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ON SOME SUBCLASSES OF STRONGLY STARLIKE ANALYTIC FUNCTIONS

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Abstract. The aim of the present article is to investigate a family of univalent analytic functions on the unit disc $\mathbb{D}$ defined for $M \geq 1$ by

$$\Re(\frac{zf'(z)}{f(z)}) > 0, \ \ \left|\left(\frac{zf'(z)}{f(z)}\right)^2 - M\right| < M, \ z \in \mathbb{D}.$$

Some proprieties, radius of convexity and coefficient bounds are obtained for classes in this family.

1. Introduction

Let $\mathcal{A}$ be the set of analytic function on the unit disc $\mathbb{D}$ with the normalization $f(0) = f'(0) - 1 = 0$. $f \in \mathcal{A}$ if f is of the form

$$f(z) = z + \sum_{n=2}^{+\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

$\mathcal{S}$ denotes the subclass of $\mathcal{A}$ of univalent functions. A function $f \in \mathcal{S}$ is said to be strongly starlike of order α, $0 < \alpha \leq 1$, if it satisfies the condition

$$\left|Arg\frac{zf'(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \ \ \forall\, z \in \mathbb{D}.$$

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This class is denoted by $\mathcal{SS}^*(\alpha)$ and was first introduced by D. A. Brannan and W. E. Kirwan [1] and independently by J. Stankiewicz [9].

$\mathcal{SS}^*(1)$ is the well known class $\mathcal{S}^*$ of starlike functions. Recall that a function $f \in \mathcal{S}$ belongs to $\mathcal{S}^*$ if the image of $\mathbb{D}$ under f is a starlike set with respect to the origin or, equivalently, if

$$\Re\Big(\frac{zf'(z)}{f(z)}\Big) > 0,\ z \in \mathbb{D}.$$

A function $f \in \mathcal{S}$ belongs to $\mathcal{SS}^*(\alpha)$ if the image of $\mathbb{D}$ under $\frac{zf'(z)}{f(z)}$ lies in the angular sector

$$\Omega_\alpha = \Big\{z \in \mathbb{C}, \big|Argz\big| < \frac{\alpha\pi}{2}\Big\}.$$

Let $\mathcal{B}$ denotes the set of Schwarz functions, i.e. $\omega \in \mathcal{B}$ if and only ω is analytic in $\mathbb{D}$, $\omega(0) = 0$ and $\big|\omega(z)\big| < 1$ for $z \in \mathbb{D}$. Given two functions f and g analytic in $\mathbb{D}$, we say that f is subordinate to g and we write $f \prec g$ if there exists $\omega \in \mathcal{B}$ such that $f = g \circ \omega$ in $\mathbb{D}$.

If g is univalent on $\mathbb{D}$, $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$.

We obtain from the Schwarz lemma that if $f \prec g$ then $\big|f'(0)\big| \leq \big|g'(0)\big|$. As a consequence of this statement, we have

$$f,\, g \in \mathcal{A}\,,\ \frac{f(z)}{z} \prec \frac{g(z)}{z} \Longrightarrow \big|a_2\big| \leq \big|b_2\big|, \tag{1.2}$$

where a_2 and b_2 are respectively the second coefficients of f and g.

W. Janowski [2] investigated the subclass

$$S^*(M) = \Big\{f \in \mathcal{S}, \frac{zf'(z)}{f(z)} \in \mathcal{D}_M, \forall z \in \mathbb{D}\Big\},$$

where

$$\mathcal{D}_M = \Big\{w \in \mathbb{C}, \big|w - M\big| < M\Big\},\ \ M \geq 1$$

J. Sókol and J. Stankiewicz [8] introduced a subclass of $\mathcal{SS}^*(\frac{1}{2})$, namely, the class $\mathcal{S}^*_L$ defined by

$$\mathcal{S}^*_L = \Big\{f \in \mathcal{S}, \frac{zf'(z)}{f(z)} \in \mathcal{L}_1, \forall z \in \mathbb{D}\Big\},$$

where

$$\mathcal{L}_1 = \Big\{w \in \mathbb{C}, \Re w > 0,\ \big|w^2 - 1\big| < 1\Big\}.$$

$\mathcal{L}_1$ is the interior of the right half of the Bernoulli's lemniscate $\big|w^2 - 1\big| = 1$.

In the present paper we are interested to the family of subclass of $\mathcal{S}$

$$\mathcal{S}^*_L(M) = \Big\{f \in \mathcal{S}, \frac{zf'(z)}{f(z)} \in \mathcal{L}_M, \forall z \in \mathbb{D}\Big\},\ \ M \geq 1, \tag{1.3}$$

where

$$\mathcal{L}_M = \left\{ w \in \mathbb{C}, \Re w > 0, \left|w^2 - M\right| < M \right\}. \tag{1.4}$$

is the interior of the right half of the Cassini's oval $\left|w^2 - M\right| = M$. For the particular case $M = 1$, $\mathcal{S}_L^*(1)$ stands for the class $\mathcal{S}_L^*$ introduced by J. Sókol and J. Stankiewicz [8]. Since $\mathcal{L}_M \subset \Omega(\frac{1}{2})$, all functions in $\mathcal{S}_L^*(M)$ are strongly starlike of order $\frac{1}{2}$.

Note that all classes above correspond to particular cases of the classes of $\mathcal{S}^*(\varphi)$ introduced by W. Ma and D. Minda [3],

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \prec \varphi \right\}.$$

where φ is Analytic univalent function with real positive part in the unit disc $\mathbb{D}$, $\varphi\big(\mathbb{D}\big)$ is symmetric with respect to the real axis and starlike with respect to $\varphi(0) = 1$ and $\varphi'(0) > 0$.

Let $m = 1 - \frac{1}{M}$ and φ_m be the function

$$\varphi_m(z) = \sqrt{\frac{1+z}{1-mz}}, \ \ z \in \mathbb{D}$$

where the branch of the square root is chosen so that $\varphi_m(0) = 1$. We have

$$\mathcal{S}_L^*(M) = \mathcal{S}^*(\varphi_m) = \left\{ f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \prec \varphi_m \right\}. \tag{1.5}$$

Observe that $\mathcal{S}_L^*$ corresponds to $m = 0$ so that $\mathcal{S}_L^* = \mathcal{S}^*(\sqrt{1+z})$.

2. Some properties of the class $\mathcal{S}_L^*(M)$

Let P the class of analytic functions p in $\mathbb{D}$ with $p(0) = 1$ and $\Re\, p(z) > 0$ in $\mathbb{D}$. For $M \geq 1$, let

$$P_{\mathcal{L}}(M) = \left\{ p \in P,\ \left|p^2(z) - M\right| < M,\ z \in \mathbb{D} \right\}.$$

It is easy to see that $P_{\mathcal{L}}(M_1) \subset P_{\mathcal{L}}(M_2)$ for $M_1 \leq M_2$.

Remark 2.1. A function $f \in \mathcal{A}$ belongs to $\mathcal{S}_L^*(M)$ if and only if there exists $p \in P_{\mathcal{L}}(M)$ such that

$$\frac{zf'(z)}{f(z)} = p(z), \ \ z \in \mathbb{D}.$$

Theorem 2.1. A function f belongs to $\mathcal{S}_L^*(M)$ if and only if there exists $p \in P_{\mathcal{L}}(M)$ such that

$$f(z) = z \ \exp \int_0^z \frac{p(\xi) - 1}{\xi} d\xi. \tag{2.1}$$

Proof. (2.1) is an immediate consequence of the Remark 2.1 □

Let $f_m \in \mathcal{A}$ be the unique function such that

$$\frac{zf_m'(z)}{f_m(z)} = \varphi_m(z), \quad z \in \mathbb{D} \tag{2.2}$$

with $m = 1 - \frac{1}{M}$. f_m belongs to $\mathcal{S}_L^*(M)$ and we have

$$f_m(z) = z \exp \int_0^z \frac{\varphi_m(\xi) - 1}{\xi} d\xi. \tag{2.3}$$

Evaluating the integral in (2.3), we get

$$f_m(z) = \frac{4z \exp \int_1^{\varphi_m(z)} H_m(t)dt}{\left(\varphi_m(z) + 1\right)^2}, \quad z \in \mathbb{D}, \tag{2.4}$$

where

$$H_m(t) = \frac{2mt + 2}{mt^2 + 1}, \quad m = 1 - \frac{1}{M}$$

For $M = 1$, H_0 is the constant function H(t) = 2 and we have

$$f_0(z) = \frac{4z \exp\left(2\sqrt{1+z} - 2\right)}{\left(\sqrt{1+z} + 1\right)^2} \quad \text{for } z \in \mathbb{D}.$$

f_0 is extremal function for problems in the class $\mathcal{S}_{\mathcal{L}}^*$ (see [8]).

It is easy to see that

$$f_m(z) = z + \frac{m+1}{2}z^2 + \frac{(m+1)(5m+1)}{16}z^3 + \frac{(m+1)(21m^2+6m+1)}{96}z^4 + \ldots \tag{2.5}$$

We need the following result by St. Ruscheweyh [5]

Lemma 2.1. [[5], Theorem 1] Let G be a convex conformal mapping of $\mathbb{D}$, $G(0) = 1$, and let

$$F(z) = z \exp \int_0^z \frac{G(\xi) - 1}{\xi} d\xi.$$

Let $f \in \mathcal{A}$. Then we have

$$\frac{zf'(z)}{f(z)} \prec G$$

if and only if for all $|s| \leq 1$, $|t| \leq 1$

$$\frac{tf(sz)}{sf(tz)} \prec \frac{tF(sz)}{sF(tz)}.$$

Theorem 2.2. If f belongs to $\mathcal{S}_L^*(M)$ then

$$\frac{f(z)}{z} \prec \frac{f_m(z)}{z}. \tag{2.6}$$

Proof. From (1.5), we obtain by applying Lemma 2.1 to the convex univalent function $G = \varphi_m$,

$$\frac{tf(z)}{f(tz)} \prec \frac{tf_m(z)}{f_m(tz)}.$$

Letting $t \longrightarrow 0$, we obtain the desired conclusion. ∎

Corollary 2.1. Let f belongs to $\mathcal{S}_L^*(M)$ and $|z| = r < 1$, then

$$-f_m(-r) \leq |f(z)| \leq f_m(r); \tag{2.7}$$

$$f_m^{'}(-r) \leq |f^{'}(z)| \leq f_m^{'}(r). \tag{2.8}$$

Proof. (2.7) follows from (2.6). Now If $M \geq 1$ we have $0 \leq m < 1$. Thus for $0 \leq r < 1$

$$\min_{|z|=r} |\varphi_m(z)| = \varphi_m(-r), \quad \max_{|z|=r} |\varphi_m(z)| = \varphi_m(r) \tag{2.9}$$

From (2.6) and (2.9) we get (2.8) by applying Theorem 2 ([3], p. 162). □

3. Radius of convexity for the class $\mathcal{S}_L^*(M)$

In the sequel $m = 1 - \frac{1}{M}$.

For $M \geq 1$, let $\mathcal{P}(M)$ be the family of analytic functions P in $\mathbb{D}$ satisfying

$$\mathrm{P(0) = 1,} \quad \mathrm{|P(z) - M| < M,} \quad \mathrm{for} \ \ \mathrm{z} \in \mathbb{D}. \tag{3.1}$$

We have

$$f \in \mathcal{S}_L^*(M) \iff \exists P \in \mathcal{P}(M) \ / \ \frac{zf'(z)}{f(z)} = \sqrt{P}. \tag{3.2}$$

We need the two following lemmas by Janowski [2]:

Lemma 3.1. [[2] , Theorem 1] For every $P(z) \in \mathcal{P}(M)$ and $|z| = r$, $0 < r < 1$, we have

$$\inf_{P \in \mathcal{P}(M)} \Re P(z) = \frac{1-r}{1+mr}. \tag{3.3}$$

The infimum is attained by

$$P(z) = \frac{1 - \epsilon z}{1 + \epsilon m z}, \quad |\epsilon| = 1. \tag{3.4}$$

Lemma 3.2. (Theorem 2, [2]) For every $P(z) \in \mathcal{P}(M)$ and $|z| = r$, $0 < r < 1$, we have

$$\inf_{P \in \mathcal{P}(M)} \Re \frac{zP^{'}(z)}{P(z)} = -\frac{(1+m)r}{(1-r)(1+mr)}. \tag{3.5}$$

The infimum is attained by

$$P(z) = \frac{1 - \epsilon z}{1 + \epsilon m z}, \quad |\epsilon| = 1. \tag{3.6}$$

Theorem 3.1. The radius of convexity of the class $\mathcal{S}_L^*(M)$ is is the unique root in $(0, 1)$ of the equation

$$4(1+mr)(1-r)^3 - (1+m)^2 r^2 = 0. \tag{3.7}$$

Proof. Let $f \in \mathcal{S}_L^*(M)$. From (3.2), there exists $P \in \mathcal{P}(M)$ such that

$$\frac{zf'(z)}{f(z)} = \sqrt{P(z)}, \quad z \in \mathbb{D}. \tag{3.8}$$

(3.8) can be written

$$zf'(z) = f(z)\sqrt{P(z)}$$

which gives

$$1 + \frac{zf''(z)}{f'(z)} = \frac{z\big(zf'\big)'(z)}{zf'(z)} = \sqrt{P(z)} + \frac{1}{2}\,\frac{zP'(z)}{P(z)}.$$

This yields for $|z| = r, \ \ 0 < r <,$

$$\Re\big(1 + \frac{zf''(z)}{f'(z)}\big) \geq \inf_{P \in \mathcal{P}(M)} \Re\sqrt{P(z)} + \frac{1}{2} \inf_{P \in \mathcal{P}(M)} \Re\frac{zP'(z)}{P(z)}. \tag{3.9}$$

Replacing (3.3) and(3.5 in (3.9), we obtain

$$\Re\big(1 + \frac{zf''(z)}{f'(z)}\big) \geq \sqrt{\frac{1-r}{1+mr}} - \frac{1}{2}\,\frac{(1+m)r}{(1-r)(1+mr)} \tag{3.10}$$

Let $h_{_M}$ be defined by

$$h_{_M} = \sqrt{\frac{1-r}{1+mr}} - \frac{1}{2}\,\frac{(1+m)r}{(1-r)(1+mr)}.$$

$h_{_M}$ is decreasing in the interval $(0,1)$, $h_{_M}(0) = 1$ and the limit of $h_{_M}$ in 1^- is $-\infty$. Let $r_{_{M-1}}$ be the unique solution of $h_{_M}(r) = 0$ in $(0,1)$, then f is convex on the disc $|z| < r_{_{M-1}}$. On the other hand,

$$1 + \frac{zf_m''(z)}{f_m'(z)} = \sqrt{\frac{1+z}{1-mz}} + \frac{1}{2}\,\frac{(1+m)z}{(1-mz)(1+z)}$$

vanishes in $z = -r_{_{M-1}}$. Thus $r_{_{M-1}}$ is the best value.

To finish, we observe that the equation $h_{_M}(r) = 0$ is equivalent in the interval $(0,1)$ to the equation

$$4\big(1+mr\big)\big(1-r\big)^3 - \big(1+m\big)^2 r^2 = 0.$$

□

Remark 3.1. As a consequence of Theorem 3.1 applying for $M = 1$, we find Theorem 4 [8] which gives r_0 the radius of convexity of the class S_L^*. $r_0 = \frac{1}{12}\left(11 + \sqrt[3]{\sqrt{44928} - 181} - \sqrt[3]{\sqrt{44928} + 181}\right) \approx 0.5679591$

Remark 3.2. As observed above, $\mathcal{S}_L^*(M)$ increases with M. Therefore $r_{_{M-1}}$ decreases when M increases. Let

$$r_\infty = \lim_{M \to +\infty} r_{_{M-1}}.$$

Substituting in (3.7), we obtain

$$\left(1+r_\infty\right)\left(1-r_\infty\right)^3-r_\infty^2=0.$$

Solving this equation in $(0,1)$, we get

$$r_\infty=\frac{1}{2}\left(1-\sqrt{2}+\sqrt{\sqrt{8}-1}\right)\approx 0.46899.$$

We have

$$r_\infty\le r_{_{M-1}}\le r_0.$$

4. Coefficient bounds for $\mathcal{S}_L^*(M)$

Theorem 4.1. Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be a function in $\mathcal{S}_L^*(M)$. Then

for $1\le M\le 2$ we have

$$\sum_{n=2}^{\infty}\left((1-m)n^2-2\right)|a_n|^2\le 1+m \tag{4.1}$$

and for $M>2$ we have

$$\sum_{n\ge\sqrt{\frac{2}{1-m}}}\left((1-m)n^2-2\right)|a_n|^2\le 1+m-\sum_{2\le k<\sqrt{\frac{2}{1-m}}}\left((1-m)k^2-2\right)|a_k|^2. \tag{4.2}$$

with $m=\frac{M-1}{M}$.

Proof. If $f\in\mathcal{S}_L^*(M)$ there exists $\omega\in\mathcal{B}$ such that

$$\left(1-m\omega(z)\right)\left(zf^{'}(z)\right)^2-f(z)^2=\omega(z)f(z)^2,\quad z\in\mathbb{D}. \tag{4.3}$$

For $0<r<1$ we have

$$\begin{aligned}2\pi\sum_{n=1}^{\infty}|a_n|^2r^2 &= \int_0^{2\pi}|f\left(re^{i\theta}\right)|^2d\theta\\ &\ge \int_0^{2\pi}|\omega\left(re^{i\theta}\right)||f\left(re^{i\theta}\right)|^2d\theta \end{aligned}\tag{4.4}$$

Replacing (4.3) in the right side of (4.5) we obtain

$$\begin{aligned}2\pi\sum_{n=1}^{\infty}|a_n|^2r^2 &\ge \int_0^{2\pi}\left|\left(1-m\omega(re^{i\theta})\right)\left(re^{i\theta}f^{'}(re^{i\theta})\right)^2-f\left(re^{i\theta}\right)^2\right|d\theta\\ &\ge \int_0^{2\pi}\left|\left(1-m\omega(re^{i\theta})\right)\left(re^{i\theta}f^{'}(re^{i\theta})\right)^2\right|d\theta-\int_0^{2\pi}\left|f\left(re^{i\theta}\right)^2\right|d\theta\\ &\ge (1-m)\int_0^{2\pi}\left|\left(re^{i\theta}f^{'}(re^{i\theta})\right)^2\right|d\theta-\int_0^{2\pi}\left|f\left(re^{i\theta}\right)^2\right|d\theta\\ &= 2\pi\sum_{n=1}^{\infty}(1-m)n^2|a_n|^2r^2-2\pi\sum_{n=1}^{\infty}|a_n|^2r^2.\end{aligned}$$

Thus

$$2\sum_{n=1}^{\infty}|a_n|^2r^2 \geq \sum_{n=1}^{\infty}(1-m)n^2|a_n|^2r^2.$$

If we let $r \to 1^-$, we obtain from le last inequality

$$2\sum_{n=1}^{\infty}|a_n|^2 \geq \sum_{n=1}^{\infty}(1-m)n^2|a_n|^2$$

which gives,

$$1+m \geq \sum_{n=2}^{\infty}\big((1-m)n^2-2\big)|a_n|^2. \tag{4.5}$$

Since $(1-m)n^2-2 \geq 0$ for all $n \geq 2$ if and only if $1 \leq M \leq 2$ then (4.5) yields (4.1) and (4.2) according to the case $1 \leq M \leq 2$ or $M > 2$. □

The following corollary is an immediate consequence of (4.2).

Corollary 4.1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in $\mathcal{S}_L^*(M)$ .Then

for $1 \leq M \leq 2$ we have

$$|a_n| \leq \sqrt{\frac{1+m}{(1-m)n^2-2}}, \quad \text{for } n \geq 2 \tag{4.6}$$

and for $M > 2$ we have

$$|a_n| \leq \sqrt{\frac{1+m-\sum_{2\leq k<\sqrt{\frac{2}{1-m}}}\big((1-m)k^2-2\big)|a_k|^2}{(1-m)n^2-2}}; \quad \text{for } n \geq \sqrt{\frac{2}{1-m}}. \tag{4.7}$$

with $m = \frac{M-1}{M}$.

Remark 4.1. For $M = 1$, (4.1) and (4.6) give respectivly Theorem 1 and Corollary 1 [6].

Theorem 4.2. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be a function in $\mathcal{S}_L^*(M)$ .Then

(i) $|a_2| \leq \frac{m+1}{2}$, for $0 \leq m \leq 1$;

(ii) $|a_3| \leq \frac{m+1}{4}$, for $0 \leq m \leq \frac{3}{5}$;

(iii) $|a_4| \leq \frac{m+1}{6}$, for $0 \leq m \leq \frac{\sqrt{3}-1}{7}$.

This estimations are sharp.

Proof. If $f \in \mathcal{S}_L^*(M)$ there exists $\omega(z) = \sum_{n=1}^{\infty} C_n z^n \in \mathcal{B}$ such that

$$\big(zf'(z)\big)^2 - f(z)^2 = \omega(z)\Big(m\big(zf'(z)\big)^2 + f(z)^2\Big), \quad z \in \mathbb{D}. \tag{4.8}$$

Let $f(z)^2 = \sum_{n=2}^{\infty} A_n z^n$, $\big(zf'(z)\big)^2 = \sum_{n=2}^{\infty} B_n z^n$. (4.8) becomes

$$\sum_{n=2}^{\infty}(B_n - A_n)z^n = \left(\sum_{n=2}^{\infty}(mB_n + A_n)z^n\right)\left(\sum_{n=1}^{\infty}C_n z^n\right) \tag{4.9}$$

Equating coefficients for $n=2$, $n=3$ in both sides of (4.9), we obtain

$$(S_m)\begin{cases} B_3-A_3=\big(mB_2+A_2\big)C_1\\ B_4-A_4=\big(mB_2+A_2\big)C_2+\big(mB_3+A_3\big)C_1\\ B_5-A_5=\big(mB_2+A2\big)C_3+\big(mB_3+A_3\big)C_2+\big(mB_4+A_4\big)C_1\end{cases}$$

A little calculation yields

$$A_2=a_1=1,\ \ A_3=2a_2,\ \ A_4=2a_3+a_2^2,\ \ A_5=2a_4+2a_2a_3$$

and

$$B_2=a_1=1,\ \ B_3=4a_2,\ \ B_4=6a_3+4a_2^2,\ \ B_5=8a_4+12a_2a_3.$$

Replacing in (S_m), we obtain

$$\begin{cases} (1)\ \ 2a_2=\big(m+1\big)C_1\\ (2)\ \ 4a_3+3a_2^2=\big(m+1\big)C_2+\big(4m+2\big)a_2C_1\\ (3)\ \ 6a_4+10a_2a_3=\big(m+1\big)C_3+\big(2m+1\big)\big(m+1\big)C_1C_2+\big((6m+2)a_3+(4m+1)a_2^2\big)C_1\end{cases}$$

Since $|C_1|\le 1$ then (1) implies that $|a_2|\le \frac{1+m}{2}$. This proves the assertion (i). On the other hand we have from (1) and (2)

$$a_3=\frac{1+m}{4}C_2+\frac{(5m+1)(m+1)}{16}C_1^2.$$

Thus

$$|a_3|\le \frac{1+m}{4}\big(|C_2|+\frac{5m+1}{4}|C_1|\big).$$

It is well known that $|C_2|\le 1-|C_1|^2$. Therefore we obtain

$$\begin{aligned} |a_3| &\le \frac{1+m}{4}\big(1-|C_1|^2+\frac{5m+1}{4}|C_1|\big)\\ &= \frac{1+m}{4}\big(1+\frac{5m-3}{4}|C_1|\big). \end{aligned} \tag{4.10}$$

Since $5m-3\le 0$ if and only if $m\le \frac{3}{5}$ then (4.10) yields the assertion (ii).

Replacing the values of a_2 and a_3 in the equation (3), we obtain

$$\begin{aligned} a_4 &= \frac{\big(m+1\big)}{6}C_3+\frac{\big(m+1\big)\big(9m+1\big)}{24}C_1C_2+\frac{\big(m+1\big)\big(21m^2+6m+1\big)}{96}C_1^3\\ &= \frac{m+1}{6}\left(C_3+\frac{9m+1}{4}C_1C_2+\frac{21m^2+6m+1}{16}C_1^3\right). \end{aligned} \tag{4.11}$$

Let $\mu=\frac{9m+1}{4}$ and $\nu=\frac{21m^2+6m+1}{16}$. Under the assumption $0\le m\le \frac{\sqrt{3}-1}{7}$, we have $(\mu,\nu)\in D_1$ (see [4], p. 127). Therefore by Lemma 2 [4] we obtain

$$\left|C_3+\frac{9m+1}{4}C_1C_2+\frac{21m^2+6m+1}{16}C_1^3\right|\le 1$$

which yields from (4.11) the assertion (iii).

The sharpness of (i) is given by the function f_m. If we take in (4.8) $\omega(z) = z^2$ and $\omega(z) = z^3$ successively, we obtain two functions in $\mathcal{S}^*_L(M)$:

$$f_{1,m}(z) = z + \frac{m+1}{4} z^3 + \dots \quad \text{and} \quad f_{2,m}(z) = z + \frac{m+1}{6} z^4 + \dots$$

which give respectively the sharpness of estimations (ii) and (iii). □

Remark 4.2. The estimation (i) can be obtained directly from (2.6).

Remark 4.3. If we take $m = 0$ in Theorem 4.2, we obtain as particular case Theorem 2 [6].

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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