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CLASS OF (n,m)-POWER-D-HYPONORMAL OPERATORS IN HILBERT SPACE

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Abstract. In this paper, we introduce a new classes of operators acting on a complex Hilbert space H, denoted by $[(n,m)DH]$, called (n,m)-power-D-hyponormal associated with a Drazin inversible operator using its Drazin inverse. Some proprieties of (n,m)-power-D-hyponormal, are investigated with some examples.

1. Introduction

Let $\mathcal{H}$ be a complex Hilbert space. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators defined in $\mathcal{H}$. Let T be an operator in $\mathcal{B}(\mathcal{H})$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$, i.e.,T commutes with T^*. The class of quasi-normal operators was first introduced and studied by A. Brown in [5] in 1953. The operator T is quasi-normal if T commutes with T^*T, i.e. $T(T^*T) = (T^*T)T$ and it is denoted by $[QN]$. A.A.S. Jibril [6, 7], in 2008 introduced the class of n power normal operators as a generalization of normal operators. The operator T is called n power normal if T^n commutes with T^* , i.e., $T^nT^* = T^*T^n$ and is denoted by $[nN]$. In the year 2011, O.A. Mahmoud Sid Ahmed introduced n power quasi normal operators [14], as a generalization of quasi normal operators. The operator T is called n power quasi normal if T^n commutes with T^*T , i.e., $T^n(T^*T) = (T^*T)T^n$ and it is denoted by $[nQN]$.
Recently in [13], the authors introduced and studied the operator $[(n,m)DN]$ and $[(n,m)DQ]$.In this search,

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we introduce a new class of operators T namely (n, m)-power-D-hyponormal operator for a positive integer n, m if

$$T^{*m}(T^D)^n \geq (T^D)^n T^{*m}, m = n = 1, 2, ...$$

denoted by $[(n, m)DH]$. And we in this work, we will try to apply the same results obtained in [8] for this new classes.

Definition 1.1. *An operator $T \in \mathcal{B}(H)$ be Drazin inversible operator. We said that T is (n, m)-power-D-hyponormal operator for a positive integer n, m if*

$$T^{*m}(T^D)^n \geq (T^D)^n T^{*m}, m = n = 1, 2, ...$$

We denote the set of all (n, m)-Power-D-hyponormal operators by $[(n, m)DH]$

Remark 1.1. *Clearly $n = m = 1$, then $(1, 1)$-Power-D-hyponormal operator is precisely Power-D-hyponormal operator.*

Definition 1.2. *An operator $T \in \mathcal{B}(\mathcal{H})^D$ is said to be (n, m)-power-D-hyponormal if $T^{*m}(T^D)^n - (T^D)^n T^{*m}$ is positive i.e: $T^{*m}(T^D)^n - (T^D)^n T^{*m} \geq 0$ or equivalently*

$$\langle \big(T^{*m}(T^D)^n - (T^D)^n T^{*m}\big)u \mid u\rangle \geq 0 \ \ for\ all\ \ u \in \mathcal{H}.$$

Example 1.1. *Let $T = \begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}, S = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in \mathcal{B}(\mathbb{R}^2)$. A simple computation shows that*

$$T^D = \frac{1}{9}\begin{pmatrix} 3 & -2 \\ 0 & -3 \end{pmatrix}, S^D = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, S^* = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, T^* = \begin{pmatrix} 3 & 0 \\ -2 & -3 \end{pmatrix}.$$

Then $T \in [(2, 2)DH]$, but $T \notin [(3, 3)DH]$ and $S \in [(3, 2)DH]$, but $S \notin [(2, 2)DH]$

Proposition 1.1. *If S, $T \in \mathcal{B}(\mathcal{H})^D$ are unitarily equivalent and if T is (n, m)-Power-D-hyponormal operators then so is S*

Proof. Let T be an (n, m)-Power-D-hyponormal operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^*$ so $S^n = UT^nU^*$

We have

$$\begin{aligned} S^{*m}(S^D)^n &= (UT^mU^*)^*\, U(T^D)^nU^* \\ &= UT^{*m}U^*U(T^D)^nU^* \\ &= UT^{*m}(T^D)^nU^* \\ &\geq U(T^D)^nT^{*m}U^* \\ &\geq U(T^D)^nU^*UT^{*m}U^* \\ &= (S^D)^nS^{*m} \end{aligned}$$

Hence, $S^{*m}(S^D)^n - (S^D)^nS^{*m} \geq 0$ □

Proposition 1.2. *Let $T \in \mathcal{B}(\mathcal{H})^D$ be an (n,m)-Power-D-hyponormal operator. Then T^* is (n,m)-Power-D-co-hyponormal operator*

Proof. Since T is (n,m)-Power-D-hyponormal operator. We have

$$T^{*m}(T^D)^n \geq (T^D)^nT^{*m} \Rightarrow \Big(T^{*m}(T^D)^n)^* \geq \big((T^D)^nT^{*m}\big)^* \quad \Rightarrow \quad (T^D)^{*n}T^m \geq T^m(T^D)^{*n}.$$

Hence, T^* is (n,m)-Power-D-co-hyponormal operator. □

Theorem 1.1. *If T, T^* are two (n,m)-Power-D-hyponormal operator, then T is an (n,m)-Power-D-normal operator.*

Proposition 1.3. *If T is $(2,2)$-power-D-hyponormal operator and $T^DT^* = -T^*T^D$. Tthen T is $(2,2)$-Power-D-normal operator.*

Proof. Since $(T^D)^2T^{*2} = T^DT^DT^*T^* = -T^DT^*T^DT^* = T^DT^*T^*T^D = -T^*T^DT^*T^D = T^{*2}(T^D)^2$

And

$T^{*2}(T^D)^2 = T^*T^*T^DT^D = -T^*T^DT^*T^D = T^DT^*T^*T^D = -T^DT^*T^DT^* = (T^D)^2T^{*2}$

So

T is $(2,2)$-Power-D-hyponormal, then

$$\begin{aligned}(T^D)^2T^{*2} \leq T^{*2}(T^D)^2 &\Rightarrow T^DT^DT^*T^* \leq T^*T^*T^DT^D \\ &\Rightarrow -T^DT^*T^DT^* \leq -T^*T^DT^*T^D \\ &\Rightarrow T^DT^*T^DT^* \geq T^*T^DT^*T^D \\ &\Rightarrow T^DT^*T^*T^D \geq T^DT^*T^*T^D \\ &\Rightarrow -T^*T^DT^*T^D \geq -T^DT^*T^DT^* \\ &\Rightarrow T^*T^DT^*T^D \leq T^DT^*T^DT^* \\ &\Rightarrow T^{*2}(T^D)^2 \geq (T^D)^2T^{*2}.\end{aligned}$$

Hence $T^{*2}(T^D)^2 = (T^D)^2T^{*2}$. □

Example 1.2. *Let* $T = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^3)$. *A simple computation, shows that ;* $T^* = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{pmatrix}, T^D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then power-D-hyponormal operator, but $T^*T \neq TT^*$ *and* $\|Tu\| \ngeq \|T^*u\|$.

Lemma 1.1. *Let* $T_k, S_k \in \mathcal{B}(\mathcal{H})^D,\ k = 1,2$ *such that* $T_1 \geq T_2 \geq 0$ *and* $S_1 \geq S_2 \geq 0$, *then*

$$\big(T_1 \otimes S_1\big) \geq \big(T_2 \otimes S_2\big) \geq 0.$$

Theorem 1.2. . *Let* $T, S \in \mathcal{B}(\mathcal{H})^D$, *such that* $(S^D)^nS^* \geq 0$ *and* $(T^D)^nT^* \geq 0$, *then .*

$T \otimes S$ *is* $(n,1)$*-Power-D-hyponormal if and only if* T *and* S *are* $(n,1)$*-Power-D-hyponormal operators*

Proof. Assume that T, S are $(n,1)$-power-D-hyponormal operators. Then

$$\begin{aligned}((T\otimes S)^D)^n(T\otimes S)^* &= (T^D\otimes S^D)^n(T^*\otimes S^*) \\ &= (T^D)^nT^*\otimes(S^D)^nS^* \\ &\leq T^*(T^D)^n\otimes S^*(S^D)^n \\ &= (T\otimes S)^*((T\otimes S)^D)^n.\end{aligned}$$

Which implies that $T \otimes S$ is $(n,1)$-power-D-hyponormal operator.

Conversely, assume that $T \otimes S$ is $(n,1)$-power-D-hyponormal operator.We aim to show that T, S are $(n,1)$-power-D-hyponormal. Since $T \otimes S$ is a $(n,1)$-power-D-hyponormal operator, we obtain

$$\begin{aligned}(T \otimes S) \text{ is } (n,1)\text{-power-}D\text{-hyponormal} \quad &\Longleftrightarrow \quad (((T \otimes S)^D)^n (T \otimes S)^* \leq (T \otimes S)^* ((T \otimes S)^D)^n \\ &\Longleftrightarrow \quad (T^D)^n T^* \otimes (S^D)^n S^* \leq T^*(T^D)^n \otimes S^*(S^D)^n.\end{aligned}$$

Then, there exists $d > 0$ such that

$$\begin{cases} d\,(T^D)^n T^* \leq T^*(T^D)^n. \\ \\ \text{and} \\ \\ d^{-1}(S^D)^n S^* \leq S^*(S^D)^n \end{cases}$$

a simple computation shows that $d = 1$ and hence

$$(T^D)^n T^* \leq T^*(T^D)^n \quad \text{and} \quad (S^D)^n S^* \leq S^*(S^D)^n.$$

Therefore, T, S are $(n,1)$-power-D-hyponormal.

□

Proposition 1.4. *If $T, S \in \mathcal{B}(\mathcal{H})^D$ are $(n,1)$-D-power-hyponormal operators commuting, such that such that $S^*(S^D)^n T^*(T^D)^n \geq (S^D)^n S^*(T^D)^n T^* \geq 0$ and $(T^D)^n T^* \geq 0$, then $TS \otimes T, TS \otimes S, ST \otimes T$ and $ST \otimes S \in \mathcal{B}(\mathcal{H}\overline{\otimes}\mathcal{H})^D$ are $(n,1)$-power-D-power-D-hyponormal if the following assertions hold:*

(1) $S^*(T^D)^n = (T^D)^n S^*$.

(2) $T^*(S^D)^n = (S^D)^n T^*$.

Proof. Assume that the conditions (1) and (2) are hold. Since T and S are $(n,1)$-power-D-hyponormal, we have

$$\begin{aligned}((TS \otimes T)^D)^n (TS \otimes T)^* &= \left((TS)^D \otimes T^D\right)^n \left((TS)^* \otimes T^*\right) \\ &= \left(((TS)^D)^n (TS)^* \otimes (T^D)^n T^*\right) \\ &= \left(((S^D)^n (T^D)^n) S^* T^* \otimes (T^D)^n T^*\right) \\ &= \left((S^D)^n S^* (T^D)^n T^* \otimes (T^D)^n T^*\right)\end{aligned}$$

$$
\begin{aligned}
&\leq \big(S^*(S^D)^nT^*(T^D)^n \otimes T^*(T^D)^n\big)\\
&= \big(S^*T^*(S^D)^n(T^D)^n \otimes T^*(T^D)^n\big)\\
&= \big((TS)^*((TS)^D)^n \otimes T^*(T^D)^n\big)\\
&= \big((TS)^* \otimes T^*\big)\big(((TS)^D)^n \otimes (T^D)^n\big)\\
&= \big(TS \otimes T\big)^*(\big(TS \otimes T\big)^D)^n
\end{aligned}
$$

Then $TS \otimes S$ is $(n,1)$-power-D-hyponormal operator.

In the same way, we may deduce the $(n,1)$-power-D-hyponormal operator of $TS \otimes S, ST \otimes T$ and $ST \otimes S$. □

Theorem 1.3. *If $T, S \in \mathcal{B}(\mathcal{H})^D$ two operators commuting. Then :*

$\big(I \otimes S\big), \big(T \otimes I\big)$ are $(n,1)$-power-D-hyponormal then $T \boxplus S$ is $(n,1)$-power-D-hyponormal.

Proof. Firstly, observe that if $\big(I \otimes S\big), \big(T \otimes I\big)$ are $(n,1)$-power-D-hyponormal, then we have following inequalities

$$(\big(T \otimes I\big)^D)^n\big(T \otimes I\big)^* \leq \big(T \otimes I\big)^*(\big(T \otimes I\big)^D)^n$$

and

$$(\big(S \otimes I\big)^D)^n\big(S \otimes I\big)^* \leq \big(S \otimes I\big)^*(\big(S \otimes I\big)^D)^n.$$

Then

$$
\begin{aligned}
&((T \boxplus S)^D)^n(T \boxplus S)^*\\
&= \big((T \otimes I + I \otimes S)^D\big)^n\big(T \otimes I + I \otimes S\big)^*\\
&= \big((T \otimes I)^D + (I \otimes S)^D\big)^n\big((T \otimes I)^* + (I \otimes S)\big)^*\\
&\leq ((T \otimes I)^D)^n\big(T \otimes I\big)^* + ((T \otimes I)^D)^n(I \otimes S\big)^*\\
&+ ((I \otimes S)^D)^n(T \otimes I)^* + ((I \otimes S)^D)^n(I \otimes S\big)^*\\
&\leq \big(T \otimes I)^*((T \otimes I)^D)^n + (I \otimes S\big)^*((T \otimes I)^D)^n\\
&+ (T \otimes I)^*((I \otimes S)^D)^n + (I \otimes S\big)^*((I \otimes S)^D)^n\\
&= (T \boxplus S)^*((T \boxplus S)^D)^n.
\end{aligned}
$$

Then $T \boxplus S$ is $(n,1)$-power-D-hyponormal.

□

Theorem 1.4. *Let $T_1, T_2, ..., T_m$ are $(n,1)$-power-D-hyponormal operator in $\mathcal{B}(\mathcal{H})^D$, such that $(T_k^D)^nT_k^* \geq 0$, $\forall k \in \{1, 2...m\}$. Then $(T_1 \oplus T_2 \oplus .... \oplus T_m)$ is $(n,1)$-power-D-hyponormal operators and $(T_1 \otimes T_2 \otimes .... \otimes T_m)$ is $(n,1)$-power-D-hyponormal operators.*

Proof. Since

$$\begin{aligned}
\left((T_1 \oplus T_2 \oplus ... \oplus T_m)^D\right)^n (T_1 \oplus T_2 \oplus ... \oplus T_m)^* &= \left((T_1^D)^n \oplus (T_2^D)^n \oplus ... \oplus (T_m^D)^n\right)(T_1^* \oplus T_2^* \oplus ... \oplus T_m^*) \\
&= ((T_1^D)^n T_1^* \oplus (T_2^D)^n T_2^* \oplus ... \oplus (T_m^D)^n T_m^*) \\
&\leq (T_1^*(T_1^D)^n \oplus T^*(T_2^D)_2^n \oplus ... \oplus T_m^*(T_m^D)^n) \\
&= (T_1^* \oplus T_2^* \oplus ... \oplus T_m^*)\left((T_1^D)^n \oplus (T_2^D)^n \oplus ... \oplus (T_m^D)^n\right) \\
&= (T_1 \oplus T_2 \oplus ... \oplus T_m)^* \left((T_1 \oplus T_2 \oplus ... \oplus T_m)^D\right)^n.
\end{aligned}$$

Then $(T_1 \oplus T_2 \oplus .... \oplus T_m)$ is $(n, 1)$-power-D-hyponormal operators.

Now,

$$\begin{aligned}
\left((T_1 \otimes T_2 \otimes ... \otimes T_m)^D\right)^n (T_1 \otimes T_2 \otimes ... \otimes T_m)^* &= \left((T_1^D)^n \otimes (T_2^D)^n \otimes ... \otimes (T_m^D)^n\right)(T_1^* \otimes T_2^* \otimes ... \otimes T_m^*) \\
&= ((T_1^D)^n T_1^* \otimes (T_2^D)^n T_2^* \otimes ... \otimes (T_m^D)^n T_m^*) \\
&\leq (T_1^*(T_1^D)^n \otimes T^*(T_2^D)_2^n \otimes ... \otimes T_m^*(T_m^D)^n) \\
&= (T_1^* \otimes T_2^* \otimes ... \otimes T_m^*)\left((T_1^D)^n \otimes (T_2^D)^n \otimes ... \otimes (T_m^D)^n\right) \\
&= (T_1 \otimes T_2 \otimes ... \otimes T_m)^* \left((T_1 \otimes T_2 \otimes ... \otimes T_m)^D\right)^n.
\end{aligned}$$

Then $(T_1 \otimes T_2 \otimes .... \otimes T_m)$ is $(n, 1)$-power-D-hyponormal operators. □

Proposition 1.5. *If T is $(2, 1)$-power-D-hyponormal and T is D-idempotent. Then T is power-D-hyponormal operator*

Proof. Since T is $(2, 1)$-power-D-hyponormal operator, then

$(T^D)^2 T^* \leq T^*(T^D)^2$

since T is D-idempotent $(T^D)^2 = T^D$, wich implies

$T^D T^* \leq T^* T^D$

Thus T is is power-D-hyponormal operator □

Proposition 1.6. *If T is $(3, 1)$-power-D-hyponormal and T is D-idempotent. Then T is power-D-hyponormal operator*

Proof. Since T is $(3, 1)$-power-D-hyponormal operator, then

$(T^D)^3 T^* \leq T^*(T^D)^3$

since T is D-idempotent $(T^D)^2 = T^D$, wich implies

$(T^D)T^* \leq T^* T^D$

Then T is power-D-hyponormal operator □

Proposition 1.7. *If T, S are $(2,1)$-power-D-hyponormal operators commuting, such that $T^DS^* = S^*T^D$ and $T^DS - ST^D = 0$, then $S+T$ is $(2,1)$-power-D-hyponormal operator.*

Proof. Since $T^DS - ST^D = 0$, hence $(T^D)^2S^2 + S^2(T^D)^2 = 0$, so $\left(S^D + T^D\right)^2 = (S^D)^2 + (T^D)^2$.

$$\begin{aligned}
\left((T+S)^D\right)^2 (S+T)^* &= \left((S^D)^2 + (T^D)^2\right)(S^* + T^*) \\
&= (S^D)^2S^* + (S^D)^2T^* + (T^D)^2S^* + (T^D)^2T^* \\
&= (S^D)^2S^* + T^*(S^D)^2 + S^*(T^D)^2 + (T^D)^2T^* \\
&\le S^*(S^D)^2 + T^*(S^D)^2 + S^*(T^D)^2 + T^*(T^D)^2 \\
&= (S+T)^*\left((T+S)^D\right)^2
\end{aligned}$$

Then $S+T$ is $(2,1)$-power-D-hyponormal operator. □

Proposition 1.8. *If T, S are $(2,1)$-power-D-hyponormal operators commuting, such that $T^DS^* = S^*T^D$ and $T^DS - ST^D = 0$, $TS = ST = S+T$ then ST is $(2,1)$-power-D-hyponormal operator.*

Proof. Since $T^DS - ST^D = 0$, hence $(T^D)^2S^2 + S^2(T^D)^2 = 0$, so $\left(S^D + T^D\right)^2 = (S^D)^2 + (T^D)^2$.
Since,

$$\begin{aligned}
\left((ST)^D\right)^2 (ST)^* &= \left((T+S)^D\right)^2 (S+T)^* \\
&= (S^D)^2S^* + (S^D)^2T^* + (T^D)^2S^* + (T^D)^2T^* \\
&= (S^D)^2S^* + T^*(S^D)^2 + S^*(T^D)^2 + (T^D)^2T^* \\
&\le S^*(S^D)^2 + T^*(S^D)^2 + S^*(T^D)^2 + T^*(T^D)^2 \\
&= (S+T)^*\left((T+S)^D\right)^2 \\
&= (ST)^*\left((TS)^D\right)^2
\end{aligned}$$

Hence
$\left((ST)^D\right)^2 (ST)^* \ge (ST)^*\left((ST)^D\right)^2$.
Then ST is $(2,1)$-power-D-hyponormal operator. □

Example 1.3. *Let* $T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $S = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \in \mathcal{B}(\mathbb{C}^2)$. *A simple computation shows that*

$$T^* = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S^* = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, T^D = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, S^D = \frac{1}{2}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then T is $(2,1)$-power-D-hyponormal operator, but

$$\left\langle \left((T^D)^2T^* - T^*(T^D)^2\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0.$$

For all $(u,v) \in (\mathbb{C}^2)$

and S is $(2,1)$-power-D-hyponormal operator, but

$$\left\langle \left((S^D)^2S^* - S^*(S^D)^2\right)\begin{pmatrix} u \\ v \end{pmatrix} \mid \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle = 0.$$

For all $(u,v) \in (\mathbb{C}^2)$

Such that $TS + ST = 0$ and $T^DS^* \neq S^*T^D$

but $S+T$ and ST are $(2,1)$-power-D-hyponormal operator

the following example shows that proposition (1.7) is not necessarily true if $T^DS^* \neq S^*T^D$

Proposition 1.9. *Let $T, S \in \mathcal{B}(\mathcal{H})^D$ are commuting and are $(n,1)$-power-D-hyponormal operators, such that $T^DS^* = S^*T^D$ and $(T+S)^*$ is commutes with*

$$\sum_{1\leq p\leq n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big).$$

Then $(T+S)$ is an $(n,1)$-power-D-hyponormal operator.

Proof. Since

$$\begin{aligned}
\big((T+S)^D\big)^n(T+S)^* &= \left[\sum_{0\leq p\leq n}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\right](T+S)^* \\
&= (S^D)^nS^* + \sum_{1\leq p\leq n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)(T+S)^* + (T^D)^nS^* + (S^D)^nT^* \\
&+ (T^D)^nT^* \\
&= (S^D)^nS^* + \sum_{1\leq p\leq n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)(T+S)^* + S^*(T^D)^n + T^*(S^D)^n \\
&+ (T^D)^nT^* \\
&\leq S^*(S^D)^n + (T+S)^*\sum_{1\leq p\leq n-1}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big) + S^*(T^D)^n + T^*(S^D)^n \\
&+ T^*(T^D)^n \\
&\leq (T+S)^*\left[\sum_{0\leq p\leq n}\binom{n}{p}\Big((T^D)^p(S^D)^{n-p}\Big)\right] \\
&= (T+S)^*\big((T+S)^D\big)^n.
\end{aligned}$$

Then $(T+S)$ is an $(n,1)$-power-D-hyponormal operator. □

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