



APPLICABLE SOLUTION FOR A CLASS OF ORDINARY DIFFERENTIAL EQUATIONS WITH SINGULARITY

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ABSTRACT. Boundary value problems arise in many real applications such as nanofluids and other areas of applied sciences. The temperature/nanoparticles concentration are usually expressed as singular 2nd-order ODEs. So, it is a challenge to obtain the exact solution of these problems due to the difficulty of the singularity encountered in the governing equations. By means of a suitable transformation, a direct approach is introduced to solve a general class of 2nd-order ODEs. The efficiency of the obtained results is validated through selected problems in the literature. It is found that several existing solutions can be deduced as special cases of our generalized one. Moreover, the present results may be invested for similar future problems in fluid mechanics, especially nanofluids.

1. INTRODUCTION

The field of nanofluid is of great importance in industry, engineering, and physics. The distributions of temperature and nanoparticles concentration of such fluids are originally governed by PDEs which can be transformed into ODEs [1-9]. Such ODEs are, basically, subjected to boundary conditions (BCs) given at infinity. In Refs. [10-16], the authors implemented several numerical/analytical methods to solve such types of problems. However, the approaches [10-16] need a massive computational work to obtain accurate solution because of the difficulty of applying the BC at infinity. In addition, it was shown in the literature [17-19] that

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some of the approximate methods have drawbacks. Therefore, solving BVPs with singularities is a challenge. Usually, the BCs at infinity are transformed to new finite ones by applying certain substitutions. Accordingly, the coefficients of the ODEs become polynomials. Hence, the temperature/nanoparticles concentration of nanofluids are usually special cases of the following class:

$$(1.1) \quad z''(t) + \left(\frac{P}{t} + Q\right) z'(t) + \left(\frac{l}{t^2} + \frac{R}{t}\right) z(t) = \alpha t^{n-1}, \quad n > -1, \alpha \in \mathbb{R},$$

under the BCs:

$$(1.2) \quad z(0) = 0, \quad z(\delta) = 1, \quad \delta \in \mathbb{R} - \{0\},$$

where P , Q , l , and R are physical parameters of the nanofluids [1-8]. The constant n takes a particular value according to the final form of the temperature equation, while $\delta \neq 0$ depends on the final BC. The main objective of this paper is to introduce a direct analysis to exactly solving Eqs. (1.1-1.2). Then, the present generalized results will be invested to construct several exact solutions for some published nanofluids problems as special cases.

2. ANALYSIS

Firstly, we rewrite Eq. (1.1) as

$$(2.1) \quad t^2 z''(t) + (Pt + Qt^2) z'(t) + (l + Rt) z(t) = \alpha t^{n+1}.$$

Suppose that

$$(2.2) \quad z(t) = t^\gamma \psi(t).$$

Accordingly,

$$(2.3) \quad z'(t) = t^{\gamma-1} (t\psi'(t) + \gamma\psi(t)),$$

and

$$(2.4) \quad z''(t) = t^{\gamma-2} (t^2\psi''(t) + 2\gamma t\psi'(t) + \gamma(\gamma-1)\psi(t)),$$

respectively. Substituting Eqs. (2.2-2.4) into Eq. (2.1), we have

$$(2.5) \quad t^2\psi''(t) + ((2\gamma + P)t + Qt^2)\psi'(t) + ((\gamma^2 - \gamma + \gamma P + l) + (\gamma Q + R)t)\psi(t) = \alpha t^{n-\gamma+1}.$$

Setting

$$(2.6) \quad \gamma^2 - \gamma + \gamma P + l = 0,$$

and solving for γ , we obtain

$$(2.7) \quad \gamma = \frac{1 - P \pm \sqrt{(1 - P)^2 - 4l}}{2}.$$

Accordingly, Eq. (2.5) becomes

$$(2.8) \quad t\psi''(t) + ((2\gamma + P) + Qt)\psi'(t) + (\gamma Q + R)\psi(t) = \alpha t^{n-\gamma}.$$

Assuming that

$$(2.9) \quad P_1 = 2\gamma + P, \quad R_1 = \gamma Q + R,$$

then Eq. (2.8) takes the form:

$$(2.10) \quad t\psi''(t) + (P_1 + Qt)\psi'(t) + R_1\psi(t) = \alpha t^{n-\gamma}.$$

3. EXACT SOLUTION AND CONVERGENCE ANALYSIS

3.1. Exact solution. Suppose that $\psi_c(t)$ and $\psi_p(t)$ are complementary and particular solution of Eq. (2.10), respectively, then

$$(3.1) \quad \psi(t) = \psi_c(t) + \psi_p(t),$$

Following [20], we have

$$(3.2) \quad \psi_c(t) = \frac{c t^{\mu_1 + \mu_2 - 1}}{\Gamma(\mu_1 + \mu_2)} {}_1F_1[\mu_1, \mu_1 + \mu_2, -Q t],$$

where ${}_1F_1$ is Kummer's function and c is a constant. μ_1 and μ_2 are defined as

$$(3.3) \quad \mu_1 = 1 - P_1 + \frac{R_1}{Q}, \quad \mu_2 = 1 - \frac{R_1}{Q}.$$

From (2.9) and (3.3), we have

$$(3.4) \quad \mu_1 = 1 - \gamma - P + \frac{R}{Q}, \quad \mu_2 = 1 - \gamma - \frac{R}{Q}.$$

$\psi_p(t)$ can be obtained as (see [20])

$$(3.5) \quad \psi_p(t) = \frac{\alpha t^{n-\gamma+1}}{(n-\gamma+1)(n-\gamma+P_1)},$$

such that

$$(3.6) \quad R_1 = -(n-\gamma+1)Q, \quad (n-\gamma+1)(n-\gamma+P_1) \neq 0.$$

From (2.9) and (3.6), we have

$$(3.7) \quad R = -(n+1)Q, \quad (n-\gamma+1)(n+\gamma+P) \neq 0,$$

and hence Eq. (2.8) reduces to

$$(3.8) \quad t\psi''(t) + ((2\gamma + P) + Qt)\psi'(t) - (n - \gamma + 1)Q\psi(t) = \alpha t^{n-\gamma}.$$

Inserting P_1 given by (2.9) into (3.5), we obtain

$$(3.9) \quad \psi_p(t) = \frac{\alpha t^{n-\gamma+1}}{(n - \gamma + 1)(n + \gamma + P)}.$$

Therefore, the general solution of Eq. (3.8) is given from (3.1) by

$$(3.10) \quad \psi(t) = \frac{c t^{\mu_1+\mu_2-1}}{\Gamma(\mu_1 + \mu_2)} {}_1F_1[\mu_1, \mu_1 + \mu_2, -Qt] + \frac{\alpha t^{n-\gamma+1}}{(n - \gamma + 1)(n + \gamma + P)},$$

where μ_1 and μ_2 are defined by (3.3). Hence, the general solution of the original equation (2.1) such that $R = -(n + 1)Q$ is obtained as

$$(3.11) \quad z(t) = \frac{c t^{\gamma+\mu_1+\mu_2-1}}{\Gamma(\mu_1 + \mu_2)} {}_1F_1[\mu_1, \mu_1 + \mu_2, -Qt] + \frac{\alpha t^{n+1}}{(n - \gamma + 1)(n + \gamma + P)},$$

for the ODE:

$$(3.12) \quad t^2 z''(t) + (Pt + Qt^2) z'(t) + (l - (n + 1)Qt) z(t) = \alpha t^{n+1}.$$

It is noted from (3.11) that the first BC $z(0) = 0$ is automatically satisfied when

$$(3.13) \quad \gamma + \mu_1 + \mu_2 > 1, \quad n > -1, \quad (n - \gamma + 1)(n + \gamma + P) \neq 0.$$

Applying the second BC in (1.2) on Eq. (3.11), yields

$$(3.14) \quad c = \frac{\delta^{1-\gamma-\mu_1-\mu_2}\Gamma(\mu_1 + \mu_2)}{{}_1F_1[\mu_1, \mu_1 + \mu_2, -Q\delta]} \left(1 - \frac{\alpha \delta^{n+1}}{(n - \gamma + 1)(n + \gamma + P)} \right).$$

In such case, the solution given by Eq. (3.11) becomes

$$(3.15) \quad z(t) = \frac{(t/\delta)^{\gamma+\mu_1+\mu_2-1} {}_1F_1[\mu_1, \mu_1 + \mu_2, -Qt]}{{}_1F_1[\mu_1, \mu_1 + \mu_2, -Q\delta]} \left(1 - \frac{\alpha \delta^{n+1}}{(n - \gamma + 1)(n + \gamma + P)} \right) + \frac{\alpha t^{n+1}}{(n - \gamma + 1)(n + \gamma + P)}.$$

Therefore

$$(3.16) \quad z(t) = \frac{(t/\delta)^{1-\gamma-P} {}_1F_1[-n - \gamma - P, 2 - 2\gamma - P, -Qt]}{{}_1F_1[-n - \gamma - P, 2 - 2\gamma - P, -Q\delta]} \times \left(1 - \frac{\alpha \delta^{n+1}}{(n - \gamma + 1)(n + \gamma + P)} \right) + \frac{\alpha t^{n+1}}{(n - \gamma + 1)(n + \gamma + P)},$$

provided that

$$(3.17) \quad 1 - \gamma - P > 0, \quad n > -1, \quad (n - \gamma + 1)(n + \gamma + P) \neq 0.$$

It can be verified by direct substitution that the solution (3.16) satisfies Eq. (3.12) and the BCs (1.2). In a sequent section, it is declared that Eq. (3.16) agrees with several existing results at prescribed values of the coefficients P, Q, l, R , and the parameter δ .

3.2. Convergence analysis. In order to prove the convergence of the solution given by Eqs. (3.16-3.17), we begin with the definition of Kummer’s function ${}_1F_1(a, b, t)$:

$$(3.18) \quad {}_1F_1(a, b, t) = \sum_{i=0}^{\infty} \frac{(a)_i t^i}{(b)_i i!},$$

where $(a)_i$ is Pochhammer symbol defined as

$$(3.19) \quad (a)_i = a(a + 1)(a + 2) \dots (a + i - 1), \quad i = 1, 2, 3, \dots, \quad (a)_0 = 1.$$

The series (3.18) is not defined for $b = 0, -1, -2, -3, \dots$, and if a is a negative integer, the series truncates.

Theorem 1: Let b is neither a negative integer nor zero, then ${}_1F_1(a, b, -Qt)$ converges for all (finite) t and finite Q .

Proofs: From the definition (3.18), ${}_1F_1(a, b, -Qt)$ is given as

$$(3.20) \quad {}_1F_1(a, b, -Qt) = \sum_{i=0}^{\infty} \frac{(a)_i (-Qt)^i}{(b)_i i!}.$$

Since b is neither a negative integer nor zero, then the series (3.20) is defined and its general term $v_i(x)$ is given by

$$(3.21) \quad v_i(x) = \frac{(a)_i (-Qt)^i}{(b)_i i!}.$$

Implementing the ratio test, we have

$$(3.22) \quad \begin{aligned} \lim_{i \rightarrow \infty} \left| \frac{v_{i+1}(x)}{v_i(x)} \right| &= \lim_{i \rightarrow \infty} \left| \frac{(a)_{i+1} (-Qt)^{i+1}}{(b)_{i+1} (i+1)!} \times \frac{(b)_i i!}{(a)_i (-Qt)^i} \right|, \\ &= \lim_{i \rightarrow \infty} \left| \frac{(a)_i (a+i) (-Q)^{i+1} (t)^{i+1}}{(b)_i (b+i) (i+1)(i!)} \times \frac{(b)_i i!}{(a)_i (-Q)^i (t)^i} \right|, \\ &= \lim_{i \rightarrow \infty} \left| \frac{(a+i)}{(b+i)(i+1)} \right| \times |-Q| |t|. \end{aligned}$$

For finite t and finite Q , we have from (3.22) that

$$(3.23) \quad \lim_{i \rightarrow \infty} \left| \frac{v_{i+1}(x)}{v_i(x)} \right| = 0,$$

and hence, ${}_1F_1(a, b, -Qt)$ is convergent.

Lemma 1: For finite δ and Q , the solution given by Eqs. (3.16-3.17) converges if $(2 - 2\gamma - P)$ is neither a negative integer nor zero.

Proofs: Let $a = -n - \gamma - P$ and $b = 2 - 2\gamma - P$, then the solution in Eqs. (3.16-3.17) takes the form:

$$(3.24) \quad z(t) = \frac{(t/\delta)^{1-\gamma-P} {}_1F_1[a, b, -Qt]}{{}_1F_1[a, b, -Q\delta]} \left(1 + \frac{\alpha \delta^{n+1}}{(n - \gamma + 1) a} \right) - \frac{\alpha t^{n+1}}{(n - \gamma + 1) a},$$

such that

$$(3.25) \quad 1 - \gamma - P > 0, \quad n > -1, \quad (n - \gamma + 1) a \neq 0.$$

Since $b = 2 - 2\gamma - P$ is neither a negative integer nor zero, then $z(t)$ in (36) is defined. Also, since Q is finite and t is finite in the domain of the problem, $t \in [0, \delta]$, then ${}_1F_1(a, b, -Qt)$ is convergent by Theorem 1. Also, ${}_1F_1(a, b, -Q\delta)$ is convergent because δ is finite and therefore the solution given by Eqs. (3.16-3.17) or its equivalent form (3.24-3.25) converges.

4. APPLICATIONS

4.1. **At** $l = 0, \alpha \neq 0, n > -1$. At $l = 0, \gamma$ in (2.7) reduces to

$$(4.1) \quad \gamma = \frac{1 - P \pm |1 - P|}{2}.$$

For $P < 1$, we have $\gamma = 1 - P$ or $\gamma = 0$. However, $\gamma = 1 - P$ doesn't satisfy the first condition in (3.25) which is $1 - \gamma - P > 0$. Also, for $P > 1$, we have $\gamma = 0$ or $\gamma = 1 - P$. Hence, $\gamma = 0$ is the only acceptable value when the special case $l = 0$ is considered. At such case ($l = 0$), Eq. (3.12) reduces to

$$(4.2) \quad tz''(t) + (P + Qt)z'(t) - ((n + 1)Q)z(t) = \alpha t^n,$$

and its solution comes by setting $\gamma = 0$ in (3.16) which is

$$(4.3) \quad z(t) = \frac{(t/\delta)^{1-P} {}_1F_1[-n - P, 2 - P, -Qt]}{{}_1F_1[-n - P, 2 - P, -Q\delta]} \left(1 - \frac{\alpha \delta^{n+1}}{(n + 1)(n + P)} \right) + \frac{\alpha t^{n+1}}{(n + 1)(n + P)}.$$

The result obtained by Eq. (4.3) agrees with the published solution in literature (Ref. [20], equation 31). Consequently, the solution in literature is a special case of our analysis when $l = 0$. In addition, the solution (40) converges if $(2 - P)$ is neither a negative integer nor zero, i.e., $P \neq 2, 3, 4, \dots$, or $P \neq (j + 1) \forall j \in \mathbb{Z}^+$.

4.2. **At** $l = 0, \alpha \neq 0, n = 1, \delta = 1$. The temperature equation for the Marangoni boundary layer was obtained by Khaled [21] as

$$(4.4) \quad tz''(t) + (l_1 - m t)z'(t) + 2mz(t) = -\lambda t,$$

subject to

$$(4.5) \quad z(0) = 0, \quad z(1) = 1.$$

At $n = 1$, Eq. (4.2), in the previous section, becomes

$$(4.6) \quad tz''(t) + (P + Qt)z'(t) - 2Qz(t) = \alpha t.$$

Comparing (4.4) with (4.2) and (4.5) with (1.2), we find that

$$(4.7) \quad P = l_1, \quad Q = -m, \quad \alpha = -\lambda, \quad \delta = 1.$$

Substituting these values of parameters into (4.3), we have

$$(4.8) \quad y(t) = \left(1 + \frac{\lambda}{2(l_1 + 1)}\right) \frac{t^{1-l_1} {}_1F_1[-1-l_1, 2-l_1, m t]}{{}_1F_1[-1-l_1, 2-l_1, m]} - \frac{\lambda t^2}{2(l_1 + 1)},$$

which is the same obtained result in Ref. [21]. As shown in the previous subsection, the solution (4.8) converges if $l_1 \neq (j + 1) \forall j \in \mathbb{Z}^+$.

4.3. **At** $l = 0, \alpha = 0, \delta = \frac{Sc}{\beta^2}$. Qasim [22] obtained the mass transfer equation:

$$(4.9) \quad t z''(t) + \left(1 - \frac{Sc}{\beta^2} + t\right) z'(t) - m z(t) = 0,$$

for a Jeffrey fluid with heat source/sink, where

$$(4.10) \quad z(0) = 0, \quad z\left(\frac{Sc}{\beta^2}\right) = 1.$$

$Sc > 0$ is the Schmidt parameter and β is a positive parameter. It then follows that

$$(4.11) \quad P = 1 - \frac{Sc}{\beta^2}, \quad Q = 1, \quad n = m - 1, \quad \alpha = 0, \quad \delta = \frac{Sc}{\beta^2}.$$

Inserting these values into (4.3), we obtain

$$(4.12) \quad z(t) = \left(\frac{\beta^2}{Sc} t\right)^{\frac{Sc}{\beta^2}} \frac{{}_1F_1\left[\frac{Sc}{\beta^2} - m, \frac{Sc}{\beta^2} + 1, -t\right]}{{}_1F_1\left[\frac{Sc}{\beta^2} - m, \frac{Sc}{\beta^2} + 1, -\frac{Sc}{\beta^2}\right]}.$$

The solution (4.12) can be verified by substitution. It should be mentioned that the solution obtained by Qasim [22] as

$$(4.13) \quad z(t) = \left(\frac{\beta^2}{Sc} t\right)^{\frac{Sc}{\beta^2}} \frac{{}_1F_1\left[\frac{Sc}{\beta^2} - m, 2\frac{Sc}{\beta^2} + 1, -t\right]}{{}_1F_1\left[\frac{Sc}{\beta^2} - m, 2\frac{Sc}{\beta^2} + 1, -\frac{Sc}{\beta^2}\right]},$$

does not satisfy Eq. (4.9). Since $\beta > 0$ and $Sc > 0$, then the magnitude $\left(\frac{Sc}{\beta^2} + 1\right)$ is never a zero or a negative integer. Hence, the solution (4.12) converges for all positive values of β and Sc .

4.4. **At** $l \neq 0, \alpha = 0, \delta = \frac{Pr}{\beta^2}$. The following heat transfer equation

$$(4.14) \quad t^2 z''(t) + \left(\left(1 - \frac{Pr}{\beta^2}\right)t + t^2\right) z'(t) - \left(mt - \frac{\gamma_1}{\beta^2}\right) z(t) = 0,$$

was also obtained by Qasim [22], where

$$(4.15) \quad z(0) = 0, \quad z\left(\frac{Pr}{\beta^2}\right) = 1.$$

$Pr > 0$ is the Prandtl number, $\beta > 0$, and γ_1 is the heat generation/absorption parameter. In this case, we have

$$(4.16) \quad P = 1 - \frac{Pr}{\beta^2}, \quad Q = 1, \quad n = m - 1, \quad \alpha = 0, \quad l = \frac{\gamma_1}{\beta^2}, \quad \delta = \frac{Pr}{\beta^2}.$$

At $l = \frac{\gamma_1}{\beta^2}$, γ is given by

$$(4.17) \quad \gamma = \frac{1}{2} \left(\frac{\text{Pr}}{\beta^2} - \sqrt{\frac{\text{Pr}^2}{\beta^4} - \frac{4\gamma_1}{\beta^2}} \right),$$

which be written as

$$(4.18) \quad \gamma = \frac{1}{2} \left(\frac{\text{Pr}}{\beta^2} - \sqrt{\frac{\text{Pr}^2}{\beta^4} - \frac{4\gamma_1}{\beta^2}} \right) = k_1 - k_2,$$

where

$$(4.19) \quad k_1 = \frac{\text{Pr}}{2\beta^2}, \quad k_2 = \frac{\sqrt{\text{Pr}^2 - 4\gamma_1\beta^2}}{2\beta^2}.$$

Hence, the solution of the present model is

$$(4.20) \quad z(t) = \left(\frac{\beta^2}{\text{Pr}} t \right)^{k_1+k_2} \frac{{}_1F_1[k_1 + k_2 - m, 2k_1 + 1, -t]}{{}_1F_1[k_1 + k_2 - m, 2k_1 + 1, -\frac{\text{Pr}}{\beta^2}]},$$

which is the same exact solution obtained by Qasim [22]. Since β and Pr are always positives, then $(2k_1 + 1) = \left(\frac{\text{Pr}}{\beta^2} + 1\right)$ is never a zero or a negative integer. Hence, the solution (4.20) converges for all positive values of β and Pr .

4.5. **At $l \neq 0$, $\alpha = 0$, $\delta = -\text{Sc}^*$.** Kameswaran et. al [2] obtained following equation for the mass transfer of nanofluids:

$$(4.21) \quad t^2 z''(t) + ((1 - \text{Sc}^*) t - t^2) z'(t) + (2t - \gamma_2 \text{Sc}^*) z(t) = 0,$$

subject to

$$(4.22) \quad z(0) = 0, \quad z(-\text{Sc}^*) = 1,$$

where γ_2 is the parameter of scaled chemical reaction and Sc^* is the modified Schmidt number. Thus

$$(4.23) \quad P = 1 - \text{Sc}^*, \quad Q = -1, \quad n = 1, \quad \alpha = 0, \quad l = -\gamma_2 \text{Sc}^*, \quad \delta = -\text{Sc}^*.$$

At $l = -\gamma_2 \text{Sc}^*$, we obtain γ as

$$(4.24) \quad \gamma = \frac{1}{2} \left(\text{Sc}^* - \sqrt{(\text{Sc}^*)^2 + 4\gamma_2 \text{Sc}^*} \right),$$

or

$$(4.25) \quad \gamma = \frac{c_1 - d_1}{2}, \quad c_1 = \text{Sc}^*, \quad d_1 = (\text{Sc}^*)^2 + 4\gamma_2 \text{Sc}^*.$$

The solution of the present model is in the form:

$$(4.26) \quad z(t) = \left(-\frac{t}{\text{Sc}^*} \right)^{\frac{c_1+d_1}{2}} \frac{{}_1F_1[\frac{c_1+d_1}{2} - 2, d_1 + 1, t]}{{}_1F_1[\frac{c_1+d_1}{2} - 2, d_1 + 1, -\text{Sc}^*]},$$

which agrees with Kameswaran et. al [2]. According to the physical values taken by the authors [2], the magnitude $d_1 + 1 = (\text{Sc}^*)^2 + 4\gamma_2\text{Sc}^* + 1$ is always positive and this admits the convergence of the solution (4.26).

5. CONCLUSION

In this paper, a general solution was obtained for a class of singular BVPs arise in the field of nanofluids. The solution was derived in terms of the hypergeometric series. The studied class reduced to several published physical models at particular choices of the involved parameters. The obtained solutions were compared with the corresponding results of several models in the literature. It was found that the results in the literature were recovered as special cases of the current ones. Furthermore, this work can be extended in the near future to deal with the recently published physical models [23-25].

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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