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NEW TYPES OF BIPOLAR FUZZY IDEALS OF BCK-ALGEBRAS

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Abstract. The notions of bipolar fuzzy closed, bipolar fuzzy positive implicative, bipolar fuzzy implicative ideals of BCK-algebras are introduced, and related properties are investigated. Characterizations of a closed, bipolar fuzzy positive implicative, bipolar fuzzy implicative ideals of BCK-algebras are given, and several properties are discussed. Finally, we prove that if T is an implicative BCK-algebra, then a fuzzy subset μ of T is a bipolar fuzzy ideal of T if and only if it is a bipolar fuzzy implicative ideal of T.

1. Introduction

BCK-algebras and BCI-algebras are two classes of non-classical logic algebras which were introduced by Y. Imai and K. Iseki in 1966 (see [9, 10]). They are algebraic formulation of BCK-system and BCI-system in combinatory logic.

Fuzzy sets, which were introduced by Zadeh [29], deal with possibilistic uncertainty, connected with imprecision of states, perceptions and preferences. After the introduction of fuzzy sets by Zadeh, fuzzy set theory has become an active area of research in various fields. These are widely scattered over many

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disciplines such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. The elements in fuzzy sets have degrees of belonging which range between 0 and 1. If the membership degree of an element is 0 then the element does not belong to the fuzzy set and it completely belongs to the correspondent fuzzy set if the membership is 1. If the membership lies between $(0, 1)$ then it belongs partially to the fuzzy set. Among such elements some have irrelevant characteristics to the property corresponding to a fuzzy set and the others have contrary characteristics to the property and the traditional fuzzy set representation cannot be described it. Moreover, Muhiuddin et al. studied the fuzzy set theoretical approach to the BCK/BCI-algebras on various aspects (see for e.g., [22–25]). Also, some related concepts of fuzzy sets in different algebras have been studied in [1, 3, 4, 6, 26–28].

As a generalization of traditional fuzzy sets, Zhang [30] first introduced the bipolar fuzzy sets concept . Also, Lee [16, 17] studied the bipolar fuzzy sets in which a negative (resp. positive) membership degree is given for each element in the fuzzy set that ranges over the interval $[-1, 0]$ (resp. $[0, 1]$). Later on, a number of research papers have been devoted to the study of bipolar fuzzy set theory in several algebraic structures (see for e.g., [7, 12, 14, 15, 20, 21].

In this paper, we introduce the notions of bipolar fuzzy closed, bipolar fuzzy positive implicative, bipolar fuzzy implicative ideals of BCK-algebras and investigate related properties. We give characterizations of a closed, bipolar fuzzy positive implicative, bipolar fuzzy implicative ideals of BCK-algebras and several properties are discussed. Finally, we prove that if T is an implicative BCK-algebra, then a fuzzy subset μ of T is a bipolar fuzzy ideal of T if and only if it is a bipolar fuzzy implicative ideal of T.

2. **Preliminaries**

We review some definitions and properties that will be useful in our results.

Definition 2.1. *An algebra $(T; *, 0)$ of kind $(2, 0)$ is called a BCK-algebra if it satisfies the following conditions:*

(K_1) $((t * u) * (t * v)) * (v * u) = 0$,

(K_2) $(t * (t * u)) * u = 0$,

(K_3) $t * t = 0$,

(K_4) $0 * t = 0$,

(K_5) $t * u = 0$ *and* $u * t = 0 \Rightarrow t = u$,

for all $t, u, v \in T$

In a BCK-algebra, the following are true.

(K_6) $t * 0 = t$,

(K_7) $(t * u) * v = (t * v) * u$.

A nonempty subset X of a BCK-algebra T is called an ideal of T if it satisfies

(I_1) $0 \in X$,

(I_2) $\forall t, u \in T, t * u \in X, u \in X \Rightarrow t \in X$.

A nonempty subset X of a BCK-algebra T is called an implicative ideal of T if it satisfies (I_1) and

(I_3) $\forall t, u, v \in T, ((t * (u * t)) * v \in X, v \in X \Rightarrow t \in X$.

A fuzzy set μ in T is called a fuzzy ideal of T if it satisfies

(F_1) $\mu(0) \geq \mu(t)$,

(F_2) $\mu(t) \geq \mu(t * u) \wedge \mu(u)$.

A fuzzy positive implicative ideal of T is a fuzzy set μ in T which satisfies

(F_1) and (F_3) $\mu(t * v) \geq \mu((t * u) * v) \wedge \mu(u * v)$.

A fuzzy implicative ideal of T is a fuzzy set μ in T which satisfies

(F_1) and (F_4) $\mu(t) \geq \mu((t * (u * t)) * v) \wedge \mu(v)$.

For more information regarding BCK-algebras, we refer the reader to [19]

Lemma 2.1. *[15] In a BCK-algebra T, every bipolar fuzzy ideal of T is a bipolar fuzzy subalgebra of T.*

3. Bipolar fuzzy ideal

In the following sections, T denotes a BCK-algebra unless otherwise specified.

For any family $\{\gamma_i \mid i \in \Gamma\}$ of real numbers, we define

$$\vee\{\gamma_i \mid i \in \Gamma\} := \begin{cases} max\{\gamma_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ sup\{\gamma_i \mid i \in \Gamma\} & \text{otherwise,} \end{cases}$$

$$\wedge\{\gamma_i \mid i \in \Gamma\} := \begin{cases} min\{\gamma_i \mid i \in \Gamma\} & \text{if } \Gamma \text{ is finite,} \\ inf\{\gamma_i \mid i \in \Gamma\} & \text{otherwise.} \end{cases}$$

Moreover, if $\Gamma = \{1, 2, ..., n\}$, then $\vee\{\gamma_i \mid i \in \Gamma\}$ and $\wedge\{\gamma_i \mid i \in \Gamma\}$ are denoted by $\gamma_1 \vee \gamma_2 \vee ... \vee \gamma_n$ and $\gamma_1 \wedge \gamma_2 \wedge ... \wedge \gamma_n$, respectively.

For a bipolar fuzzy set $\mu = (T; \mu_n, \mu_p)$ and $(\alpha, \beta) \in [-1, 0) \times (0, 1]$, we define (N) $N(\mu; \alpha) := \{t \in T : \mu_n(t) \leq \alpha\}$ is called the *negative* $\alpha - cut$ of μ.

(P) $P(\mu; \beta) := \{t \in T : \mu_p(t) \geq \beta\}$ is called *positive* $\beta - cut$ of μ.

The set

$$C(\mu;(\alpha,\beta)) := N(\mu;\alpha) \cap P(\mu;\beta)$$

is called the $(\alpha,\beta)-cut$ of $\mu = (T;\mu_n,\mu_p)$.

For every $k \in (0,1)$, if $(\alpha,\beta) = (-k,k)$, then the set

$$C(\mu;k) := N(\mu;-k) \cap P(\mu;k)$$

is called the $k-cut$ of $\mu = (T;\mu_n,\mu_p)$.

Definition 3.1. *[15] A bipolar fuzzy set $\mu = (T;\mu_n,\mu_p)$ in a BCK-algebra T is called a bipolar fuzzy ideal of T if it satisfies the following assertions:*

(BF_1) $(\forall t \in T)$ $(\mu_n(0) \leq \mu_n(t),\ \mu_p(0) \geq \mu_p(t))$,

(BF_2) $(\forall t,u \in T)$ $(\mu_n(t) \leq \mu_n(t*u) \vee \mu_n(u),\ \mu_p(t) \geq \mu_p(t*u) \wedge \mu_p(u))$.

Example 3.1. *Consider the BCK-algebra $(T;*,0)$ given in Table 1.*

*	0	c	d	e
0	0	0	0	0
c	c	0	0	c
d	d	c	0	d
e	e	e	e	0

Table 1: Cayley table

Define a bipolar fuzzy set $\mu = (T;\mu_n,\mu_p)$ in T by

*	0	c	d	e
μ_n	-0.2	-0.3	-0.3	-0.8
μ_p	0	0.5	0.5	1

Then by routine calculations, $\mu = (T;\mu_n,\mu_p)$ is a bipolar fuzzy ideal of T.

Example 3.2. *Consider the BCK-algebra $(T;*,0)$ given in Table 2.*

*	0	a	c	d	e
0	0	0	0	0	0
a	a	0	a	0	0
c	c	c	0	0	0
d	d	d	d	0	0
e	e	d	e	a	0

Table 2: Cayley table

Define μ by

*	0	a	c	d	e
μ_n	-0.7	-0.6	-0.7	-0.6	-.06
μ_p	0.6	0.1	0.6	0.1	0.1

Then by routine calculations, $\mu = (T;\mu_n,\mu_p)$ is a bipolar fuzzy ideal of T.

Theorem 3.1. *If $\mu = (T;\mu_n,\mu_p)$ is a bipolar fuzzy ideal in T and*

$$I(0) = \{t \in T : \mu_n(t) = \mu_n(0), \mu_p(t) = \mu_p(0)\}.$$

Then $I(0)$ is an ideal of T.

Proof. Let $t,u \in T$ be such that $t * u \in I(0)$ and $u \in I(0)$. Then we have $\mu_n(t*u) = \mu_n(0), \mu_p(t*u) = \mu_p(0), \mu_n(u) = \mu_n(0)$ and $\mu_p(u) = \mu_p(0)$. Thus using Definition 3.1 (BF_2), we get $\mu_n(t) \leq \mu_n(0) \vee \mu_n(0) = \mu_n(0)$ and $\mu_p(t) \geq \mu_n(0) \wedge \mu_n(0) = \mu_n(0)$. On the other hand, we know from Definition 3.1 (BF_1) that $\mu_n(0) \leq \mu_n(t)$, $\mu_p(0) \geq \mu_p(t)$ and so $\mu_n(t) = \mu_n(0)$, $\mu_p(t) = \mu_p(0)$. Hence, $t \in I(0)$. It is obvious that $0 \in I(0)$. Therefore, $I(0)$ is an ideal of T. □

If $\mu = (T;\mu_n,\mu_p)$ is a bipolar fuzzy set in T which satisfies Definition 3.1 (BF_1) then the following example shows that Definition 3.1 (BF_2) is sufficient for $I(0)$ to be an ideal of T.

Example 3.3. *Consider the BCK-algebra $(T;*,0)$ given in Table 3.*

*	0	a	c	d	e
0	0	0	0	0	0
a	a	0	a	0	0
c	c	c	0	c	0
d	d	d	d	0	d
e	e	e	e	e	0

Table 3: Cayley table

Define μ *by*

$*$	0	a	c	d	e
μ_n	-0.7	-0.7	-0.5	-0.3	-.07
μ_p	0.8	0.8	0.4	0.2	0.8

Note that $\mu = (T; \mu_n, \mu_p)$ *is a bipolar fuzzy set in* T *which satisfies Definition 3.1* (BF_1) *and not* (BF_2) *as there exist* $c, e \in T$ *where*

$$\mu_n(c) = -0.5 \not\leq -0.7 = \mu_n(c * e) \vee \mu_n(e).$$

Then $I(0) = \{0, a, e\}$ *is not an ideal of* T *as* $c * e = 0 \in I(0)$ *and* $e \in I(0)$*, but* $c \notin I(0)$*.*

Theorem 3.2. *Let* $\mu = (T; \mu_n, \mu_p)$ *be a bipolar fuzzy set in* T *and let* $\emptyset \neq I \subseteq T$ *such that* $0 \in I$*. Define* $\mu_n(t) = \alpha_1$ *or* α_2 *whenever* $t \in I$ *or* $t \notin I$ *respectively, and* $\mu_p(t) = \beta_1$ *or* β_2 *whenever* $t \in I$ *or* $t \notin I$ *respectively where* $\alpha_1, \alpha_2 \in [-1, 0]$ *such that* $\alpha_1 \leq \alpha_2$ *and* $\beta_1, \beta_2 \in [0, 1]$ *such that* $\beta_1 \geq \beta_2$*. Then Definition 3.1* (BF_2) *and the implication* $(\forall t, u \in T)$ $(t * u \in I$ *and* $u \in I \Rightarrow t \in I)$ *are equivalent.*

Proof. Let $t * u \in I$ and $u \in I$. Then $\mu_n(t * u) = \mu_n(u) = \alpha_1$, and $\mu_p(t * u) = \mu_p(u) = \beta_1$. Therefore, $\mu_n(t) \leq \mu_n(t * u) \vee \mu_n(u) = \alpha_1 \vee \alpha_1 = \alpha_1$ and so $\mu_n(t) = \alpha_1$ as $\alpha_1 \leq \alpha_2$. Similarly, $\mu_p(t) \geq \beta_1$ and so $\mu_p(t) = \beta_1$ as $\beta_1 \geq \beta_2$. So whenever $t * u \in I$ and $u \in I$ we have $t \in I$.
Conversely, if the implication is valid then for $t, u \in T$ we have $\mu_n(t * u) = \mu_n(u) = \mu_n(t) = \alpha_1$, $\mu_p(t * u) = \mu_p(u) = \mu_p(t) = \beta_1$. Then it is obvious to see that Definition 3.1 (BF_2) is satisfied. □

Having $0 \in I$ in Theorem 6.1 and as $(\forall t \in T)$ $(\mu_n(0) = \mu_n(t * t) \leq \mu_n(t)$, $\mu_p(0) = \mu_p(t * t) \geq \mu_p(t))$. Thus $(\forall t \in T)$ $(\mu_n(0) \leq \mu_n(t)$, $\mu_p(0) \geq \mu_p(t))$. That is Definition 3.1 (BF_1) is satisfied. Hence, we have the following theorem.

Theorem 3.3. *Let* $\mu = (T; \mu_n, \mu_p)$ *be a bipolar fuzzy set in* T *and let* $\emptyset \neq I \subseteq T$ *such that* $0 \in I$*. Define* $\mu_n(t) = \alpha_1$ *or* α_2 *whenever* $t \in I$ *or* $t \notin I$ *respectively. Define* $\mu_p(t) = \beta_1$ *or* β_2 *whenever* $t \in I$ *or* $t \notin I$ *respectively where* $\alpha_1, \alpha_2 \in [-1, 0]$ *such that* $\alpha_1 \leq \alpha_2$ *and* $\beta_1, \beta_2 \in [0, 1]$ *such that* $\beta_1 \geq \beta_2$*. Then* $\mu = (T; \mu_n, \mu_p)$ *is a bipolar fuzzy ideal of* T *if and only if* I *is an ideal of* T*.*

Theorem 3.4. *[15] A bipolar fuzzy set* $\mu = (T; \mu_n, \mu_p)$ *in* T *is a bipolar fuzzy ideal of* T *if both the nonempty negative* α*-cut and the nonempty positive* β*-cut of* $\mu = (T; \mu_n, \mu_p)$ *are ideals of* T *for all* $(\alpha, \beta) \in [-1, 0) \times (0, 1]$*.*

Corollary 3.1. *If* $\mu = (T; \mu_n, \mu_p)$ *in* T *is a bipolar fuzzy ideal of* T*, then the* k*-cut of* $\mu = (T; \mu_n, \mu_p)$ *is an ideal of* T *for all* $k \in (0, 1)$*.*

The following example shows the converse of corollary 3.1 may not be true.

Example 3.4. *Let $T = \{0, 1, 2, 3, 4\}$ be a set in which the operation $*$ is defined by the following Cayley table which is given in Table 4.*

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	2	0
3	3	3	3	0	0
4	4	4	3	2	0

Table 4: Cayley table

Then $(T; *, 0)$ is a BCK-algebra. Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T given by

*	0	1	2	3	4
μ_n	-0.7	-0.4	-0.6	-0.4	-0.2
μ_p	0.8	0.2	0.6	0.3	0.4

By routine calculations, we know that

$$C(\mu; 0.4) := N(\mu; -0.4) \cap P(\mu; 0.4) = \{0, 2\}.$$

is an ideal of T, but $\mu = (T; \mu_n, \mu_p)$ is not an bipolar fuzzy ideal of T because

$$\mu_n(4) = -0.2 \not\leq -0.4 = \mu_n(4 * 2) \vee \mu_n(2).$$

Lemma 3.1. *[15] Every bipolar fuzzy ideal $\mu = (T; \mu_n, \mu_p)$ of T satisfies the following implication.*

$$(\forall t, u \in T)(t \leq u \Rightarrow \mu_n(t) \leq \mu_n(u), \mu_p(t) \geq \mu_p(u)).$$

Proposition 3.1. *[15] A bipolar fuzzy set $\mu = (T; \mu_n, \mu_p)$ in T is a bipolar fuzzy ideal of T if and only if for all $t, u, v \in T$, $(t * u) * v = 0$ implies $\mu_n(t) \leq \mu_n(u) \vee \mu_n(v)$ and $\mu_p(t) \geq \mu_p(u) \vee \mu_p(v)$.*

4. Bipolar fuzzy closed ideal

Definition 4.1. *A bipolar fuzzy ideal $\mu = (T; \mu_n, \mu_p)$ in T is said to a bipolar fuzzy closed ideal if $\mu_n(0*t) \leq \mu_n(t)$ and $\mu_p(0 * t) \geq \mu_p(t)$ for all $t \in T$.*

Example 4.1. *Consider the BCI-algebra $(T; *, 0)$ given in Table 5.*

$*$	0	a	c	d	e
0	0	0	c	d	e
a	a	0	c	d	e
c	c	c	0	e	d
d	d	d	e	0	c
e	e	e	d	c	0

Table 5: Cayley table

Define μ by

$*$	0	a	c	d	e
μ_n	-0.8	-0.6	-0.8	-0.4	-.03
μ_p	0.6	0.6	0.4	0.3	0.1

Then $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy closed ideal of T.

As an extension to Theorem 3.3 we give the following theorem.

Theorem 4.1. *Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T and let $\emptyset \neq I \subseteq T$ such that $0 \in I$. Define $\mu_n(t), \mu_p(t)$ as defined in Theorem 3.3. Then $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy closed ideal of T if and only if I is a closed ideal of T.*

Proof. Assume $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy closed ideal of T and we show that $0 * t \in I$, $\forall t \in I$. For $t \in I$, we have $\mu_n(t) = \alpha_1, \mu_p(t) = \beta_1$. As $0 * t \leq 0$ then $\mu_n(0 * t) \leq \mu_n(0)$, $\mu_p(0 * t) \geq \mu_p(0)$. Therefore, $\mu_n(0 * t) \leq \mu_n(0) = \alpha_1$ and so $\mu_n(0 * t) = \alpha_1$ as $\alpha_1 \leq \alpha_2$. Also $\mu_n(0 * t) = \beta_1$ as $\beta_1 \geq \beta_2$. That is, I is a closed ideal. For the converse, let I be a closed ideal of T and show that $\mu_n(0 * t) \leq \mu_n(t)$, $\mu_p(0 * t) \geq \mu_p(t)$. For $t \in T$ we know that $0 * t \leq 0$ and so $\mu_n(0 * t) \leq \mu_n(0) \leq \mu_n(t)$. Also $\mu_p(0 * t) \geq \mu_p(0) \geq \mu_p(t)$. This proves that $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy closed ideal of T. □

Theorem 4.2. *Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T. If one of the following assertion is satisfied*

(1) $(\forall t, u \in T)$ $(\mu_n(t) \leq \mu_n(t * u) \vee \mu_n(u), \mu_p(t) \geq \mu_p(t * u) \wedge \mu_p(u))$

(2) $(\forall t, u \in T)$ $(\mu_n(t) \leq \mu_n(u * t) \vee \mu_n(u), \mu_p(t) \geq \mu_p(u * t) \wedge \mu_p(u))$

then $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy closed ideal of T.

Proof. We will show that using any of the assertions above we get $\mu_n(0 * t) \leq \mu_n(t)$, $\mu_p(0 \ * t) \geq \mu_p(t)$. Using (1), let $u = 0$ then we have $\mu_n(0 * t) \leq \mu_n((0 * t) * 0) \vee \mu_n(0) = \mu_n(0)$. As $0 * t = 0$ we know that $0 \leq t$ and so $\mu_n(0) \leq \mu_n(t)$. Therefore, $\mu_n(0 * t) \leq \mu_n(t)$. Also knowing that $\mu_p(0) \geq \mu_p(t)$ and $\mu_p(0 * t) \geq \mu_p((0 * t) * 0) \wedge \mu_p(0) = \mu_p(0)$ we get $\mu_p(0 * t) \geq \mu_p(t)$.

Use the same approach with assertion (2) and let $u = 0$ to have $\mu_n(0 * t) \leq \mu_n(0 * (0 * t)) \vee \mu_n(0) = \mu_n(0) \leq \mu_n(t)$. Therefore, $\mu_n(0 * t) \leq \mu_n(t)$, $\mu_p(0 * t) \geq \mu_p(t)$. □

5. **Bipolar fuzzy positive implicative ideal**

Definition 5.1. *A bipolar fuzzy set $\mu = (T; \mu_n, \mu_p)$ in T is called bipolar fuzzy positive implicative ideal of T if it satisfies (BF_1) and the following assertions:*
(BF_3) $(\forall t, u, v \in T)$ $\mu_n(t * v) \leq \mu_n((t * u) * v) \vee \mu_n(u * v)$
(BF_4) $(\forall t, u, v \in T)$ $\mu_p(t * v) \geq \mu_p((t * u) * v) \wedge \mu_p(u * v)$.

Example 5.1. *Let $T = \{0, 1, 2, 3\}$ be a set in which the operation $*$ is given by the following Cayley table in Table 6.*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 6: Cayley table

Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T defined as follows

$*$	0	1	2	3
μ_n	-0.6	-0.6	-0.6	-0.4
μ_p	0.8	0.8	0.8	0.6

Then by routine calculations $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy positive implicative ideal of T.

Note that every bipolar fuzzy positive implicative ideal is a bipolar fuzzy ideal, but converse is not true.

Example 5.2. *Consider the BCK-algebra $(T, *, 0)$ given in Example 5.2.*
Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T given by:

$*$	0	1	2	3
μ_n	-0.6	-0.5	-0.5	-0.4
μ_p	0.8	0.7	0.7	0.6

By direct calculations we know that $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy ideal of T but is not a bipolar fuzzy positive implicative ideal as

$$\mu_n(2 * 1) = \mu(1) = -0.5 \not\leq -0.6 = \mu_n((2 * 1) * 1) \vee \mu_n(1 * 1).$$

We provide a condition for a bipolar fuzzy ideal to be a bipolar fuzzy positive implicative ideal.

Theorem 5.1. *If $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy ideal of T satisfying for all $t, u, v \in T$*

(BF_5) $\mu_n(t * v) \leq \mu_n(((t * u) * u) * v) \vee \mu_n(v)$,

(BF_6) $\mu_p(t * v) \geq \mu_p(((t * u) * u) * v) \wedge \mu_p(v)$,

then μ is a bipolar fuzzy positive implicative ideal of T.

Proof. Using (K_1) and (K_7), we have

$((t * v) * v) * (u * v) \leq (t * v) * u = (t * u) * v,\ \forall t, u, v \in T$. Since μ is order reversing, it follows from (BF_3) and (BF_4) that

$$\begin{aligned}\mu_n(t * v) &\leq \mu_n(((t * v) * v) * (u * v)) \vee \mu_n(u * v)\\ &\leq \mu_n((t * u) * v) \vee \mu_n(u * v)\end{aligned}$$

and

$$\begin{aligned}\mu_p(t * v) &\geq \mu_p(((t * v) * v) * (u * v)) \wedge \mu_p(u * v)\\ &\geq \mu_p((t * u) * v) \wedge \mu_p(u * v)\end{aligned}$$

for all $t, u, v \in T$. Hence, μ is a bipolar fuzzy positive implicative ideal of T. □

Theorem 5.2. *Let $w \in T$. If $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy positive implicative ideal of T, then $I(w)$ is a positive implicative ideal of T.*

Proof. Recall that $0 \in I(w)$. Let $t, u \in T$ be such that $(t * u) * v \in I(w)$ and $u * v \in I(w)$. Then $\mu_n(w) \geq \mu_n((t*u)*v)$, $\mu_p(w) \leq \mu_p((t*u)*v)$, $\mu_n(w) \geq \mu_n(u*v)$ and $\mu_p(w) \leq \mu_p(u*v)$. Since $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy positive implicative ideal of T, it follows from (BF_3) and (BF_4) that

$$\mu_n(t * v) \leq \mu_n((t * u) * v) \vee \mu_n(u * v) \leq \mu_n(w)$$

and

$$\mu_p(t * v) \geq \mu_p((t * u) * v) \wedge \mu_p(u * v) \geq \mu_p(w)$$

so that $t * v \in I(w)$. Therefore $I(w)$ is a positive implicative ideal of T. □

Lemma 5.1. *Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy ideal of T. Then*

- *μ is a bipolar fuzzy positive implicative ideal of T if and only if it satisfies*

(BF_7) $\forall t, u \in T, \mu_n(t * u) \leq \mu_n((t * u) * u)$ *and* $\mu_p(t * u) \geq \mu_p((t * u) * u)$.

- *μ is a bipolar fuzzy positive implicative ideal of T if and only if it satisfies*

(BF_8) $\forall t, u, v \in T, \mu_n((t * v) * (u * v)) \leq \mu_n((t * u) * v)$ *and*

$\mu_p((t * v) * (u * v)) \geq \mu_p((t * u) * v)$.

Theorem 5.3. *If μ is a bipolar fuzzy positive implicative ideal of T, then*

(BF_9) $\forall t, u, a, b \in T, ((t * u) * u) * a \leq b \Rightarrow \mu_n(t * u) \leq \mu_n(a) \vee \mu_n(b)$ *and*

$\mu_p(t * u) \geq \mu_p(a) \wedge \mu_p(b)$.

(BF_{10}) $\forall t, u, v, a, b \in T, ((t * u) * v) * a \leq b \Rightarrow \mu_n((t * v) * (u * v)) \leq \mu_n(a) \vee \mu_n(b)$

and $\mu_p((t*v)*(u*v)) \geq \mu_p(a) \wedge \mu_p(b)$.

Proof. Let $t, u, a, b \in T$ be such that $((t*u)*u)*a \leq b$. Using Proposition 3.1, we have

$$\mu_n((t*u)*u) \leq \mu_n(a) \vee \mu_n(b) \text{ and } \mu_p((t*u)*u) \geq \mu_p(a) \wedge \mu_p(b).$$

It follows that from (BF_3), (BF_4), (K_3) and (BF_1),

$$\begin{aligned}\mu_n(t*u) &\leq \mu_n((t*u)*u) \vee \mu_n(u*u)\\ &= \mu_n((t*u)*u) \vee \mu_n(0)\\ &= \mu_n((t*u)*u)\\ \mu_n(t*u) &\leq \mu_n(a) \vee \mu_n(b)\end{aligned}$$

and

$$\begin{aligned}\mu_p(t*u) &\geq \mu_p((t*u)*u) \wedge \mu_p(u*u)\\ &= \mu_p((t*u)*u) \wedge \mu_p(0)\\ &= \mu_p((t*u)*u)\\ \mu_p(t*u) &\geq \mu_p(a) \wedge \mu_p(b).\end{aligned}$$

Now let $t, u, v, a, b \in T$ be such that $((t*u)*v)*a \leq b$, that is, $(((t*u)*v)*a)*b = 0$.

Since μ is a bipolar fuzzy positive implicative ideal of T, it follows from Proposition 3.1 and Lemma 5.1 that

$$\mu_n((t*v)*(u*v)) \leq \mu_n((t*u)*v) \leq \mu_n(a) \vee \mu_n(b)$$

and

$$\mu_p((t*v)*(u*v)) \geq \mu_p((t*u)*v) \geq \mu_p(a) \wedge \mu_p(b).$$

This completes the proof. □

We now give conditions for a bipolar fuzzy set to be a bipolar fuzzy positive implicative ideal.

Theorem 5.4. *Let μ be a bipolar fuzzy set in T satisfying the condition (BF_9). Then μ is a bipolar fuzzy positive implicative ideal of T.*

Proof. We first prove that μ is a bipolar fuzzy ideal of T. Let $t, u, v \in T$ be such that $t*u \leq v$. Then

$$(((t*0)*0)*u)*v = (t*u)*v = 0, \text{ that is, } ((t*0)*0)*u \leq v,$$

which implies from (K_6) and (BF_9) that

$$\mu_n(t) = \mu_n(t*0) \leq \mu_n(u) \vee \mu_n(v)$$

and

$$\mu_p(t) = \mu_p(t*0) \geq \mu_p(u) \wedge \mu_p(v).$$

Therefore, by Proposition 3.1, we know that μ is a bipolar fuzzy ideal of T. Note that $(((t*u)*u)*((t*u)*u))*0 = 0$ for all $t, u \in T$. Using (BF_9) and (BF_1), we have

$$\mu_n(t*u) \leq \mu_n((t*u)*u) \vee \mu_n(0) = \mu_n((t*u)*u)$$

and

$$\mu_p(t*u) \geq \mu_p((t*u)*u) \wedge \mu_p(0) = \mu_p((t*u)*u).$$

So μ is a bipolar fuzzy positive implicative ideal of T by Lemma 5.1. □

Theorem 5.5. *Let μ be a fuzzy set in T satisfying the condition (BF_{10}). Then μ is a bipolar fuzzy positive implicative ideal of T.*

Proof. Let $t, u, a, b \in T$ be such that $((t*u)*u)*a = b$, that is $(((t*u)*u)*a)*b = 0$, which implies from (K_6), (K_3) and (BF_{10}) that

$$\mu_n(t*u) = \mu_n((t*u)*0) = \mu_n((t*u)*(u*u)) \leq \mu_n(a) \vee \mu_n(b)$$

and

$$\mu_p(t*u) = \mu_p((t*u)*0) = \mu_p((t*u)*(u*u)) \geq \mu_n(a) \wedge \mu_p(b).$$

So μ is a bipolar fuzzy positive implicative ideal of T by Theorem 5.4. □

Theorem 5.6. *Let μ and λ be bipolar fuzzy ideals of T such that $\mu_n(0) = \lambda_n(0), \mu_p(0) = \lambda_p(0)$ and $\mu \subseteq \lambda$, that is $\mu_n(t) \leq \lambda_n(t)$ and $\mu_p(t) \geq \lambda_p(t)$ for all $t \in T$. If μ is bipolar fuzzy positive implicative ideal of T, then so is λ.*

Proof. Assume that μ is a bipolar fuzzy positive implicative ideal of T. For any $t, u, v \in T$, we have

$\lambda_n(((t*v)*(u*v))*((t*u)*v))$

$= \lambda_n(((t*v)*((t*u)*v))*(u*v))$ [*by* (K_7)]

$= \lambda_n(((t*((t*u)*v))*v)*(u*v))$ [*by* (K_7)]

$\geq \mu_n(((t*((t*u)*v))*v)*(u*v))$ [*since* $\mu \subseteq \lambda$]

$\geq \mu_n(((t*((t*u)*v))*u)*v)$

$= \mu_n(((t*u)*((t*u)*v))*v)$ [*by* (K_7)]

$= \mu_n(((t*u)*v)*((t*u)*v))$ [*by* (K_7)]

$= \mu_n(0) = \lambda_n(0).$ [*by* (K_3) *and assumption*]

It follows from (BF_1) and (BF_2) that

$\lambda_n((t*v)*(u*v))$

$\leq \lambda_n(((t*v)*(u*v))*((t*u)*v)) \vee \lambda_n((t*u)*v)$

$\leq \lambda_n(0) \vee \lambda_n((t*u)*v)$

$= \lambda_n((t*u)*v)$

and

$\lambda_p(((t*v)*(u*v))*((t*u)*v))$

$= \lambda_p(((t*v)*((t*u)*v))*(u*v))$ [*by* (K_7)]

$= \lambda_p(((t*((t*u)*v))*v)*(u*v))$ [*by* (K_7)]

$\leq \mu_p(((t*((t*u)*v))*v)*(u*v))$ [*since* $\mu \subseteq \lambda$]

$\leq \mu_p(((t*((t*u)*v))*u)*v)$

$= \mu_p(((t*u)*((t*u)*v))*v)$ [*by* (K_7)]

$= \mu_p(((t*u)*v)*((t*u)*v))$ [*by* (K_7)]

$= \mu_p(0) = \lambda_p(0).$ [*by* (K_3) *and assumption*]

It follows from (BF_1) and (BF_2) that

$\lambda_p((t*v)*(u*v))$

$\geq \lambda_p(((t*v)*(u*v))*((t*u)*v)) \wedge \lambda_p((t*u)*v)$

$\geq \lambda_p(0) \wedge \lambda_p((t*u)*v)$

$= \lambda_p((t*u)*v)$

for all $t, u, v \in T$.

Hence, by Lemma 5.1, λ is a bipolar fuzzy positive implicative ideal of T. □

6. Bipolar fuzzy implicative ideal

Definition 6.1. *A bipolar fuzzy set* $\mu = (T; \mu_n, \mu_p)$ *in* T *is called bipolar fuzzy implicative ideal of* T *if it satisfies* (BF_1) *and the following assertions:*

(BF_{11}) $(\forall t, u, v \in T)$ $\mu_n(t) \leq \mu_n((t*(u*t))*v) \vee \mu_n(v)$

(BF_{12}) $(\forall t, u, v \in T)$ $\mu_p(t) \geq \mu_p((t*(u*t))*v) \wedge \mu_p(v)$.

Example 6.1. *Let* $T = \{0, 1, 2, 3\}$ *be a set in which the operation* $*$ *is defined by Table 7:*

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	1	0	2
3	3	3	3	0

Table 7: Cayley table

Hence, $(T; *, 0)$ *is a* BCK*-algebra. Let* $\mu = (T; \mu_n, \mu_p)$ *be a bipolar fuzzy set in* T *defined by*

	0	1	2	3
μ_n	-0.5	-0.5	-0.5	-0.4
μ_p	0.7	0.7	0.7	0.6

By routine calculations, $\mu = (T; \mu_n, \mu_p)$ *is a bipolar fuzzy implicative ideal of* T.

Now, we give a relation between a bipolar fuzzy ideal and a bipolar fuzzy implicative ideal.

Theorem 6.1. *Every bipolar fuzzy implicative ideal of* T *is a bipolar fuzzy ideal of* T.

Proof. Let μ be a bipolar fuzzy implicative ideal of T.Then for all $t, u, v \in T$,

$$\mu_n(t) \leq \mu_n((t*(u*t))*v) \vee \mu_n(v)$$

and

$$\mu_p(t) \geq \mu_p((t*(u*t))*v) \wedge \mu_p(v).$$

Putting $u = t$ and $v = u$,

$$\mu_n(t) \leq \mu_n((t*(t*t))*u) \vee \mu_n(u)$$
$$\mu_n(t) \leq \mu_n((t*0)*u) \vee \mu_n(u)$$
$$\mu_n(t) \leq \mu_n(t*u) \vee \mu_n(u)$$

and

$$\mu_p(t) \geq \mu_p((t*(t*t))*u) \wedge \mu_p(u)$$
$$\mu_p(t) \geq \mu_p((t*0)*u) \wedge \mu_p(u)$$
$$\mu_p(t) \geq \mu_p(t*u) \wedge \mu_p(u).$$

Thus μ satisfies (BF_2). Consequently, μ is a bipolar fuzzy ideal of T by (BF_1). □

In view of Lemma 2.1 and Theorem 6.1, we conclude the following Corollary.

Corollary 6.1. *Every bipolar fuzzy implicative ideal of T is a bipolar fuzzy subalgebra of T.*

The following example shows that the converse of Theorem 6.1 is not true in general.

Example 6.2. *Let $T = \{0, a, c, d, e\}$ be a set in which the operation $*$ is defined by Table 8:*

$*$	0	a	c	d	e
0	0	0	0	0	0
a	a	0	a	0	0
c	c	c	0	0	0
d	d	d	d	0	0
e	e	d	e	a	0

Table 8: Cayley table

Hence, $(T; *, 0)$ is a BCK-algebra. Let $\mu = (T; \mu_n, \mu_p)$ be a bipolar fuzzy set in T defined by

$*$	0	a	c	d	e
μ_n	-0.8	-0.7	-0.8	-0.7	-.07
μ_p	0.7	0.2	0.7	0.2	0.2

By routine calculations, we know that $\mu = (T; \mu_n, \mu_p)$ is a bipolar fuzzy ideal of T, but not a bipolar fuzzy implicative ideal of T since $P(\mu; 0.5) = \{0,\ c\}$ is not an implicative ideal of T.

We provide a condition for bipolar fuzzy ideal to be a bipolar fuzzy implicative ideal.

Theorem 6.2. *If T is an implicative BCK-algebra, then every bipolar fuzzy ideal of T is a bipolar fuzzy implicative ideal of T.*

Proof. Since T is an implicative BCK-algebra, it follows that $t = t * (u * t)$ for all $t, u \in T$. Let μ be a bipolar fuzzy ideal of T. Then by (BF_2)

$$\mu_n(t) \leq \mu_n(t * v) \vee \mu_n(v)$$
$$\mu_n(t) \leq \mu_n((t * (u * t) * v)) \vee \mu_n(v)$$

and

$$\mu_p(t) \geq \mu_p(t * v) \wedge \mu_p(v)$$
$$\mu_p(t) \geq \mu_p((t * (u * t) * v)) \wedge \mu_p(v).$$

Hence, μ is a bipolar fuzzy implicative ideal of T. The proof is complete. □

In view of Theorem 6.1 and Theorem 6.2, we have the following theorem.

Theorem 6.3. *If T is an implicative BCK-algebra, then a fuzzy subset μ of T is a bipolar fuzzy ideal of T if and only if it is a bipolar fuzzy implicative ideal of T.*

Theorem 6.4. *Let μ be a bipolar fuzzy implicative ideal of T. Then the set $T_\mu = \{t \in T : \mu_n(t) = \mu_n(0)$ and $\mu_p(t) = \mu_p(0)\}$ is an implicative ideal of T.*

Proof. Clearly $0 \in T$. Let $t, u, v \in T_\mu$ be such that $(t * (u * t)) * v \in T_\mu$ and $v \in T_\mu$. Then, we have

$$\mu_n((t * (u * t)) * v) = \mu_n(v) = \mu_n(0)$$

and

$$\mu_p((t * (u * t)) * v) = \mu_p(v) = \mu_p(0).$$

It follows that

$$\mu_n(t) \leq \mu_n((t * (u * t)) * v) \vee \mu_n(v)$$
$$\leq \mu_n(0) \vee \mu_n(0)$$
$$\mu_n(t) \leq \mu_n(0)$$

and

$$\mu_p(t) \geq \mu_p((t * (u * t)) * v) \wedge \mu_p(v)$$
$$\geq \mu_p(0) \vee \mu_p(0)$$
$$\mu_p(t) \geq \mu_p(0).$$

By using (BF_1), we get $\mu_n(t) = \mu_n(0)$ and $\mu_p(t) = \mu_p(0)$ and hence $t \in T_\mu$. Consequently, T_μ is an implicative ideal of T. □

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