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APPROXIMATE SOLUTION OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY LEAST SQUARES METHOD

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Abstract. In this paper, least squares approximation method is developed for solving a class of linear fractional integro-differential equations comprising Volterra and Fredhlom cases. This method is based on a polynomial of degree n to compute an approximate solution of these equations. The convergence analysis of the proposed method is proved. In addition, to show the accuracy and the efficiency of the proposed method, some examples are presented.

1. Introduction

Fractional calculus is a significant branch of mathematics that is used in many fields of science and engineering [2–4]. Many researchers have investigated the analytic results on the existence and uniqueness of solutions of the fractional differential equations [5–8]. As we know, for most fractional differential equations, there are not method to obtain analytic solutions, so numerical techniques must be used. During the past years, methods for solving fractional differential equations are developed. Additionally, some methods have recently been emerged, such as the Adomian decomposition method [10, 11], the operational matrix [12, 13], the collocation method [14, 16], etc.

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In this paper, by using least squares approximation, a numerical method has been shown to linear fractional integro-differential equations in the following form

$$D^{\alpha}y(t) = p(t)y(t) + f(t) + \lambda_1 \int_0^1 k_1(t,x)y(x)\,dx + \lambda_2 \int_0^t k_2(t,x)y(x)\,dx \quad t \in I = [0,1] \tag{1.1}$$

with the initial conditions

$$y^{(k)}(0) = d_k \; i = 0,..,m-1, \; m-1 < \alpha \leq m, \tag{1.2}$$

where $y^k(t)$ stands for the kth-order derivative of $y(t)$ and D^{α} denotes the Riemann-Liouville fractional derivative of order α. Clearly, when $\lambda_1 = 0$, $\lambda_2 = 0$, the above equation reduces to a linear fractional differential equation.

The rest of the paper is organized as follows: In section 2, we will briefly review some notations and definitions of the fractional calculus theory are used in the paper. In Section 3, we introduce the least squares approximation method for solving Eq. (1.1), and discuss its convergence. In Section 4, we show the efficiency of the proposed method with some numerical examples. Section 5, as the final section, presents a conclusion.

2. Brief review of fractional calculus

In this section, notations and definitions of the fractional calculus theory, which are going to be used in this paper, are presented [1].

2.1. **Definition.** The Riemann-Liouville fractional integral operator I^{α} of order $\alpha \geq 0$ of a function $f(x)$, is defined as

$$I^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{(\alpha-1)} f(t)\,dt \;, \alpha > 0, \tag{2.1}$$

where $x > 0$ and $\Gamma(.)$ is the Euler gamma function.

The Riemann-Liouville fractional derivative of order α will be denoted by D^{α} and defined by

$$D^{\alpha}f(x) = \frac{d^m}{dx^m}(I^{m-\alpha}f(x)), \tag{2.2}$$

where $m-1 < \alpha \leq m, m \in \mathbb{N}$ and m is the smallest integer order greater than α. We just mention the following property

$$D^{\alpha}x^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha}, \beta > -1. \tag{2.3}$$

3. Method of solution

In this section, we apply the least squares approximation method for solving Eq. (1.1). We define the following operator

$$T(t,y(t)) = D^{\alpha}y(t) - p(t)y(t) - f(t) - \lambda_1 \int_0^1 k_1(t,x)y(x)\,dx - \lambda_2 \int_0^t k_2(t,x)y(x)\,dx. \tag{3.1}$$

We construct Taylor-series expansion for the solution $y(t)$ in Eq.(1.1) as

$$y(t) \simeq y_n(t) = \sum_{r=0}^{n} \frac{y^{(r)}(0)}{r!}t^r = \sum_{r=0}^{n} \frac{d_r}{r!}t^r. \tag{3.2}$$

Substituting (3.2) into (3.1), we have

$$\begin{aligned}
T(t,y_n(t)) &= D^{\alpha}y_n(t) - p(t)y_n(t) - f(t) - \lambda_1 \int_0^1 k_1(t,x)y_n(x)\,dx - \lambda_2 \int_0^t k_2(t,x)y_n(x)\,dx \\
&= D^{\alpha}(\sum_{r=0}^{n} \frac{d_r}{r!}t^r) - p(t)\sum_{r=0}^{n} \frac{d_r}{r!}t^r - f(t) - \sum_{r=0}^{n} \frac{d_r}{r!}\lambda_1 \int_0^1 k_1(t,x)x^r\,dx - \\
&\quad \sum_{r=0}^{n} \frac{d_r}{r!}\lambda_2 \int_0^t k_2(t,x)x^r\,dx \\
&= \sum_{r=0}^{n} \frac{d_r\Gamma(r+1)}{r!\Gamma(r-\alpha+1)}t^{r-\alpha} - p(t)\sum_{r=0}^{n} \frac{d_r}{r!}t^r - f(t) - \sum_{r=0}^{n} \frac{d_r}{r!}\lambda_1 \int_0^1 k_1(t,x)x^r\,dx - \\
&\quad \sum_{r=0}^{n} \frac{d_r}{r!}\lambda_2 \int_0^t k_2(t,x)x^r\,dx \\
&= \sum_{r=0}^{n} d_r\gamma_r(t) - f(t),
\end{aligned}$$

where

$$\gamma_r(t) = \frac{\Gamma(r+1)}{r!\Gamma(r-\alpha+1)}t^{r-\alpha} - p(t)\frac{t^r}{r!} - \frac{\lambda_1}{r!}\int_0^1 k_1(t,x)x^r\,dx - \frac{\lambda_2}{r!}\int_0^t k_2(t,x)x^r\,dx.$$

Let $R_n(t) := T(t,y_n(t)) - T(t,y(t)), t \in [0,1]$.

Remark 3.1. If $R_n(t) = 0$, then $y(t) = y_n(t)$; if $lim_{n\to\infty}R_n(t) = 0$, then $lim_{n\to\infty}y_n(t) = y(t)$.

Remark 3.2. For any $t \in [0,1]$, if $R_n(t) \equiv 0$, then $y_n(t)$ is an exact solution of Eqs. (1.1) and (1.2); if $lim_{n\to\infty}R_n(t) = 0$, then $y_n(t)$ converges to the exact solution of Eqs. (1.1) and (1.2).

Let

$$J = J(d_m, d_{m+1}, ..., d_n) = \int_0^1 T^2(t,y_n(t))dt. \tag{3.3}$$

The problem is to find real constants $d_m, d_{m+1}, ..., d_n$ such that these constants will minimize J. A necessary condition for the constants $d_m, d_{m+1}, ..., d_n$ to minimize J is that

$$\frac{\partial J}{\partial d_j} = 0,$$

for each $j = m, m+1, ..., n$. By referring (3.3), we get

$$\frac{\partial J}{\partial d_j} = 2[\sum_{r=0}^{n} d_r \int_0^1 \gamma_r(t)\gamma_j(t)dt - \int_0^1 f(t)\gamma_j(t)dt] = 0. \tag{3.4}$$

Thus, we have

$$\sum_{r=m}^{n} d_r \int_0^1 \gamma_r(t)\gamma_j(t)dt = \int_0^1 f(t)\gamma_j(t)dt - \beta_j, \tag{3.5}$$

where

$$\beta_j = \sum_{r=0}^{m-1} d_r \int_0^1 \gamma_r(t)\gamma_j(t)dt \tag{3.6}$$

for each $j = m, m+1, ..., n$.

In order to find $y_n(t)$, we have to solve $(n-m)$ a system of linear equations while assuming $(n-m)$ unknowns d_r. The system (3.5) can be written in the form:

$$Gd = F \tag{3.7}$$

where

$$G = \begin{pmatrix} (\gamma_m, \gamma_m) & (\gamma_m, \gamma_{m+1}) & . \quad . \quad . & (\gamma_m, \gamma_n) \\ (\gamma_{m+1}, \gamma_m) & (\gamma_{m+1}, \gamma_{m+1}) & . \quad . \quad . & (\gamma_{m+1}, \gamma_n) \\ \vdots & \vdots & . \quad . \quad . & \vdots \\ (\gamma_n, \gamma_m) & (\gamma_n, \gamma_{m+1}) & . \quad . \quad . & (\gamma_n, \gamma_n) \end{pmatrix}, \tag{3.8}$$

$$d = [d_m, d_{m+1}, ..., d_n]^T,$$

and

$$F = [(\gamma_m, f) - \beta_m, (\gamma_{m+1}, f) - \beta_{m+1}, ..., (\gamma_n, f) - \beta_n]^T$$

Definition 3.3. If Eq. (3.7) has a unique solution d, then $y_n(t) = \sum_{r=1}^{n} \frac{d_r}{r!} t^r$ is called an optimal squared approximation solution of Eqs. (1.1)-(1.2) defined on a set as $span\{1, t, t^2, ...t^n\}$, $t \in [0, 1]$.

Remark 3.4. If $lim_{n\to\infty} \int_0^1 T^2(t, y_n(t))dt = 0$, then the optimal squared approximation solution $y_n(t)$ converges to the exact solution $y(t)$ of Eqs. (1.1) and (1.2).

We are interested to know that as $n \to \infty$ the optimal squared approximation solution $y_n(t)$ will converge to the exact solution $y(t)$ of Eqs. (1.1) and (1.2). This conception is proven in Theorem 3.5.

Theorem 3.5. Suppose $y(t)$, $t \in [0,1]$ is an exact solution and $y_n(t)$ is an optimal squared approximation solution of Eqs. (1.1) and (1.2). If $\exists p_n(t) = \sum_{r=1}^{n} d_r t^r$ such that $\forall t \in [0,1]$, $\lim_{n\to\infty} p_n(t) = y(t)$ then

$$\lim_{n\to\infty} \int_0^1 T^2(t, y_n(t))dt = 0.$$

Proof. The proof is similar to proof of Theorem 3 in [19].

4. Illustrative examples

In this section, we use the presented method in Section 3 for solving two examples.

Example 4.1. For first example, consider the fractional integro-differential equation

$$D^{0.75}y(t) + \frac{1}{5}t^2e^ty(t) - \int_0^t xe^ty(x)\,dx = \frac{6t^{2.25}}{\Gamma(3.25)},$$

$$y(0) = 0,$$

where the exact solution is given by $y(t) = t^3$.

We applied the presented method with $n = 3$ for solving this example and achieved the corresponding absolute errors in Table 1.

Table 1: Absolute errors for Example 1 for $n = 3$.

$t = 0$	$t = 0.2$	$t = 0.4$	$t = 0.6$	$t = 0.8$	$t = 1.0$
0	1.314×10^{-15}	1.045×10^{-15}	1.914×10^{-16}	2.475×10^{-16}	6.717×10^{-16}

Example 4.2. Consider the equation

$$D^{\alpha}y(t) + \frac{7t^2}{12}y(t) - \int_0^1 txy(x)\,dx - \int_0^t (x+t)y(x)\,dx = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} - \frac{t}{4},$$

$$y(0) = 0,$$

with the exact solution $y(t) = t^2$. By the presented method in section 3 for $n = 2$ and different values of α absolute errors are reported in Table 2.

Example 4.3. Consider the equation [20]

$$D^{\alpha}y(t) + y(t) = t^4 - \frac{1}{2}t^3 - \frac{3}{\Gamma(4-\alpha)}t^{3-\alpha} + \frac{24}{\Gamma(5-\alpha)}t^{4-\alpha},$$

$$y(0) = 0, \quad 0 \le \alpha \le 1$$

whose exact solution is given by $y(t) = t^4 - \frac{1}{2}t^3$.

Table 2: Absolute errors for Example 2.

t	$\alpha = 0.2$	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$
0.0	0	0	0	0	0
0.1	0	1.734×10^{-17}	1.387×10^{-17}	0	0
0.2	0	3.469×10^{-17}	2.775×10^{-17}	0	0
0.3	0	4.163×10^{-17}	4.163×10^{-17}	0	0
0.4	0	5.551×10^{-17}	5.551×10^{-17}	0	0
0.5	0	5.551×10^{-17}	5.551×10^{-17}	0	0
0.6	0	5.551×10^{-17}	5.551×10^{-17}	0	0
0.7	0	5.551×10^{-17}	1.110×10^{-16}	0	0
0.8	0	0	1.110×10^{-16}	0	0
0.9	0	1.110×10^{-16}	1.110×10^{-16}	0	0
1.0	0	0	2.220×10^{-16}	0	0

By taking different values of α, we solved the above problem by means of the presented method. The maximum absolute error with the presented method and SCT method [20] for $n = 4$ are compared in Table 3.

Table 3: Comparison of maximum absolute error for example 3.

α	n=4	Present method	Method of [20]
0.01		1.26×10^{-14}	1.2×10^{-5}
0.1		1.41×10^{-14}	1.3×10^{-4}
0.5		5.30×10^{-15}	7.8×10^{-4}
0.99		1.30×10^{-15}	8.6×10^{-4}
1		1.14×10^{-15}	8.6×10^{-4}

Example 4.4. Consider the equation

$$D^{\frac{3}{2}} y(t) + y(t) = \frac{6}{\Gamma(2.5)} t^{1.5} + \frac{6}{\Gamma(1.5)} t^{0.5} + t^3 + t^2,$$

$$y(0) = 0, y'(t) = 0,$$

whose exact solution is given by $y(t) = t^3 + t^2$.

By applying the technique described in section 3 with m=4, we approximate solution as

$$y(t) = \sum_{r=0}^{4} \frac{y^{(r)}(0)}{r!} t^r = \sum_{r=0}^{4} \frac{d_r}{r!} t^r$$

Here, by using Eq. (3.7), we obtain

$$\begin{pmatrix} 1.00901 & 0.422206 & 0.123762 \\ 0.422206 & 0.19103 & 0.0586763 \\ 0.123762 & 0.0586763 & 0.0186265 \end{pmatrix} \begin{pmatrix} d_2 \\ d_3 \\ d_4 \end{pmatrix} = \begin{pmatrix} 4.55127 \\ 1.9906 \\ 0.599582 \end{pmatrix} \tag{4.1}$$

Finally by solving Eq.(4.14), we get

$$d_2 = 2, d_3 = 6, d_4 = 0.$$

Thus we can write

$$y(t) = d_0 + d_1 t + d_2 \frac{t^2}{2!} + d_3 \frac{t^3}{3!} + d_4 \frac{t^4}{4!} = t^3 + t^2,$$

which is the exact solution.

5. Conclusion

In this paper, we proposed least squares approximation method to solve a class of linear fractional integro-differential equations comprising of Fredholm and Volterra cases based on a polynomial of degree n. The numerical experiments show that the proposed method can be suitable method for solving these equations.

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