda+\lambda n)\delta_n|a_n||z|^{n-1}\right\}} \\ &<\frac{\left|(1+\lambda)-2\alpha e^{-i\beta}\right|+\sum_{n=2}^{\infty}n^{\Omega}(1-\lambda+\lambda n)\left|(1+\lambda)n-2\alpha e^{-i\beta}\right||a_n|}{(1+\lambda)\left\{1-\sum_{n=2}^{\infty}n^{\Omega+1}(1-\lambda+\lambda n)|a_n|\right\}}. \end{aligned}$$

Therefore, if

$$\sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)n+\left|(1+\lambda)n-2\alpha e^{-i\beta}\right|\right\}|a_n|\leq(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{1+\lambda}<\cos\beta$, then

$$\begin{aligned} &\sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left|(1+\lambda)n-2\alpha e^{-i\beta}\right||a_n| \\ \leq\ &(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|-\sum_{n=2}^{\infty}(1-\lambda+\lambda n)(1+\lambda)n^{\Omega+1}|a_n|. \end{aligned}$$

Using the inequality in (2.4), we obtain

$$\left|\frac{2\alpha e^{-i\beta}-(1+\lambda)G(z)}{(1+\lambda)G(z)}\right|<1.$$

Therefore, $f(z)\in S_\beta(\alpha,\lambda)$ for some $|\beta|<\frac{\pi}{2}$ and $0<\frac{\alpha}{1+\lambda}<\cos\beta$. □

Taking $\lambda=0$ and $\Omega=0$ in Theorem 2.2, we get Corollary 2.1 given by Owa et al. [9].

Corollary 2.1. *If $f(z)\in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{n+\left|n-2\alpha e^{-i\beta}\right|\right\}|a_n|\leq 1-\left|1-2\alpha e^{-i\beta}\right|$$

for some $|\beta|<\frac{\pi}{2}$ and $0<\alpha<\cos\beta$, then $f(z)\in S_\beta(\alpha)$.

Taking $\beta = \frac{\pi}{4}$ in Theorem 2.2, we have Corollary 2.2.

Corollary 2.2. *[9] If $f(z) \in A$ satisfies*

$$\begin{aligned}&\sum_{n=2}^{\infty}(1-\lambda+\lambda n)\left\{(1+\lambda)n+\sqrt{(1+\lambda)^2 n^2 - 2\sqrt{2}(1+\lambda)n\alpha+4\alpha^2}\right\}n^{\Omega}|a_n| \\ \leq\ & (1+\lambda)-\sqrt{(1+\lambda)^2-2\sqrt{2}(1+\lambda)\alpha+4\alpha^2}\end{aligned}$$

for some $0<\frac{\alpha}{1+\lambda}<\frac{\sqrt{2}}{2}$, then $f(z)\in S_{\frac{\pi}{4}}(\alpha,\lambda)$.

3. Sufficient Conditions for Starlikeness of Reciprocal Order

First we give following example.

Example 3.1. *Let us define the function $f(z)$ by*

$$f(z) = ze^{\left(1-\frac{\alpha}{1+\lambda}\right)z} \quad (z \in U) \tag{3.1}$$

with $0<\alpha<1, 0\leq\lambda<1$.

From (3.1) after taking the logarithmical differentiation we have that

$$\ln f(z) = \ln z + \left(1-\frac{\alpha}{1+\lambda}\right)z \Rightarrow \frac{f'(z)}{f(z)} = \frac{1}{z}+\left(1-\frac{\alpha}{1+\lambda}\right)$$

$$\Rightarrow \frac{zf'(z)}{f(z)} = 1+\left(1-\frac{\alpha}{1+\lambda}\right)z.$$

This gives us that

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \operatorname{Re}\left\{1+\left(1-\frac{\alpha}{1+\lambda}\right)z\right\} > \frac{\alpha}{1+\lambda}\,(z\in U).$$

Therefore , we see that $f(z)\in S^*\left(\frac{\alpha}{1+\lambda}\right)$.

Furthermore, we have that

$$\frac{zf'(z)}{f(z)} = 1+\left(1-\frac{\alpha}{1+\lambda}\right)z \Rightarrow \frac{f(z)}{zf'(z)} = \frac{1}{1+\left(1-\frac{\alpha}{1+\lambda}\right)z}.$$

It follows that

$$\frac{f(z)}{zf'(z)} = 1 \quad (z=0)$$

and

$$\operatorname{Re}\left\{\frac{f(z)}{zf'(z)}\right\} = \operatorname{Re}\left\{\frac{1}{1+\left(1-\frac{\alpha}{1+\lambda}\right)e^{i\theta}}\right\} > \frac{1+\lambda}{2(1+\lambda)-\alpha}\left(z=e^{i\theta}\right). \tag{3.2}$$

Therefore, we conclude that $f(z)\in S^*\left(\frac{\alpha}{1+\lambda}\right)$ and starlike of reciprocal order $\frac{1+\lambda}{2(1+\lambda)-\alpha}$ in U.

In order to establish our main results, we require the following lemma due to Nunokama et al. [6].

Lemma 3.1. *Let $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be analytic in U and suppose that there exists a point $z_0 \in U$ such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and*

$$\operatorname{Re}\{p(z_0)\} = 0.$$

Then we have

$$z_0 p'(z_0) \le -\frac{1}{2}\left(1 + |p(z_0)|^2\right), \tag{3.3}$$

where $z_0 p'(z_0)$ is a negative real number.

Theorem 3.1. *Let $F(z) = (1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right) \in A$ satisfies $F(z)F'(z) \neq 0$ in $0 < |z| < 1$ and*

$$\begin{aligned} &\operatorname{Re}\left[\frac{(1-\lambda)\left(D^{\Omega} f(z)\right)+\lambda\left(D^{\Omega+1} f(z)\right)}{(1-\lambda)(D^{\Omega+1} f(z))+\lambda(D^{\Omega+2} f(z))}\right. \\ &\left.\times\left\{1-\frac{\alpha}{1+\lambda}\frac{\lambda\left(D^{\Omega+3} f(z)\right)+(1-2\lambda)\left(D^{\Omega+2} f(z)\right)-(1-\lambda)\left(D^{\Omega+1} f(z)\right)}{(1-\lambda)(D^{\Omega+1} f(z))+\lambda(D^{\Omega+2} f(z))}\right\}\right] \\ &> -\frac{\alpha}{2(1+\lambda)}\left\{3+\left|\frac{(1-\lambda)\left(D^{\Omega} f(z)\right)+\lambda\left(D^{\Omega+1} f(z)\right)}{(1-\lambda)(D^{\Omega+1} f(z))+\lambda(D^{\Omega+2} f(z))}\right|^2\right\} \quad (\alpha \ge 0). \end{aligned} \tag{3.4}$$

Then $F(z)$ is starlike of reciprocal order 0 in U and thus, $F(z)$ is starlike in U.

Proof. Let us define the function $p(z)$ by

$$p(z) = \frac{F(z)}{zF'(z)}. \tag{3.5}$$

Then $p(z)$ is analytic in U and $p(0) = 1$. Differentiating (3.5) logarithmically we obtain

$$\begin{aligned} \frac{p'(z)}{p(z)} &= \frac{F'(z)}{F(z)} - \frac{1}{z} - \frac{F''(z)}{F'(z)} \Rightarrow \frac{\alpha}{1+\lambda} z \frac{P'(z)}{P(z)} = \left\{\frac{F'(z)}{F(z)} - \frac{1}{z} - \frac{F''(z)}{F'(z)}\right\}\frac{\alpha}{1+\lambda} z \\ &\Rightarrow \frac{\alpha}{1+\lambda} zP'(z) + \frac{\alpha}{1+\lambda} P(z) = \left\{\frac{zF'(z)}{F(z)} - \frac{zF''(z)}{F'(z)}\right\}\frac{\alpha}{1+\lambda} P(z) \\ &\Rightarrow \frac{\alpha}{1+\lambda} zP'(z) + \left(\frac{\alpha}{1+\lambda} + 1\right) P(z) - \frac{\alpha}{1+\lambda} \\ &= \left\{1 - \frac{\alpha}{1+\lambda}\frac{zF''(z)}{F'(z)}\right\} P(z) = \left\{1 - \frac{\alpha}{1+\lambda}\frac{z^2 F''(z)}{zF'(z)}\right\} P(z). \end{aligned}$$

Furthermore, we can write

$$F(z) = (1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right) \Rightarrow zF'(z) = (1-\lambda) z\left(D^{\Omega} f(z)\right)' + \lambda z\left(D^{\Omega+1} f(z)\right)'$$

$$\Rightarrow zF'(z) = (1-\lambda)\left(D^{\Omega+1} f(z)\right) + \lambda\left(D^{\Omega+2} f(z)\right)$$

and

$$\left(zF^{'}(z)\right)^{'} = (1-\lambda)\left(D^{\Omega+1}f(z)\right)^{'} + \lambda\left(D^{\Omega+2}f(z)\right)^{'}$$
$$\Rightarrow F^{'}(z) + zF^{''}(z) = (1-\lambda)\left(D^{\Omega+1}f(z)\right)^{'} + \lambda\left(D^{\Omega+2}f(z)\right)^{'}$$
$$\Rightarrow zF^{'}(z) + z^{2}F^{''}(z) = (1-\lambda)z\left(D^{\Omega+1}f(z)\right)^{'} + \lambda z\left(D^{\Omega+2}f(z)\right)^{'}$$
$$\Rightarrow z^{2}F^{''}(z) = (1-\lambda)\left(D^{\Omega+2}f(z)\right) + \lambda\left(D^{\Omega+3}f(z)\right) - (1-\lambda)\left(D^{\Omega+1}f(z)\right) - \lambda\left(D^{\Omega+2}f(z)\right)$$
$$\Rightarrow z^{2}F^{''}(z) = \lambda\left(D^{\Omega+3}f(z)\right) + (1-2\lambda)\left(D^{\Omega+2}f(z)\right) - (1-\lambda)\left(D^{\Omega+1}f(z)\right)$$

Thus,we obtain

$$\begin{aligned} & \frac{\alpha}{1+\lambda}zP^{'}(z) + \left(\frac{\alpha}{1+\lambda}+1\right)P(z) - \frac{\alpha}{1+\lambda} \\ = & \frac{(1-\lambda)\left(D^{\Omega}f(z)\right) + \lambda\left(D^{\Omega+1}f(z)\right)}{(1-\lambda)\left(D^{\Omega+1}f(z)\right) + \lambda\left(D^{\Omega+2}f(z)\right)} \\ & \times\left\{1 - \frac{\alpha}{1+\lambda}\frac{\lambda\left(D^{\Omega+3}f(z)\right) + (1-2\lambda)\left(D^{\Omega+2}f(z)\right) - (1-\lambda)\left(D^{\Omega+1}f(z)\right)}{(1-\lambda)\left(D^{\Omega+1}f(z)\right) + \lambda\left(D^{\Omega+2}f(z)\right)}\right\}. \end{aligned} \tag{3.6}$$

Suppose that there exists a point $z_0 \in U$ such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and

$$\operatorname{Re}\{p(z_0)\} = 0,$$

then from Lemma 3.1,we have,

$$z_0 p^{'}(z_0) \leq -\frac{1}{2}\left(1 + |p(z_0)|^{2}\right).$$

Therefore from (3.6), we have

$$\begin{aligned} & \operatorname{Re}\left[\frac{(1-\lambda)\left(D^{\Omega}f(z_0)\right) + \lambda\left(D^{\Omega+1}f(z_0)\right)}{(1-\lambda)\left(D^{\Omega+1}f(z_0)\right) + \lambda\left(D^{\Omega+2}f(z_0)\right)}\right. \\ & \left.\times\left\{1 - \frac{\alpha}{1+\lambda}\frac{\lambda\left(D^{\Omega+3}f(z_0)\right) + (1-2\lambda)\left(D^{\Omega+2}f(z_0)\right) - (1-\lambda)\left(D^{\Omega+1}f(z_0)\right)}{(1-\lambda)\left(D^{\Omega+1}f(z_0)\right) + \lambda\left(D^{\Omega+2}f(z_0)\right)}\right\}\right] \\ = & \operatorname{Re}\left\{\frac{\alpha}{1+\lambda}z_0P^{'}(z_0) + \left(\frac{\alpha}{1+\lambda}+1\right)P(z_0) - \frac{\alpha}{1+\lambda}\right\} \\ \leq & \frac{\alpha}{1+\lambda}\left\{-\frac{1}{2}\left(1+|p(z_0)|^{2}\right)\right\} - \frac{\alpha}{1+\lambda} = -\frac{\alpha}{2(1+\lambda)}\left\{3 + |p(z_0)|^{2}\right\} \\ = & -\frac{\alpha}{2(1+\lambda)}\left\{3 + \left|\frac{(1-\lambda)\left(D^{\Omega}f(z_0)\right) + \lambda\left(D^{\Omega+1}f(z_0)\right)}{(1-\lambda)\left(D^{\Omega+1}f(z_0)\right) + \lambda\left(D^{\Omega+2}f(z_0)\right)}\right|^{2}\right\}. \end{aligned}$$

which contradicts the hypothesis (3.4) of Theorem 3.1.Thus we complete the proof of Theorem 3.1. □

Taking $\lambda = 0$ and $\Omega = 0$ in Theorem 3.1, we get Corollary 3.1 given by B.A.Frasin and M.Ab.Sabri [1].

Corollary 3.1. *[1] Let $f(z) \in A$ satisfies $f(z)f^{'}(z) \neq 0$ in $0 < |z| < 1$ and*

$$\operatorname{Re}\left[\frac{zf^{'}(z)}{f(z)}\left\{1 - \alpha\frac{zf^{''}(z)}{f^{'}(z)}\right\}\right] > -\frac{\alpha}{2}\left\{3 + \left|\frac{f(z)}{zf^{'}(z)}\right|^{2}\right\} \quad (z \in U;\ \alpha \geq 0).$$

Then $f(z)$ is starlike of reciprocal order 0 in U and thus, $f(z)$ is starlike in U.

Theorem 3.2. *Let* $F(z) = (1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right) \in A$ *satisfies*

$$\begin{aligned} &\operatorname{Re}\left[\frac{F(z)}{zF'(z)}\left\{1-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\frac{zF''(z)}{F'(z)}\right\}\right] \\ &> -\frac{1}{2}\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\left|\frac{F(z)}{zF'(z)}-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right|^{2} \\ &+\frac{1}{2}\left\{3\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)^{2}-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right\} \end{aligned} \tag{3.7}$$

Then $F(z)$ *is starlike of reciprocal order* $\frac{1+\lambda}{2(1+\lambda)-\alpha}$ *in* U*.*

Proof. Let us define the function $\frac{F(z)}{zF'(z)}$ by

$$\frac{F(z)}{zF'(z)} = \left(\frac{(1+\lambda)-\alpha}{2(1+\lambda)-\alpha}\right)P(z) + \frac{1+\lambda}{2(1+\lambda)-\alpha};\ p(0)=1. \tag{3.8}$$

Differentiating (3.8) we obtain

$$\frac{1}{z} - \frac{F(z)}{z^2F'(z)} - \frac{F(z).F''(z)}{z\left[F'(z)\right]^2} = \left(\frac{(1+\lambda)-\alpha}{2(1+\lambda)-\alpha}\right)P'(z)$$

$$\begin{aligned} \Rightarrow\ &\left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right) - \left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\frac{F(z)}{zF'(z)} - \left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\frac{F(z).F''(z)}{\left[F'(z)\right]^2} \\ =\ &\left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\left(\frac{(1+\lambda)-\alpha}{2(1+\lambda)-\alpha}\right)zP'(z) \end{aligned}$$

$$\begin{aligned} \Rightarrow\ &\frac{F(z)}{zF'(z)}\left\{1-\left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\frac{zF''(z)}{F'(z)}\right\} \\ =\ &\left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\left(\frac{(1+\lambda)-\alpha}{2(1+\lambda)-\alpha}\right)zP'(z) \\ &+\left\{1+\left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)\right\}\left(\frac{(1+\lambda)-\alpha}{2(1+\lambda)-\alpha}\right)p(z) + \left(\frac{(1+\lambda)}{2(1+\lambda)-\alpha}\right)^{2}. \end{aligned} \tag{3.9}$$

Suppose that there exists a point $z_0 \in U$ such that $\operatorname{Re}\{p(z)\} > 0$ for $|z| < |z_0|$ and

$$\operatorname{Re}\{p(z_0)\} = 0,$$

then from Lemma 3.1,we have,

$$z_0 p'(z_0) \leq -\frac{1}{2}\left(1+|p(z_0)|^2\right).$$

Therefore from (3.9), we have

$$\begin{aligned} &\operatorname{Re}\left[\frac{F(z_0)}{z_0F'(z_0)}\left\{1-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\frac{z_0F''(z_0)}{F'(z_0)}\right\}\right] \\ \leq\ &-\frac{1}{2}\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\left(\frac{1+\lambda-\alpha}{2(1+\lambda)-\alpha}\right)\left\{1+|p(z_0)|^2\right\} + \left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)^{2} \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad Re\left[\frac{F(z_0)}{z_0F'(z_0)}\left\{1-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\frac{z_0F''(z_0)}{F'(z_0)}\right\}\right]\\
&\leq \quad -\frac{1}{2}\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\left(\frac{1+\lambda-\alpha}{2(1+\lambda)-\alpha}\right)|p(z_0)|^2\\
&\quad\quad +\frac{1}{2}\left\{3\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)^2-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right\}
\end{aligned}$$

and thus we write

$$\begin{aligned}
&\mathrm{Re}\left[\frac{F(z_0)}{z_0F'(z_0)}\left\{1-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\frac{z_0F''(z_0)}{F'(z_0)}\right\}\right]\\
\leq \quad &-\frac{1}{2}\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\left|\frac{F(z_0)}{z_0F'(z_0)}-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right|^2\\
&+\frac{1}{2}\left\{3\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)^2-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right\}
\end{aligned}$$

which contradicts the hypothesis (3.7) of Theorem 3.2. It follow that

$$\begin{aligned}
&\mathrm{Re}\left[\frac{F(z)}{zF'(z)}\left\{1-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\frac{zF''(z)}{F'(z)}\right\}\right]\\
> \quad &-\frac{1}{2}\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\left|\frac{F(z)}{zF'(z)}-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right|^2\\
&+\frac{1}{2}\left\{3\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)^2-\left(\frac{1+\lambda}{2(1+\lambda)-\alpha}\right)\right\}.
\end{aligned}$$

Thus we complete the proof of Theorem 3.2. □

Taking $\lambda = 0$ and $\Omega = 0$ in Theorem 3.2, we get the following Corollary 3.2.

Corollary 3.2. *Let $f(z) \in A$ satisfies*

$$\begin{aligned}
&\mathrm{Re}\left[\frac{f(z)}{zf'(z)}\left\{1-\left(\frac{1}{2-\alpha}\right)\frac{zf''(z)}{f'(z)}\right\}\right]\\
> \quad &-\frac{1}{2}\left(\frac{1}{2-\alpha}\right)\left|\frac{f(z)}{zf'(z)}-\left(\frac{1}{2-\alpha}\right)\right|^2+\frac{1}{2}\left\{3\left(\frac{1}{2-\alpha}\right)^2-\left(\frac{1}{2-\alpha}\right)\right\}
\end{aligned}$$

Then $f(z)$ is starlike of reciprocal order $\frac{1}{2-\alpha}$ in U.

4. Subordination Results and Coefficient Inequality for Functions in the Class $S^{-1}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)$

Definition 4.1. *(Hadamard product or Convolution)*. *The Hadamard product of two power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ analytic in U, is defined as then their Hadamard product (or convolution), $f * g$ is defined by the power series*

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n = (g * f)(z). \tag{4.1}$$

The functionf $*\, g$ is also analytic in U.

Tariq Al-Hawary and B.A.Frasin [10] introduce the following subclass of A by making use of the Hadamard product.

Definition 4.2. *[10] Let $\phi(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ be analytic in U, such that $\delta_n \geq 0, \mu_n \geq 0$ and $\delta_n \geq \mu_n$ for $n \geq 2$, we say that $f(z) \in A$ is in the class $S^{-1}(\phi, \psi; \alpha, \beta)$ if $f(z) * \phi(z) \neq 0$, $f(z) * \psi(z) \neq 0$ and*

$$\left| \frac{1}{e^{i\beta}\left(\frac{f(z)*\phi(z)}{f(z)*\psi(z)}\right)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha} \quad (\beta \in \mathbb{R}, 0 < \alpha < 1, \; z \in U). \tag{4.2}$$

By making use of the Hadamard product (4.1), we now introduce the following subclasses of A.

Definition 4.3. *Let $F(z) = (1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)$. Furthermore, let $\phi(z) = z + \sum_{n=2}^{\infty} \delta_n z^n$ and $\psi(z) = z + \sum_{n=2}^{\infty} \mu_n z^n$ be analytic in U, such that $\delta_n \geq 0, \mu_n \geq 0$ and $\delta_n \geq \mu_n$ for $n \geq 2$, we say that $f(z) \in A$ is in the class $S^{-1}_{(\Omega,\lambda)}(\phi, \psi; \alpha, \beta)$ if $F(z) * \phi(z) \neq 0$, $F(z) * \psi(z) \neq 0$ and*

$$\left| \frac{1}{e^{i\beta}\left(\frac{F(z)*\phi(z)}{F(z)*\psi(z)}\right)} - \frac{1+\lambda}{2\alpha} \right| < \frac{1+\lambda}{2\alpha} \quad (\beta \in \mathbb{R}, 0 < \alpha < 1, 0 \leq \lambda < 1, \; \Omega \in \mathbb{N}_0, \; z \in U). \tag{4.3}$$

Theorem 4.1. *Let $F(z) = (1-\lambda)\left(D^{\Omega} f(z)\right) + \lambda\left(D^{\Omega+1} f(z)\right)$. If $f(z) \in A$ satisfies*

$$\begin{aligned} & \sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)\delta_n + \left|(1+\lambda)\delta_n - 2\alpha e^{-i\beta}\mu_n\right|\right\}|a_n| \\ \leq \; & (1+\lambda) - \left|(1+\lambda) - 2\alpha e^{-i\beta}\right| \end{aligned} \tag{4.4}$$

for some that $|\beta| < \frac{\pi}{2}$ and that $0 < \frac{\alpha}{1+\lambda} < \cos\beta$, then $f(z) \in S^{-1}_{(\Omega,\lambda)}(\phi, \psi; \alpha, \beta)$.

Proof. It suffices to show that

$$\left| \frac{2\alpha}{(1+\lambda)e^{i\beta}\left(\frac{F(z)*\phi(z)}{F(z)*\psi(z)}\right)} - 1 \right| < 1 \Rightarrow \left| \frac{2\alpha e^{-i\beta}(F(z)*\psi(z)) - (1+\lambda)(F(z)*\phi(z))}{(1+\lambda)(F(z)*\phi(z))} \right| < 1.$$

We observe that

$$\begin{aligned} & \left| \frac{2\alpha e^{-i\beta}(F(z)*\psi(z)) - (1+\lambda)(F(z)*\phi(z))}{(1+\lambda)(F(z)*\phi(z))} \right| \\ = & \left| \frac{(1+\lambda)(F(z)*\phi(z)) - 2\alpha e^{-i\beta}(F(z)*\psi(z))}{(1+\lambda)(F(z)*\phi(z))} \right| \\ = & \left| \frac{\left\{(1+\lambda) - 2\alpha e^{-i\beta}\right\} z + \sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left[(1+\lambda)\delta_n - 2\alpha e^{-i\beta}\mu_n\right] a_n z^n}{(1+\lambda)\left\{z + \sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\delta_n a_n z^n\right\}} \right| \\ \leq & \frac{\left|(1+\lambda) - 2\alpha e^{-i\beta}\right| + \sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left|(1+\lambda)\delta_n - 2\alpha e^{-i\beta}\mu_n\right| |a_n| |z|^{n-1}}{(1+\lambda)\left\{1 - \sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\delta_n |a_n| |z|^{n-1}\right\}}. \end{aligned}$$

It follows that the last term is bounded by 1 if

$$\begin{aligned}&\sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)\delta_n+\left|(1+\lambda)\delta_n-2\alpha e^{-i\beta}\mu_n\right|\right\}|a_n| \\ \leq\ & (1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\end{aligned}$$

for some that $|\beta|<\frac{\pi}{2}$ and that $0<\frac{\alpha}{1+\lambda}<\cos\beta$. □

Taking $\lambda=0$ and $\Omega=0$ in Theorem 4.1, we get Corollary 4.1 given by Tariq Al-Hawary et al. [10].

Corollary 4.1. *[10] If $f(z)\in A$ satisfies*

$$\sum_{n=2}^{\infty}\left\{\delta_n+\left|\delta_n-2\alpha e^{-i\beta}\mu_n\right|\right\}|a_n|\leq 1-\left|1-2\alpha e^{-i\beta}\right|$$

for some that $|\beta|<\frac{\pi}{2}$ and that $0<\alpha<\cos\beta$, then $f(z)\in S^{-1}(\phi,\psi;\alpha,\beta)$.

Now, to proceed our subordination results in this section, let us first recall the following definition and Lemma.

Definition 4.4. *(Subordination Principle) Given two functions $f(z)$, $g(z)\in A$ in U, g be univalent in U ,$f(0)=g(0)$ and $f(U)\subset g(U)$, then we say that the function $f(z)$ is subordinate to $g(z)$ in U, and write $f(z)\prec g(z), z\in U$. Moreover, we say that $g(z)$ is superordinate to $f(z)$ in U.*

Definition 4.5. *A sequence $\{b_n\}_{n=1}^{\infty}$ of complex numbers is said to be a subordinating factor sequence if, whenever $f(z)$ of the form 1.1, $a_1=1$ is analytic, univalent and convex in U ,we have the subordination given by*

$$z+\sum_{k=1}^{\infty}a_k b_k z^k\prec f(z),\ (z\in U).$$

The following lemma is due to Wilf [3].

Lemma 4.1. *([3], [7]) The sequence $\{b_n\}_{n=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$\operatorname{Re}\left\{1+2\sum_{n=1}^{\infty}b_n z^n\right\}>0\ \ (z\in U).$$

Let $S^{*^{-1}}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)\subseteq S^{-1}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)$ is denote the subclass of functions $f\in S$ whose coefficients a_n satisfy the inequalities (4.4).

Employing the techniques used by T.Al-Hawary and B.A.Frasin [10], we state and prove the following theorem.

Theorem 4.2. *Let $f(z)\in S^{*^{-1}}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)$ and $n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)\delta_n+\left|(1+\lambda)\delta_n-2\alpha e^{-i\beta}\mu_n\right|\right\}$ is increasing function for $n\geq 2$, $|\beta|<\frac{\pi}{2}$,$0<\frac{\alpha}{1+\lambda}<\cos\beta$. Then*

$$\frac{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|}{2\left[2^{-\Omega}+(1+\lambda)\delta_2-\left|2^{-\Omega}-\frac{2^{1-\Omega}}{1+\lambda}\alpha e^{-i\beta}\right|+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right]}(f*g)(z)\prec g(z) \tag{4.5}$$

for every function $g(z)$ in the class K and

$$\operatorname{Re} f(z) > -\frac{\left[2^{-\Omega} + (1+\lambda)\delta_2 - \left|2^{-\Omega} - \frac{2^{1-\Omega}}{1+\lambda}\alpha e^{-i\beta}\right| + \left|(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|\right]}{(1+\lambda)\delta_2 + |(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2|} \tag{4.6}$$

for $z \in U$. The constant

$$\frac{(1+\lambda)\delta_2 + \left|(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|}{2\left[2^{-\Omega} + (1+\lambda)\delta_2 - \left|2^{-\Omega} - \frac{2^{1-\Omega}}{1+\lambda}\alpha e^{-i\beta}\right| + |(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2|\right]} \tag{4.7}$$

cannot be replace by any larger one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S^{*^{-1}}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)$ and suppose that $g(z) = z + \sum_{n=2}^{\infty} d_n z^n \in K$.Then

$$\begin{aligned}
&\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}(f*g)(z)\\
&= \frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}\left(z+\sum_{n=2}^{\infty} a_n d_n z^n\right);\\
&= \sum_{n=1}^{\infty}\left\{\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}\right\} a_n d_n z^n; (a_1 = 1, d_1 = 1).
\end{aligned}$$

Thus, by Definition 4.5, the assertion of our theorem will hold if the sequence

$$\left\{\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} a_n\right\}_{n=1}^{\infty} \tag{4.8}$$

is a subordinating factor sequence with $a_1 = 1$. In view of Lemma 4.1, this will be the case if and only if

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty}\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} a_n z^n\right\} > 0$$

for $z \in U$. Since, $n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)\delta_n + \left|(1+\lambda)\delta_n - 2\alpha e^{-i\beta}\mu_n\right|\right\}$ is increasing for all $n \geq 2, |\beta| < \frac{\pi}{2}$, $0 < \frac{\alpha}{1+\lambda} < \cos\beta$, we obtain

$$\begin{aligned}
&\operatorname{Re}\left\{1 + \sum_{n=1}^{\infty}\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} a_n z^n\right\}\\
&= \operatorname{Re}\left\{\begin{array}{l}
1 + \frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} z +\\
\frac{1}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}\\
\sum_{n=2}^{\infty} 2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2 + \left|(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|\right\} a_n z^n
\end{array}\right\}\\
&\geq \left\{\begin{array}{l}
1 - \frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} r -\\
\frac{1}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}\\
\sum_{n=2}^{\infty} n^{\Omega}(1-\lambda+\lambda n)\left\{(1+\lambda)\delta_2 + \left|(1+\lambda)\delta_2 - 2\alpha e^{-i\beta}\mu_2\right|\right\} |a_n| r^n
\end{array}\right\}\\
&> 1 - \frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} r\\
&- \frac{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} r = 1 - r
\end{aligned}$$

Since $|z| = r < 1$, therefore we obtain

$$\operatorname{Re}\left\{1 + 2\sum_{n=1}^{\infty}\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]} a_n z^n\right\} > 0,$$

which by Lemma 4.1 shows that

$$\left\{\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}a_n\right\}_{n=1}^{\infty}(a_1=1)$$

is a subordinating factor sequence and hence also the subordination result (4.5).

The inequality (4.6) follows from (4.5) by taking $g(z)=\frac{z}{1-z}$.

To prove the sharpness of the constant

$$\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]},$$

we consider the function

$$f_0(z)=z-\frac{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|}{2^{\Omega}(1+\lambda)[(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|]}z^2(|\beta|<\tfrac{\pi}{2},0<\tfrac{\alpha}{1+\lambda}<\cos\beta),$$

which is a member of the class $S^{*^{-1}}_{(\Omega,\lambda)}(\phi,\psi;\alpha,\beta)$. Thus from the relation (4.5), we obtain

$$\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}f_0(z)\prec\frac{z}{1-z}.$$

Since $\operatorname{Re}\left(\frac{z}{1-z}\right)>-\frac{1}{2}$, $|z|=r$, this implies

$$\begin{aligned}(4.9)\qquad &\operatorname{Re}\left\{\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}\right.\\ &\left.\times f_0(z)*\frac{z}{1-z}\right\}>-\frac{1}{2}.\end{aligned}$$

Therefore, we have

$$\operatorname{Re}\{f(z)\}>-\frac{\left[(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|+2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right]}{2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}}$$

which is equation (4.6).

Now to show that sharpness of the constant factor

$$\frac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{[(1+\lambda)-|(1+\lambda)-2\alpha e^{-i\beta}|+2^{\Omega}(1+\lambda)\{(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\}]}.$$

We consider the function

$$f_0(z)=z-\frac{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|}{2^{\Omega}(1+\lambda)\left[(1+\lambda)\delta_2+|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2|\right]}z^2.$$

Let $\zeta=\left[(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|+2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right]$.

Applying equation (4.5) with $g(z)=\frac{z}{1-z}$, we have

$$\frac{1}{2\zeta}\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\prec\frac{z}{1-z}.$$

Using the fact that $|\operatorname{Re}(z)|\le|z|$, we now show that the

$$(4.10)\qquad \min_{z\in U}\left\{\operatorname{Re}\frac{1}{2\zeta}.\left[\begin{array}{l}\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z-\\ \left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\end{array}\right]\right\}=-\frac{1}{2}$$

We have

$$\begin{aligned}&\left|\mathrm{Re}\,\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\right|\\ &\le\left|\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\right|\\ &\le\left|\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}\right]\right|+\tfrac{1}{2\zeta}.\left|\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}\right|\\ &=\tfrac{1}{2}\left\{\tfrac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}+\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}}{\zeta}\right\}=\tfrac{1}{2}\end{aligned}$$

for $|z|=1$. This implies that

$$\begin{aligned}&\left|\mathrm{Re}\,\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z\right.\right.\\ &\left.\left.-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\right|\le\tfrac{1}{2}\end{aligned}$$

and therefore

$$\begin{aligned}&-\tfrac{1}{2}\le\mathrm{Re}\,\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z\right.\\ &\left.-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\le\tfrac{1}{2}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}&\min_{z\in U}\left\{\mathrm{Re}\,\tfrac{1}{2\zeta}.\left[\left\{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right\}z\right]\right.\\ &\left.-\left\{(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|\right\}z^2\right]\Big\}=-\tfrac{1}{2}.\end{aligned}$$

That is

$$\begin{aligned}&\mathrm{Re}\left\{\tfrac{2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}}{2\left[(1+\lambda)-\left|(1+\lambda)-2\alpha e^{-i\beta}\right|+2^{\Omega}(1+\lambda)\left\{(1+\lambda)\delta_2+\left|(1+\lambda)\delta_2-2\alpha e^{-i\beta}\mu_2\right|\right\}\right]}\left(f_0*g\right)(z)\right\}\\ &=-\tfrac{1}{2}\end{aligned}$$

which shows the equation (4.10). □

Taking $\lambda=0$ and $\Omega=0$ in Theorem 4.2, we get Corollary 4.2 given by Tariq Al-Hawary et al. [10].

Corollary 4.2. *Let $f(z)\in S^{*^{-1}}(\phi,\psi;\alpha,\beta)$ and $\delta_n+\left|\delta_n-2\alpha e^{-i\beta}\mu_n\right|$ is increasing function for $n\ge 2, |\beta|<\frac{\pi}{2}$,$0<\alpha<\cos\beta$. Then*

$$\frac{\delta_2+\left|\delta_2-2\alpha e^{-i\beta}\mu_2\right|}{2\left\{1+\delta_2-|1-2\alpha e^{-i\beta}|+|\delta_2-2\alpha e^{-i\beta}\mu_2|\right\}}\left(f*g\right)(z)\prec g(z)$$

for every function $g(z)$ in the class K and

$$\mathrm{Re}\,f(z)>-\frac{1+\delta_2-\left|1-2\alpha e^{-i\beta}\right|+\left|\delta_2-2\alpha e^{-i\beta}\mu_2\right|}{\delta_2+|\delta_2-2\alpha e^{-i\beta}\mu_2|}$$

for $z\in U$. The constant

$$\frac{\delta_2+\left|\delta_2-2\alpha e^{-i\beta}\mu_2\right|}{2\left\{1+\delta_2-|1-2\alpha e^{-i\beta}|+|\delta_2-2\alpha e^{-i\beta}\mu_2|\right\}}$$

cannot be replace by any larger one.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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