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SURFACES AS GRAPHS OF FINITE TYPE IN $\mathbb{H}^2 \times \mathbb{R}$

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Abstract. In this paper, we prove that $\Delta X = 2\mathrm{H}$ where Δ is the Laplacian operator, $r = (x, y, z)$ the position vector field and H is the mean curvature vector field of a surface $\mathcal{S}$ in $\mathbb{H}^2 \times \mathbb{R}$ and we study surfaces as graphs in $\mathbb{H}^2 \times \mathbb{R}$ which has finite type immersion.

1. Introduction

The $\mathbb{H}^2 \times \mathbb{R}$ geometry is one of eight homogeneous Thurston 3-geometries

$$E^3,\ S^3,\ H^3,\ S^2 \times \mathbb{R},\ H^2 \times \mathbb{R},\ \widetilde{SL(2, \mathbb{R})},\ Nil,\ Sol.$$

The Riemannian manifold (M, g) is called homogeneous if for any $x,\ y \in M$ there exists an isometry $\phi : M \to M$ such that $y = \phi(x)$. The two and three-dimensional homogeneous geometries are discussed in detail in [6] .

A Euclidean submanifold is said to be of finite Chen-type if its coordinate functions are a finite sum of eigenfunctions of its Laplacian [3]. B. Y. Chen posed the problem of classifying the finite type surfaces in the

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3-dimensional Euclidean space $\mathbb{E}^3$. Further, the notion of finite type can be extended to any smooth function on a submanifold of a Euclidean space or a pseudo-Euclidean space.

Let $\mathcal{S}$ be a 2-dimensional surface of the Euclidean 3-space $\mathbb{E}^3$. If we denote by r, H and Δ the position vector field, the mean curvature vector field and the Laplace operator of $\mathcal{S}$ respectively, then it is well-known that [3]

$$\Delta r = -2H. \tag{1.1}$$

A well-known result due to Takahashi states that minimal surfaces and spheres are the only surfaces in $\mathbb{E}^3$ satisfying the condition $\Delta r = \lambda r$ for a real constant λ. From (1.1), we know that minimal surfaces and spheres also verify the condition

$$\Delta H = \lambda H,\ \lambda \in \mathbb{R}. \tag{1.2}$$

Equation (1.1) shows that $\mathcal{S}$ is a minimal surface of $\mathbb{E}^3$ if and only if its coordinate functions are harmonic. In [9], D. W. Yoon studied surfaces invariant under the 1-parameter subgroup in Sol_3.

In 2012, M. Bekkar and B. Senoussi [1] studied the translation surfaces in the 3-dimensional Euclidean and Lorentz-Minkowski space under the condition

$$\Delta^{III} r_i = \mu_i r_i,\ \mu_i \in \mathbb{R},$$

where Δ^{III} denotes the Laplacian of the surface with respect to the third fundamental form III.

A surface $\mathcal{S}$ in the Euclidean 3-space $\mathbb{E}^3$ is called minimal when locally each point on the surface has a neighborhood which is the surface of least area with respect to its boundary [5]. In 1775, J. B. Meusnier showed that the condition of minimality of a surface in $\mathbb{E}^3$ is equivalent with the vanishing of its mean curvature function, $H = 0$.

Let $z = f(x, y)$ define a graph $\mathcal{S}$ in the Euclidean 3-space $\mathbb{E}^3$. If $\mathcal{S}$ is minimal, the function f satisfies

$$(1 + (f_y)^2)f_{xx} - 2f_{xy}f_xf_y + (1 + (f_x)^2)f_{yy} = 0,$$

which was obtained by J. L. Lagrange in 1760.

In 1835, H. F. Scherk studied translation surfaces in $\mathbb{E}^3$ and proved that, besides the planes, the only minimal translation surfaces are given by

$$z(x,\ y) = \frac{1}{\lambda}\log|\cos(\lambda x)| - \frac{1}{\lambda}\log|\cos(\lambda y)|,$$

where λ is a non-zero constant. In 1991, F. Dillen, L. Verstraelen and G. Zafindratafa. [4] generalized this result to higher-dimensional Euclidean space.

In 2015, D. W. Yoon [8] studied translation surfaces in the product space $\mathbb{H}^2 \times \mathbb{R}$ and classified translation surfaces with zero Gaussian curvature in $\mathbb{H}^2 \times \mathbb{R}$.

In 2019, B. Senoussi, M. Bekkar [7] studied translation surfaces of finite type in H_3 and Sol_3 and the authors gived some theorems.

A surface $\mathcal{S}(\gamma_1,\ \gamma_2)$ in $\mathbb{H}^2 \times \mathbb{R}$ is a surface parametrized by

$$\mathcal{S} : \Omega \subseteq \mathbb{R}^2 \to \mathbb{H}^2 \times \mathbb{R}, \qquad X(s,\ t) = \gamma_1(s) * \gamma_2(t) \ or \ X(s,\ t) = \gamma_2(t) * \gamma_1(s),$$

where γ_1 and γ_2 are any generating curves in $\mathbb{R}^3$. Since the multiplication $*$ is not commutative.

In this work we study the surfaces as graphs of functions $\varphi = f(s,\, t))$ in $\mathbb{H}^2 \times \mathbb{R}$ satisfy the condition

$$\Delta x_i = \lambda_i x_i,\ \ \lambda_i \in \mathbb{R}. \tag{1.3}$$

2. **Preliminaries**

Let $\mathbb{H}^2$ be represented by the upper half-plane model $\{(x,\, y) \in \mathbb{R} \mid \ y > 0\}$ equipped with the metric $g_{\mathbb{H}} = \dfrac{(dx^2 + dy^2)}{y^2}$. The space $\mathbb{H}^2$, with the group structure derived by the composition of proper affine maps, is a Lie group and the metric $g_{\mathbb{H}}$ is left invariant.

Therefore, the product space $\mathbb{H}^2 \times \mathbb{R}$ is a Lie group with the left invariant product metric

$$g = \frac{(dx^2 + dy^2)}{y^2} + dz^2,$$

we can define the multiplication law on $\mathbb{H}^2 \times \mathbb{R}$ as follows

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (y\bar{x} + x, y\bar{y}, z + \bar{z}).$$

The left identity is $(0, 1, 0)$ and the inverse of (x, y, z) is $(-\dfrac{x}{y}, \dfrac{1}{y}, -z)$, on $\mathbb{H}^2 \times \mathbb{R}$ a left-invariant metric

$$ds^2 = (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2,$$

where

$$\omega^1 = \frac{dx}{y}, \ \ \omega^2 = \frac{dy}{y}, \ \ \omega^3 = dz,$$

is the orthonormal coframe associated with the orthonormal frame

$$e_1 = y\frac{\partial}{\partial x}, \ \ e_2 = y\frac{\partial}{\partial y}, \ \ e_3 = \frac{\partial}{\partial z},$$

The corresponding Lie brackets are

$$[e_1,\ e_2] = -e_1, \quad [e_i,\ e_i] = [e_3,\ e_1] = [e_2,\ e_3] = 0, \forall i = 1, 2, 3.$$

The Levi-Civita connection ∇ of $\mathbb{H}^2\times\mathbb{R}$ is given by

$$\begin{pmatrix} \nabla_{e_1}e_1 \\ \nabla_{e_1}e_2 \\ \nabla_{e_1}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \quad \nabla_{e_2}e_i = \nabla_{e_3}e_i = 0, \quad \forall i = 1,2,3.$$

Let $\mathcal{S}$ be an immersed surface in $\mathbb{H}^2\times\mathbb{R}$ given as the graph of the function $z = f(x,y)$. Hence, the position vector is described by $r(x,y) = (x, y, f(x,y))$ and the tangent vectors $r_x = \dfrac{\partial r}{\partial x}$ and $r_y = \dfrac{\partial r}{\partial y}$ in terms of the orthonormal frame (e_1, e_2, e_3) are described by

$$r_x = \frac{\partial}{\partial x} + f_r\frac{\partial}{\partial z} = \frac{1}{y}e_1 + f_x e_3, \tag{2.1}$$

$$r_y = \frac{\partial}{\partial y} + f_y\frac{\partial}{\partial z} = \frac{1}{y}e_2 + f_y e_3. \tag{2.2}$$

Definition 2.1. *[3] The immersion (S,r) is said to be of finite Chen-type k if the position vector X admits the following spectral decomposition*

$$r = r_0 + \sum_{i=1}^{k} r_i,$$

where r_i are $\mathbb{E}^3$-valued eigenfunctions of the Laplacian of (S,r) : $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$, $i = 1,2,..,k$. If λ_i are different, then $\mathcal{S}$ is said to be of k-type.

For the matrix $G = (g_{ij})$ consisting of the components of the induced metric on $\mathcal{S}$, we denote by $G^{-1} = (g^{ij})$ (resp. $D = \det(g_{ij})$) the inverse matrix (resp. the determinant) of the matrix (g_{ij}). The Laplacian Δ on $\mathcal{S}$ is, in turn, given by

$$\Delta = \frac{-1}{\sqrt{|D|}}\sum_{ij}\frac{\partial}{\partial r^i}\big(\sqrt{|D|}g^{ij}\frac{\partial}{\partial r^j}\big). \tag{2.3}$$

If $r = r(x,y) = (r_1 = r_1(x,y), r_2 = r_2(x,y), r_3 = r_3(x,y))$ is a function of class C^2 then we set

$$\Delta r = (\Delta r_1, \Delta r_2, \Delta r_3).$$

3. Surfaces as graphs of finite type in $\mathbb{H}^2\times\mathbb{R}$

Let $\mathcal{S}$ be a graph of a smooth function

$$f : \Omega \subset \mathbb{R}^2 \to \mathbb{R}.$$

We consider the following parametrization of $\mathcal{S}$

$$r(x,y) = (x, y,\ f(x,\ y)), \quad (x,y) \in \Omega.$$

Theorem 3.1. *A Beltrami formula in $\mathbb{H}^2 \times \mathbb{R}$ is given by the following:*

$$\Delta r = 2\mathbf{H},$$

where Δ is the Laplacian of the surface and $\mathbf{H}$ is the mean curvature vector field of $\mathcal{S}$.

Proof. A basis of the tangent space $T_p\mathcal{S}$ associated to this parametrization is given by

$$\begin{aligned} r_x = & \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial z} = & \frac{1}{y} e_1 + f_x e_3, \\ r_y = & \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial z} = & \frac{1}{y} e_2 + f_y e_3, \end{aligned}$$

The coefficients of the first fundamental form of $\mathcal{S}$ are given by

$$E = g(r_x,\ r_x) = \frac{1}{y^2} + f_x^2, \quad F = g(r_x,\ r_y) = f_x f_y, \quad G = g(r_y,\ r_y) = \frac{1}{y^2} + f_y^2.$$

The unit normal vector field $\mathbf{N}$ on $\mathcal{S}$ is given by

$$\mathbf{N} = \frac{1}{W}\left(-\frac{1}{y} f_x e_1 - \frac{1}{y} f_y e_2 + \frac{1}{y^2} e_3 \right),$$

where $W = \sqrt{\frac{1}{y^4} + \frac{1}{y^2} f_x^2 + \frac{1}{y^2} f_y^2}$.

To compute the second fundamental form of $\mathcal{S}$, we have to calculate the following

$$\begin{aligned} r_{xx} &= \nabla_{r_x} r_x = \frac{1}{y^2} e_2 + f_{xx} e_3, \\ r_{xy} &= \nabla_{r_x} r_y = \nabla_{r_y} r_x = -\frac{1}{y^2} e_1 + f_{xy} e_3, \\ r_{yy} &= \nabla_{r_y} r_y = -\frac{1}{y^2} e_2 + f_{yy} e_3. \end{aligned} \tag{3.1}$$

So, the coefficients of the second fundamental form of $\mathcal{S}$ are given by

$$\begin{aligned} L &= g(\nabla_{r_x} r_x,\ \mathbf{N}) = \frac{1}{W y^2}\left(f_{xx} - \frac{1}{y} f_y \right), \\ M &= g(\nabla_{r_x} r_y,\ \mathbf{N}) = \frac{1}{W y^2}\left(f_{xy} + \frac{1}{y} f_x \right), \\ N &= g(\nabla_{r_y} r_y,\ \mathbf{N}) = \frac{1}{W y^2}\left(f_{yy} + \frac{1}{y} f_y \right), \end{aligned}$$

where $W = \sqrt{\frac{1}{y^4} + \frac{1}{y^2} f_x^2 + \frac{1}{y^2} f_y^2}$.

Thus, the mean curvature H of $\mathcal{S}$ is given by

$$H = \frac{EN - 2FM + GL}{2W^2}.$$

$$H = \frac{1}{2W^3 y^2}\left[\frac{1}{y^2}(f_{xx} + f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2 f_x f_y f_{xy} \right].$$

By (2.3), the Laplacian operator Δ of r can be expressed as

$$\Delta = -\frac{1}{W^4}\left[W^2\left(G\frac{\partial^2}{\partial x^2} - 2F\frac{\partial^2}{\partial x\partial y} + E\frac{\partial^2}{\partial y^2}\right) + \Delta_1\frac{\partial}{\partial x} + \Delta_2\frac{\partial}{\partial y}\right], \tag{3.2}$$

where

$$\begin{aligned}\Delta_1 =& \frac{2}{y^2}f_y f_x^2 f_{xy} - \frac{1}{y^4}f_x f_{xx} - \frac{1}{y^2}f_x f_y^2 f_{xx} - \frac{1}{y^4}f_x f_{yy} - \frac{1}{y^2}f_x^3 f_{yy} \\ & - \frac{2}{y^5}f_x f_y - \frac{1}{y^3}f_x^3 f_y - \frac{1}{y^3}f_x f_y^3,\end{aligned}$$

and

$$\begin{aligned}\Delta_2 =& \frac{2}{y^2}f_x f_y^2 f_{xy} - \frac{1}{y^4}f_y f_{yy} - \frac{1}{y^2}f_x^2 f_y f_{yy} - \frac{1}{y^4}f_y f_{xx} - \frac{1}{y^2}f_y^3 f_{xx} \\ & - \frac{1}{y^5}f_y^2 + \frac{1}{y^5}f_x^2 + \frac{1}{y^3}f_x^4 + \frac{1}{y^3}f_x^2 f_y^2.\end{aligned}$$

By a straightforward computation, the Laplacian operator Δ of r with the help of (3.1) and (3.2) turns out to be

$$\Delta r = -\frac{1}{W^4}\left[\begin{array}{c}\left(\frac{2}{y^3}f_x^2 f_y f_{xy} - \frac{1}{y^5}f_x f_{xx} - \frac{1}{y^3}f_x f_y^2 f_{xx} - \frac{1}{y^5}f_x f_{yy} - \frac{1}{y^3}f_x^3 f_{yy} + \frac{1}{y^4}f_x^3 f_y + \frac{1}{y^4}f_x f_y^3\right)e_1 \\ +\left(\frac{2}{y^3}f_x f_y^2 f_{xy} - \frac{1}{y^5}f_y f_{yy} - \frac{1}{y^3}f_x^2 f_y f_{yy} - \frac{1}{y^5}f_y f_{xx} - \frac{1}{y^3}f_y^3 f_{xx} + \frac{1}{y^4}f_x^2 f_y^2 + \frac{1}{y^4}f_y^4\right)e_2 \\ +\left(-\frac{2}{y^4}f_x f_y f_{xy} - \frac{1}{y^5}f_x^2 f_y - \frac{1}{y^5}f_y^3 + \frac{1}{y^6}f_{xx} + \frac{1}{y^4}f_y^2 f_{xx} + \frac{1}{y^6}f_{yy} + \frac{1}{y^4}f_x^2 f_{yy}\right)e_3\end{array}\right],$$

$$\Delta r = \left[\begin{array}{c}\left(\frac{-f_x}{Wy}\right)\frac{1}{W^3y^2}\left(\frac{1}{y^2}(f_{xx}+f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2f_x f_y f_{xy}\right)e_1 \\ +\left(\frac{-f_y}{Wy}\right)\frac{1}{W^3y^2}\left(\frac{1}{y^2}(f_{xx}+f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2f_x f_y f_{xy}\right)e_2 \\ +\left(\frac{1}{Wy^2}\right)\frac{1}{W^3y^2}\left(\frac{1}{y^2}(f_{xx}+f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2f_x f_y f_{xy}\right)e_3\end{array}\right],$$

$$\Delta r = \frac{1}{W^3y^2}\left(\frac{1}{y^2}(f_{xx}+f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2f_x f_y f_{xy}\right)\left(\begin{array}{c}\left(\frac{-f_x}{Wy}\right)e_1 \\ +\left(\frac{-f_y}{Wy}\right)e_2 \\ +\left(\frac{1}{Wy^2}\right)e_3\end{array}\right),$$

thus we get

$$\begin{aligned}\Delta r &= 2HN, \\ &= 2\mathbf{H},\end{aligned} \tag{3.3}$$

where $\mathbf{H}$ is the mean curvature vector field of $\mathcal{S}$.

$\mathcal{S}$ is a minimal surfaces in $\mathbb{H}^2 \times \mathbb{R}$ if and only if its coordinate functions are harmonic . □

4. Surfaces as graphs in $\mathbb{H}^2 \times \mathbb{R}$ satisfying $\triangle x_i = \lambda_i x_i$

Let $\mathcal{S}$ be an immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ given as the graph of function $z = f(x, y)$. Hence, the vector position is described by $r(x, y) = (x, y, f(x, y))$.

We have

$$r_x = \frac{1}{y} e_1 + f_x e_3, \quad r_y = \frac{1}{y} e_2 + f_y e_3,$$

where $r_x = \dfrac{\partial r}{\partial x}$, $r_y = \dfrac{\partial r}{\partial x}$, *and* $f_x = \dfrac{\partial f}{\partial x}$, $f_y = \dfrac{\partial f}{\partial y}$.

From an earlier results the mean curvature H of $\mathcal{S}$ and the unit normal vector field $\mathbf{N}$ on $\mathcal{S}$ are given by

$$H = \frac{1}{2W^3 y^2} \left[\frac{1}{y^2}(f_{xx} + f_{yy}) + (f_x^2 f_{yy} + f_y^2 f_{xx}) - \frac{1}{y}(f_x^2 f_y + f_y^3) - 2 f_x f_y f_{xy} \right],$$

and

$$\mathbf{N} = \frac{1}{W} \left(-\frac{1}{y} f_x e_1 - \frac{1}{y} f_y e_2 + \frac{1}{y^2} e_3 \right), \tag{4.1}$$

where $W = \sqrt{\dfrac{1}{y^4} + \dfrac{1}{y^2} f_x^2 + \dfrac{1}{y^2} f_y^2}$.

If the vector position on the tangent space $T_p\mathcal{S}$ is described by $r = (x, y, f(x, y))$

$$r(x, y) = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + f(x, y) \frac{\partial}{\partial z},$$

then

$$r(x, y) = \frac{x}{y} e_1 + e_2 + f(x, y) e_3. \tag{4.2}$$

The equation (1.3) by means of (3.3), (4.1) and (4.2) gives rise to the following system of ordinary differential equations

$$\left(\frac{2H}{W} \right) f_x = -\lambda_1 x, \tag{4.3}$$

$$\left(\frac{2H}{W} \right) f_y = -\lambda_2 y, \tag{4.4}$$

$$\frac{2H}{W} = \lambda_3 y^2 f. \tag{4.5}$$

Therefore, the problem of classifying the surfaces $\mathcal{S}$ of (1.3) is reduced to the integration of this system of ordinary differential equations.

Next we study it according to the constants λ_1, λ_2 and λ_3.

Case 1. Let $\lambda_3 = 0$. In this case the system (4.3), (4.4) and (4.5) is reduced equivalently to

$$\left(\frac{2H}{W}\right) f_x = -\lambda_1 x, \tag{4.6}$$

$$\left(\frac{2H}{W}\right) f_y = -\lambda_2 y, \tag{4.7}$$

$$\frac{2H}{W} = 0. \tag{4.8}$$

The equation (4.8) implies that the mean curvature H is identically zero. Thus, the surface $\mathcal{S}$ is minimal; and we get also $\lambda_1 = \lambda_2 = 0$.

Case 2. Let $\lambda_3 \neq 0$. in this case we study the general system (4.3), (4.4) and (4.5).

2-i): If $\lambda_1 = \lambda_2 = 0$, then $H = 0$. From (4.5) we obtain $\lambda_3 = 0$, so we get a contradiction.

2-ii): If $\lambda_1 = 0$ and $\lambda_2 \neq 0$., from (4.3) we obtain $Hf_x = 0$.

2-ii-a: If $H = 0$ (4.4), (4.5) implies that $\lambda_2 = \lambda_3 = 0$. So we get a contradiction.

2-ii-b: if $f_x = 0$, then $f(x, y) = \varphi(y)$, where φ is smooth function of y.

The mean curvature H turns to

$$H = \frac{1}{2Wy^3}\left(\frac{1}{y}\varphi'' - \varphi'^3\right), \tag{4.9}$$

where $\varphi' = \dfrac{d\varphi}{dy}$.

Using (4.4) and (4.5) we obtain

$$\varphi' = \frac{-\lambda_2}{\lambda_3 y \varphi},$$

which leads to,

$$\lambda_3 \varphi' \varphi = \frac{-\lambda_2}{y}.$$

After integrating with respect to y, we obtain

$$\frac{\lambda_3}{2}\varphi^2(y) = -\lambda_2 \ln y + \phi(x); \quad y > 0,$$

where ϕ is smooth function of x,

and hence

$$f(x,y) = \varphi(y) = \pm\sqrt{\frac{\lambda_2}{\lambda_3}\ln\frac{1}{y^2} + \phi(x)}.$$

Using the condition $f_x = 0$ we get $\phi(x) = a,\ a \in \mathbb{R}$.

Thus,

$$f(x,y) = \varphi(y) = \pm\sqrt{\frac{\lambda_2}{\lambda_3}\ln\frac{1}{y^2} + c}; \quad c = \frac{2}{\lambda_3}a,$$

in this subcase, the surfaces $\mathcal{S}$ are given by

$$r(x,y) = \left(x, y, \pm\sqrt{\frac{\lambda_2}{\lambda_3}\ln\frac{1}{y^2} + c}\right); \quad \lambda_2 \neq 0,\ \lambda_3 \neq 0, \quad c \in \mathbb{R}.$$

2-iii): If $\lambda_1 \neq 0$ and $\lambda_2 = 0.$, from (4.4) we obtain $Hf_y = 0$.

2-iii-a: If $H = 0$, (4.3) and (4.5) implies that $\lambda_2 = \lambda_3 = 0$. So we get a contradiction.

2-iii-b: If $f_y = 0$,then $f(x,y) = \psi(x)$, where ψ is smooth function of x.

The mean curvature H turns to

$$H = \frac{1}{2Wy^4}\psi'', \tag{4.10}$$

where $\psi' = \dfrac{d\psi}{dx}$.

Using (4.3) and (4.5) we get

$$\psi' = \frac{-\lambda_1 x}{\lambda_3 y^2 \psi},$$

so we can write

$$\lambda_3 y^2 + \lambda_1 \frac{x}{\psi\psi'} = 0, \tag{4.11}$$

A differentiation with respect to y gives

$$\lambda_3 y = 0,$$

this implies that $\lambda_3 = 0$ and from (4.8) we get the mean curvature H is identically zero. From (4.6) and (4.7) we obtain $\lambda_1 = \lambda_2 = 0$, which leads to a contradiction.

2-iv): If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ From (4.3), we have

$$\frac{2H}{W} = -\frac{\lambda_1 x}{\psi'}. \tag{4.12}$$

Substituting (4.12) into (4.5), we get

$$-\frac{\lambda_1 x}{\psi'} = \lambda_3 y^2 \psi,$$

A differentiation with respect to x gives

$$-\lambda_1 \left(\frac{\psi - x\psi''}{\psi'^2} \right) = \lambda_3 y^2 \psi',$$

this equation gives

$$\lambda_1 \left(\frac{\psi' - x\psi''}{\psi'^3} \right) + \lambda_3 y^2 = 0. \tag{4.13}$$

A differentiation with respect to y gives

$$\lambda_3 y = 0,$$

this implies that $\lambda_3 = 0$ and from (4.8) we get the mean curvature H is identically zero. From (4.6) and (4.7) we obtain $\lambda_1 = \lambda_2 = 0$, which leads to a contradiction.

Therefore, we have the following theorem,

Theorem 4.1. *Let $\mathcal{S}$ be a surface as graph of function parametrized by $r(x,y) = (x, y, f(x,y))$ in $\mathbb{H}^2 \times \mathbb{R}$ Then, $\mathcal{S}$ satisfies the equation $\Delta r_i = \lambda_i r_i$, $\lambda_i \in \mathbb{R}$ if and only if $\mathcal{S}$ is minimal surfaces or parametrized as*

$$S: \ r(x,y) = \left(x, y, \pm\sqrt{\frac{\lambda_2}{\lambda_3} \ln \frac{1}{y^2} + c} \right); \quad \lambda_2 \neq 0, \ \lambda_3 \neq 0, \ c \in \mathbb{R}.$$

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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