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GROWTH ANALYSIS OF FUNCTIONS ANALYTIC IN THE UNIT POLYDISC

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Abstract. In this paper we study some growth properties of composite functions analytic in the unit polydisc. Some results related to the generalised n variables based p-th Nevanlinna order (generalised n variables based p-th Nevanlinna lower order) and the generalised n variables based p-th Nevanlinna relative order (generalised n variables based p-th Nevanlinna relative lower order) of an analytic function with respect to an entire function are established in this paper where n and p are any two positive integers. In fact in this paper we extend some results of [3] and [4].

1. Introduction, Definitions and Notations.

A function f analytic in the unit disc $U = \{z : |z| < 1\}$ is said to be of finite Nevanlinna order [6] if there exists a number μ such that the Nevanlinna characteristic function

$$T_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

satisfies $T_f(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$. The greatest lower bound of all such numbers μ is called the Nevanlinna order of f. Thus the Nevanlinna order ρ_f of f is given by

$$\rho_f = \limsup_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order λ_f of f are given by

$$\lambda_f = \liminf_{r\to 1} \frac{\log T_f(r)}{-\log(1-r)}.$$

L. Bernal introduced the relative order between two entire functions of single variables to avoid comparing growth just with the exponential function $\exp z$. In this connection, Banerjee and Dutta [2] gave the following definition in a unit disc:

Definition 1. [2] *If f be analytic in U and g be entire , then the relative order of f with respect to g denoted by $\rho_g(f)$ is defined by*

$$\rho_g(f) = \inf\left\{\mu > 0 : T_f(r) < T_g\left[\left(\frac{1}{1-r}\right)^{\mu}\right] \text{ for all } 0 < r_0(\mu) < r < 1\right\}.$$

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Similarly, one may define $\lambda_g(f)$, the relative lower order of f with respect to g. With $g(z) = \exp z$, the definition coincides with the definition of Nevanlinna order of f.
Analogously,

$$\lambda_g(f) = \liminf_{r\to 1} \frac{\log T_g^{-1} T_f(r)}{-\log(1-r)} .$$

Extending the notion of single variables to several variables, let $f(z_1, z_2, \cdots, z_n)$ be a non-constant analytic function of n complex variables $z_1, z_2, \cdots z_{n-1}$ and z_n in the unit polydisc

$$U = \{(z_1, z_2, \cdots, z_n) : |z_j| \leq 1,\ j = 1, 2, \cdots, n;\ r_1 > 0, r_2 > 0, \cdots r_n > 0\} .$$

Now in the line of Nevanlinna order [6], in this paper we introduce the generalised n variables based p-th Nevanlinna order and the generalised n variables p-th Nevanlinna lower order for functions of n complex variables analytic in a unit polydisc as follows :

$$\rho_f^{[p]}(r_1, r_2, ..., r_n) = \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_f(r_1, r_2, ..., r_n)}{-\log[(1-r_1)(1-r_2)...(1-r_n)]}$$

and

$$\lambda_f^{[p]}(r_1, r_2, ..., r_n) = \liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_f(r_1, r_2, ..., r_n)}{-\log[(1-r_1)(1-r_2)...(1-r_n)]}$$

where $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.
When $n = p = 1$, the above definition reduces to the definition of Juneja and Kapoor [6].

Likewise, one may introduce the generalised n variables based p-th relative Nevanlinna order (generalised n variables based p-th relative Nevanlinna lower order) for functions of n complex variables analytic in a unit polydisc in the following manner :

Definition 2. *Let $T_f(r_1, r_2, ..., r_n)$ denote the Nevanlinna's characteristic function of f of n variables. The generalised n variables based p-th relative Nevanlinna order $\rho_g^{[p]f}(r_1, r_2, ..., r_n)$ and generalised n variables based p-th relative Nevanlinna lower order $\lambda_g^{[p]f}(r_1, r_2, ..., r_n)$ of an analytic function f in U with respect to another entire function g in n complex variables are defined in the following way :*

$$\rho_g^{[p]f}(r_1, r_2, ..., r_n) = \limsup_{r_1, r_2 \to \infty} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{-\log[(1-r_1)(1-r_2)...(1-r_n)]}$$

and

$$\lambda_g^{[p]f}(r_1, r_2, ..., r_n) = \liminf_{r_1, r_2 \to \infty} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{-\log[(1-r_1)(1-r_2)...(1-r_n)]}$$

where n and p are any two positive integers .
If we consider $p = n = 1$ in Definition 2, then it coincides with Definition 1.

In the paper we establish some results relating to the composition of two non-constant analytic functions, of n complex variables in the unit polydisc

$$U = \{(z_1, z_2, \cdots, z_n) : |z_j| \leq 1,\ j = 1, 2, \cdots, n;\ r_1 > 0, r_2 > 0, \cdots r_n > 0\} .$$

Also we prove a few theorems related to generalised n variables based p-th relative Nevanlinna order $\rho_g^{[p]f}(r_1, r_2, ..., r_n)$ (generalised n variables based p-th relative

Nevanlinna lower order $\lambda_g^{[p]f}(r_1, r_2, ..., r_n)$) of an analytic function f with respect to an entire function g of n complex variables which are in fact some extensions of earlier results as proved in [3] and [4]. We do not explain the standard definitions and notations in the theory of entire functions of severable variables as those are available in [1], [5] and [7].

2. Theorems.

In this section we present the main results of the paper.

Theorem 1. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that* $0 < \lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) \leq \rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) < \infty$ *and* $0 < \lambda_g^{[q]}(r_1, r_2, ..., r_n) \leq \rho_g^{[q]}(r_1, r_2, ..., r_n) < \infty$. *Then*

$$\begin{aligned}\frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\rho_g^{[q]}(r_1, r_2, ..., r_n)} &\leq \liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\lambda_g^{[q]}(r_1, r_2, ..., r_n)} \\ &\leq \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\lambda_g^{[q]}(r_1, r_2, ..., r_n)}\end{aligned}$$

where p and q are any two positive integers .

Proof. From the definition of generalised n variables based p-th Nevanlinna order and generalised n variables based p-th Nevanlinna lower order of analytic functions in the unit polydisc U, we have for arbitrary positive ϵ and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\begin{aligned}(1) \qquad & \log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n) \\ \geq\ & \left(\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) - \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right]\end{aligned}$$

and

$$\begin{aligned}(2) \qquad & \log^{[q]} T_g(r_1, r_2, ..., r_n) \\ \leq\ & \left(\rho_g^{[q]}(r_1, r_2, ..., r_n) + \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].\end{aligned}$$

Now from (1) and (2) , it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) - \epsilon}{\rho_g^{[q]}(r_1, r_2, ..., r_n) + \epsilon}.$$

As $\epsilon\,(> 0)$ is arbitrary, we obtain that

$$(3) \qquad \liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\rho_g^{[q]}(r_1, r_2, ..., r_n)}.$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\begin{aligned}(4) \qquad & \log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n) \\ \leq\ & \left(\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) + \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right]\end{aligned}$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$,

(5)
$$\log^{[q]} T_g(r_1, r_2, ..., r_n) \geq \left(\lambda_g^{[q]}(r_1, r_2, ..., r_n) - \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].$$

So combining (4) and (5), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) + \epsilon}{\lambda_g^{[q]}(r_1, r_2, ..., r_n) - \epsilon}.$$

Since $\epsilon\,(>0)$ is arbitrary, it follows that

(6)
$$\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\lambda_g^{[q]}(r_1, r_2, ..., r_n)}.$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity, we get that

(7)
$$\log^{[q]} T_g(r_1, r_2, ..., r_n) \leq \left(\lambda_g^{[q]}(r_1, r_2, ..., r_n) + \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].$$

Now from (1) and (7), we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) - \epsilon}{\lambda_g^{[q]}(r_1, r_2, ..., r_n) + \epsilon}.$$

Choosing $\epsilon \to 0$, we get that

(8)
$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\lambda_g^{[q]}(r_1, r_2, ..., r_n)}.$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$,

(9)
$$\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n) \leq \left(\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) + \epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].$$

So from (5) and (9), it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) + \epsilon}{\lambda_g^{[q]}(r_1, r_2, ..., r_n) - \epsilon}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain that

(10)
$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \leq \frac{\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n)}{\lambda_g^{[q]}(r_1, r_2, ..., r_n)}.$$

Thus the theorem follows from (3), (6), (8) and (10). ∎

The following theorem can be proved in the line of Theorem 1 and so its proof is omitted.

Theorem 2. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U with $0<\lambda_{f\circ g}^{[p]}(r_1,r_2,...,r_n)\le\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)<\infty$ and $0<\lambda_f^{[l]}(r_1,r_2,...,r_n)\le\rho_f^{[l]}(r_1,r_2,...,r_n)<\infty$ where p and l are any two positive integers. Then*

$$\begin{aligned}\frac{\lambda_{f\circ g}^{[p]}(r_1,r_2,...,r_n)}{\rho_f^{[l]}(r_1,r_2,...,r_n)} &\le \liminf_{r_1,r_2,...r_n\to 1}\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[l]}T_f(r_1,r_2,...,r_n)}\le\frac{\lambda_{f\circ g}^{[p]}(r_1,r_2,...,r_n)}{\lambda_f^{[l]}(r_1,r_2,...,r_n)}\\ &\le \limsup_{r_1,r_2,...r_n\to 1}\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[l]}T_f(r_1,r_2,...,r_n)}\le\frac{\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)}{\lambda_f^{[l]}(r_1,r_2,...,r_n)}.\end{aligned}$$

Theorem 3. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0<\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)<\infty$ and $0<\rho_g^{[q]}(r_1,r_2,...,r_n)<\infty$. Then*

$$\begin{aligned}\liminf_{r_1,r_2,...r_n\to 1}\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[q]}T_g(r_1,r_2,...,r_n)} &\le \frac{\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)}{\rho_g^{[q]}(r_1,r_2,...,r_n)}\\ &\le \limsup_{r_1,r_2,...r_n\to 1}\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[q]}T_g(r_1,r_2,...,r_n)}\end{aligned}$$

where p and q are any two positive integers .

Proof. From the definition of generalised n variables based p-th Nevanlinna order, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\begin{aligned}(11)\qquad &\log^{[q]}T_g(r_1,r_2,...,r_n)\\ &\ge \left(\rho_g^{[q]}(r_1,r_2,...,r_n)-\epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].\end{aligned}$$

Now from (9) and (11) , it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[q]}T_g(r_1,r_2,...,r_n)}\le\frac{\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)+\epsilon}{\rho_g^{[q]}(r_1,r_2,...,r_n)-\epsilon}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain that

$$(12)\qquad \liminf_{r_1,r_2,...r_n\to 1}\frac{\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[q]}T_g(r_1,r_2,...,r_n)}\le\frac{\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)}{\rho_g^{[q]}(r_1,r_2,...,r_n)}.$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\begin{aligned}(13)\qquad &\log^{[p]}T_{f\circ g}(r_1,r_2,...,r_n)\\ &\ge \left(\rho_{f\circ g}^{[p]}(r_1,r_2,...,r_n)-\epsilon\right)\left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right].\end{aligned}$$

So combining (2) and (13) , we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) - \epsilon}{\rho^{[q]}_g(r_1, r_2, ..., r_n) + \epsilon}.$$

Since $\epsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \geq \frac{\rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\rho^{[q]}_g(r_1, r_2, ..., r_n)}. \tag{14}$$

Thus the theorem follows from (12) and (14) . ∎

The following theorem can be carried out in the line of Theorem 3 and therefore we omit its proof.

Theorem 4. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U with $0 < \rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) < \infty$ and $0 < \rho^{[l]}_f(r_1, r_2, ..., r_n) < \infty$ where p and l are any two positive integers. Then*

$$\begin{aligned}\liminf_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[l]} T_f(r_1, r_2, ..., r_n)} &\leq \frac{\rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\rho^{[l]}_f(r_1, r_2, ..., r_n)} \\ &\leq \limsup_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[l]} T_f(r_1, r_2, ..., r_n)}.\end{aligned}$$

The following theorem is a natural consequence of Theorem 1 and Theorem 3:

Theorem 5. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $0 < \lambda^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) \leq \rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) < \infty$ and $0 < \lambda^{[q]}_g(r_1, r_2, ..., r_n) \leq \rho^{[q]}_g(r_1, r_2, ..., r_n) < \infty$. Then*

$$\begin{aligned}&\liminf_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} \\ \leq\ & \min\left\{\frac{\lambda^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\lambda^{[q]}_g(r_1, r_2, ..., r_n)}, \frac{\rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\rho^{[q]}_g(r_1, r_2, ..., r_n)}\right\} \\ \leq\ & \max\left\{\frac{\lambda^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\lambda^{[q]}_g(r_1, r_2, ..., r_n)}, \frac{\rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n)}{\rho^{[q]}_g(r_1, r_2, ..., r_n)}\right\} \\ \leq\ & \limsup_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)}\end{aligned}$$

where p and q are any two positive integers .

Analogously one may state the following theorem without its proof.

Theorem 6. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U with $0 < \lambda^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) \leq \rho^{[p]}_{f\circ g}(r_1, r_2, ..., r_n) < \infty$ and $0 < \lambda^{[l]}_f(r_1, r_2, ..., r_n) \leq \rho^{[l]}_f(r_1, r_2, ..., r_n) < \infty$ where p and l are any two* *positive integers .Then*

$$\begin{aligned}
&\liminf_{r_1,r_2,\ldots r_n\to 1} \frac{\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[l]} T_f(r_1,r_2,...,r_n)} \\
\leq\ & \min\left\{\frac{\lambda^{[p]}_{f\circ g}(r_1,r_2,...,r_n)}{\lambda^{[l]}_f(r_1,r_2,...,r_n)}, \frac{\rho^{[p]}_{f\circ g}(r_1,r_2,...,r_n)}{\rho^{[l]}_f(r_1,r_2,...,r_n)}\right\} \\
\leq\ & \max\left\{\frac{\lambda^{[p]}_{f\circ g}(r_1,r_2,...,r_n)}{\lambda^{[l]}_f(r_1,r_2,...,r_n)}, \frac{\rho^{[p]}_{f\circ g}(r_1,r_2,...,r_n)}{\rho^{[l]}_f(r_1,r_2,...,r_n)}\right\} \\
\leq\ & \limsup_{r_1,r_2,\ldots r_n\to 1} \frac{\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[l]} T_f(r_1,r_2,...,r_n)}.
\end{aligned}$$

Theorem 7. *Let f and g be any two non-constant analytic functions of n complex variables in the unit polydisc U such that $\rho^{[l]}_f(r_1,r_2,...,r_n) < \infty$ and $\lambda^{[p]}_{f\circ g}(r_1,r_2,...,r_n)$ $= \infty$. Then*

$$\lim_{r_1,r_2,\ldots r_n\to 1} \frac{\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n)}{\log^{[l]} T_f(r_1,r_2,...,r_n)} = \infty$$

where p and l are any two positive integers .

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\beta > 0$ such that for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n) \leq \beta \log^{[l]} T_f(r_1,r_2,...,r_n)\ . \tag{15}$$

Again from the definition of $\rho^{[l]}_f(r_1,r_2,...,r_n)$, it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\begin{aligned}
&\log^{[l]} T_f(r_1,r_2,...,r_n) \\
\leq\ & \left[\rho^{[l]}_f(r_1,r_2,...,r_n) + \epsilon\right]\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right]\ .
\end{aligned} \tag{16}$$

Thus from (15) and (16), we have for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\begin{aligned}
&\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n) \\
\leq\ & \beta\left[\rho^{[l]}_f(r_1,r_2,...,r_n) + \epsilon\right]\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right] \\
& i.e.,\ \frac{\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n)}{\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right]} \\
\leq\ & \frac{\beta\left[\rho^{[l]}_f(r_1,r_2,...,r_n) + \epsilon\right]\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right]}{\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right]}
\end{aligned}$$

$$i.e.,\ \liminf_{r_1,r_2,\ldots r_n\to 1} \frac{\log^{[p]} T_{f\circ g}(r_1,r_2,...,r_n)}{\left[-\log\left[(1-r_1)(1-r_2)\ldots(1-r_n)\right]\right]} = \lambda^{[p]}_{f\circ g}(r_1,r_2,...,r_n) < \infty.$$

This is a contradiction.
Hence the theorem follows. ∎

Remark 1. *Theorem 7 is also valid with "limit superior" instead of "limit" if* $\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) = \infty$ *is replaced by* $\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) = \infty$ *and the other conditions remain the same.*

Corollary 8. *Under the assumptions of Theorem 7 and Remark 1,*

$$\lim_{r_1, r_2, ... r_n \to 1} \frac{T_{f\circ g}(r_1, r_2, ..., r_n)}{T_f(r_1, r_2, ..., r_n)} = \infty \text{ and } \limsup_{r_1, r_2, ... r_n \to 1} \frac{T_{f\circ g}(r_1, r_2, ..., r_n)}{T_f(r_1, r_2, ..., r_n)} = \infty$$

respectively hold if $p = l$.

The proof is omitted.

Analogously one may also state the following theorem and corollaries without their proofs as those may be carried out in the line of Remark 1, Theorem 7 and Corollary 8 respectively.

Theorem 9. *Let* f *and* g *be any two non-constant analytic functions of* n *complex variables in the unit polydisc* U *with* $\rho_g^{[q]}(r_1, r_2, ..., r_n) < \infty$ *and* $\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) = \infty$ *where* p *and* q *are any two positive integers. Then*

$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[q]} T_g(r_1, r_2, ..., r_n)} = \infty .$$

Corollary 10. *Theorem 9 is also valid with "limit" instead of "limit superior" if* $\rho_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) = \infty$ *is replaced by* $\lambda_{f\circ g}^{[p]}(r_1, r_2, ..., r_n) = \infty$ *and the other conditions remain the same.*

Corollary 11. *Under the assumptions of Theorem 7 and Corollary 10,*

$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{T_{f\circ g}(r_1, r_2, ..., r_n)}{T_g(r_1, r_2, ..., r_n)} = \infty \text{ and } \lim_{r_1, r_2, ... r_n \to 1} \frac{T_{f\circ g}(r_1, r_2, ..., r_n)}{T_g(r_1, r_2, ..., r_n)} = \infty$$

respectively hold if $p = q$.

In the next three theorems we establish some comparative growth properties related to the generalised n variables based p-th relative Nevanlinna order (generalised n variables based p-th relative Nevanlinna lower order) of an analytic function with respect to an entire function in the unit poly disc U.

Theorem 12. *Let* f, h *be any two analytic functions of* n *complex variables in* U *and* g *be entire in* n *complex variables such that* $0 < \lambda_g^{[p]f}(r_1, r_2, ..., r_n) \leq \rho_g^{[p]f}(r_1, r_2, ..., r_n) < \infty$ *and* $0 < \lambda_g^{[p]h}(r_1, r_2, ..., r_n) \leq \rho_g^{[p]h}(r_1, r_2, ..., r_n) < \infty$. *Then*

$$\begin{aligned} \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)} &\leq \liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)} \\ &\leq \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)} \end{aligned}$$

where p *is any positive integer.*

Proof. From the definition of generalised n variables based p-th relative Nevanlinna order and generalised n variables based p-th relative Nevanlinna lower order of an analytic function with respect to an entire function in an unit polydisc U, we have for arbitrary positive ϵ and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\begin{aligned} &\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n) \\ \geq\ & \left[\lambda_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right] \end{aligned} \tag{17}$$

and

$$\begin{aligned} &\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n) \\ \leq\ & \left[\rho_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right]. \end{aligned} \tag{18}$$

Now from (17) and (18), it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon}{\rho_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{19}$$

Again we have for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n) \leq \left[\lambda_g^{[p]f}(r_1, r_2, ..., r_n) + \epsilon\right] \left[-\log(1-r)\right] \tag{20}$$

and for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$,

$$\begin{aligned} &\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n) \\ \geq\ & \left[\lambda_g^{[p]h}(r_1, r_2, ..., r_n) - \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right]. \end{aligned} \tag{21}$$

So combining (20) and (21), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$, tending to infinity that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n) + \epsilon}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n) - \epsilon}.$$

Since $\epsilon\,(>0)$ is arbitrary, it follows that

$$\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{22}$$

Also for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\begin{aligned} &\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n) \\ \leq\ & \left[\lambda_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2)...(1-r_n)\right]\right]. \end{aligned} \tag{23}$$

Now from (17) and (23), we obtain for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$, tending to infinity that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon}.$$

Choosing $\epsilon\,(>0)$, we get that

$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \geq \frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{24}$$

Also for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$,

$$\begin{aligned} & \log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n) \\ \leq \; & \left[\rho_g^{[p]f}(r_1, r_2, ..., r_n) + \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2) ... (1-r_n)\right]\right]. \end{aligned} \tag{25}$$

So from (21) and (25), it follows for all sufficiently large values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n) + \epsilon}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n) - \epsilon}.$$

As $\epsilon\,(>0)$ is arbitrary, we obtain from above that

$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \leq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{26}$$

Thus the theorem follows from (19), (22), (24) and (26). ∎

Theorem 13. *Let f, h be any two analytic functions of n complex variables in U and g be entire in n complex variables with $0 < \rho_g^{[p]f}(r_1, r_2, ..., r_n) < \infty$ and $0 < \rho_g^{[p]h}(r_1, r_2, ..., r_n) < \infty$ where p is any positive integer. Then*

$$\begin{aligned} \liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} & \leq \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)} \\ & \leq \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)}. \end{aligned}$$

Proof. From the definition of generalised n variables based p-th relative Nevanlinna order, we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\begin{aligned} & \log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n) \\ \geq \; & \left[\rho_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2) ... (1-r_n)\right]\right]. \end{aligned} \tag{27}$$

Now from (25) and (27), it follows for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \le \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n) + \epsilon}{\rho_g^{[p]h}(r_1, r_2, ..., r_n) - \epsilon}.$$

As $\epsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \le \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{28}$$

Again for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity,

$$\begin{aligned} &\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n) \\ \ge\ & \left[\rho_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon\right] \left[-\log\left[(1-r_1)(1-r_2)\dots(1-r_n)\right]\right]. \end{aligned} \tag{29}$$

So combining (18) and (29), we get for a sequence of values of $\left(\frac{1}{1-r_1}\right)$, $\left(\frac{1}{1-r_2}\right)$, ... and $\left(\frac{1}{1-r_n}\right)$ tending to infinity that

$$\frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \ge \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n) - \epsilon}{\rho_g^{[p]h}(r_1, r_2, ..., r_n) + \epsilon}.$$

Since $\epsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \ge \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}. \tag{30}$$

Thus the theorem follows from (28) and (30). ∎

In view of Theorem 12 and Theorem 13, we may state the following theorem without its proof.

Theorem 14. *Let f, h be any two analytic functions of n complex variables in U and g be entire in n complex variables such that $0 < \lambda_g^{[p]f}(r_1, r_2, ..., r_n) \le \rho_g^{[p]f}(r_1, r_2, ..., r_n) < \infty$ and $0 < \lambda_g^{[p]h}(r_1, r_2, ..., r_n) \le \rho_g^{[p]h}(r_1, r_2, ..., r_n) < \infty$. Then*

$$\begin{aligned} &\liminf_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \\ \le\ & \min\left\{\frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}, \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}\right\} \\ \le\ & \max\left\{\frac{\lambda_g^{[p]f}(r_1, r_2, ..., r_n)}{\lambda_g^{[p]h}(r_1, r_2, ..., r_n)}, \frac{\rho_g^{[p]f}(r_1, r_2, ..., r_n)}{\rho_g^{[p]h}(r_1, r_2, ..., r_n)}\right\} \\ \le\ & \limsup_{r_1, r_2, ... r_n \to 1} \frac{\log^{[p]} T_g^{-1} T_f(r_1, r_2, ..., r_n)}{\log^{[p]} T_g^{-1} T_h(r_1, r_2, ..., r_n)} \end{aligned}$$

where p is any positive integer.

Theorem 15. *Let f, h be any two analytic functions of n complex variables in U and g be entire in n complex variables such that $\rho_h^{[p]f}(r_1, r_2, ..., r_n) < \infty$ and $\lambda_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$ where p is any positive integer. Then*

$$\lim_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[p]} T_h^{-1} T_f(r_1, r_2, ..., r_n)} = \infty \ .$$

The proof is omitted because it can be carried out using the same technique as involved in Theorem 7.

Remark 2. *Theorem 15 is also valid with "limit superior" instead of "limit" if $\lambda_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$ is replaced by $\rho_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$ and the other conditions remain the same.*

Corollary 16. *Under the assumptions of Theorem 15 and Remark 2,*

$$\lim_{r_1, r_2, \ldots r_n \to 1} \frac{T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{T_h^{-1} T_f(r_1, r_2, ..., r_n)} = \infty \text{ and } \limsup_{r_1, r_2, \ldots r_n \to 1} \frac{T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{T_h^{-1} T_f(r_1, r_2, ..., r_n)} = \infty$$

respectively hold.

The proof is omitted.

Similarly, one may also state the following theorem and corollaries without their proofs as they may be carried out in the line of Remark 2, Theorem 15 and Corollary 16 respectively.

Theorem 17. *Let f, h be any two analytic functions of n complex variables in U and g be entire in n complex variables such that $\rho_h^{[p]g}(r_1, r_2, ..., r_n) < \infty$ and $\rho_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$. Then*

$$\limsup_{r_1, r_2, \ldots r_n \to 1} \frac{\log^{[p]} T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{\log^{[p]} T_h^{-1} T_g(r_1, r_2, ..., r_n)} = \infty$$

where p is any positive integer.

Corollary 18. *Theorem 17 is also valid with "limit" instead of "limit superior" if $\rho_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$ is replaced by $\lambda_h^{[p]f\circ g}(r_1, r_2, ..., r_n) = \infty$ and the other conditions remain the same.*

Corollary 19. *Under the assumptions of Theorem 15 and Corollary 18,*

$$\limsup_{r_1, r_2, \ldots r_n \to 1} \frac{T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{T_h^{-1} T_g(r_1, r_2, ..., r_n)} = \infty \text{ and } \lim_{r_1, r_2, \ldots r_n \to 1} \frac{T_h^{-1} T_{f\circ g}(r_1, r_2, ..., r_n)}{T_h^{-1} T_g(r_1, r_2, ..., r_n)} = \infty$$

respectively hold.

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