International Journal of Analysis and Applications
Volume 17, Number 2 (2019), 208-225
URL: https://doi.org/10.28924/2291-8639
DOI: 10.28924/2291-8639-17-2019-208

FIXED POINTS FOR TRIANGULAR $\alpha-$ADMISSIBLE GERAGHTY CONTRACTION TYPE MAPPINGS IN PARTIAL b-METRIC SPACES

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Abstract. In this paper, we introduce the notion of generalized $C-$class functions for Geraghty contraction type mappings on a set X. We utilize our new notion to prove fixed point results in the setting of triangular weak $\alpha-$admissible mappings with respect to η in Partial b-Metric Spaces. Our results modify and improve many exciting results in the literature. Also, we introduce an example and an application to show the validity of our main result.

Received 2018-09-26; accepted 2018-11-20; published 2019-03-01.

2010 *Mathematics Subject Classification.* 47H10, 54H25.

Key words and phrases. C-class functions; $\alpha-$admissible mapping; fixed point; $b-$metric spaces; partial metric spaces; partial $b-$metric spaces.

1. Introduction and preliminaries

One of the most important tools in fixed point theory is Banach contraction principle. A lot of authors have extended or generalized this contraction and proved the existence of fixed and common fixed point theorems (for example see [19]- [28]). In this sequel, Bakhtin [7] and Czerwik [10] introduced b-metric spaces as a generalization of metric spaces. They proved the contraction mapping principle in $b-$metric spaces that generalized the famous Banach contraction principle in such spaces. After that, several papers have dealt with fixed point theory for single-valued and multi-valued operators in b-metric spaces (for example see [11], [27], [29], [32]).

On the other hand, Matthews [21] introduced the notion of partial metric space as a part of the study of denotational semantics of dataflow networks, showing that the contraction mapping principle [8] can be generalized to the partial metric context for applications in program verifications.

$b-$metric spaces [7] and Partial metric spaces [21] are two well known generalizations of usual metric spaces. Also, the Banach contraction principle is a fundamental result in the fixed point theory, which has been used and extended in many different directions. Recently, Shukla [35] introduced a generalization and unification of partial metric and b-metric spaces as the concept of partial b-metric space.

In this section, we recall some useful definitions and auxiliary results that will be needed in the sequel. Throughout this paper, $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the set of real numbers, respectively.

Definition 1.1. *([7], [10]) Let X is a nonempty set and let $s \geq 1$ be a given real number. A function $d : X \times X \to [0, \infty)$ is said to be a $b-$metric space on X if and only if for all $x, y, z \in X$, the following conditions hold:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq s[d(x, y) + d(y, z)]$.

The triplet (X, d, s) is called a $b-$metric space.

It is well known that the class of b-metric spaces is larger than the class of metric spaces when $s = 1$, the concept of b-metric space coincides with the concept of metric space.

Example 1.1. *Consider the set $X = [0, 1]$ endowed with the function $d : X \times X \to [0, \infty)$ defined by $d(x, y) = |x - y|^2$ for all $x, y \in X$. Clearly, $(X, d, 3)$ is a $b-$metric space but it is not a metric space.*

Example 1.2. *Let $X = \mathbb{R}$ and let the mapping $d : X \times X \to [0, \infty)$ be defined by*

$$d(x, y) = | x - y |^2 \text{ for all } x, y \in X.$$

Then (X, d) is a b-metric space with coefficient $s = 2$.

Definition 1.2. *[21] Let X be a nonempty set. A function $p : X \times X \to [0, \infty)$ is called a partial metric space if for all $x, y, z \in X$, the following conditions are satisfied:*

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$,

(p_2) $p(x, x) \leq p(x, y)$,

(p_3) $p(x, y) = p(y, x)$,

(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space(PMS). The sequence $\{x_n\}$ in X converges to a point $x \in X$ if $\lim_{n\to\infty} p(x_n, x) = p(x, x)$. Also the sequence $\{x_n\}$ is called $p-$Cauchy if the $lim_{n,m\to\infty} p(x_n, y_m)$ exists. The partial metric space (X, p) is called complete if for every p-Cauchy sequence $\{x_n\}^n_\infty$, there is some $x \in X$ such that

$$p(x, x) = \lim_{n\to\infty} p(x_n, x) = \lim_{n,m\to\infty} p(x_n, x_m).$$

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x, y) = \max\{x, y\}$ for all $x, y \in \mathbb{R}^+$.

Definition 1.3. *[35] Let X be a nonempty set. A function $b : X \times X \to [0, \infty)$ is called a $b-$partial metric space if for all $x, y, z \in X$, the following conditions are satisfied:*

(p_{b1}) *$x = y$ if and only if $b(x, x) = b(x, y) = b(y, y)$,*

(p_{b2}) $b(x, x) \leq b(x, y)$,

(p_{b3}) $b(x, y) = b(y, x)$,

(p_{b4}) *there exists a real number $s \geq 1$ such that $b(x, y) \leq s[b(x, z) + b(z, y)] - b(z, z)$.*

Remark 1.1. *[35] In a partial $b-$metric space (X, b) if $x, y \in X$ and $b(x, y) = 0$, then $x = y$, but the converse may not be true.*

Remark 1.2. *[35] It is clear that every partial metric space is a partial $b-$metric space with coefficient $s = 1$ and every $b-$metric space is a partial $b-$metric space with the same coefficient and zero self-distance. However, the converse of this fact need not hold.*

Example 1.3. *[35] Let $X = \mathbb{R}^+$, $p > 1$ is a constant and $b : X \times X \to \mathbb{R}^+$ be defined by*

$$b(x, y) = [\max\{x, y\}]^p - |x - y|^p$$

for all $x, y \in X$. Then, (X, b) is a partial $b-$metric space with coefficient $s = 2p > 1$, but it is neither a b-metric nor a partial metric space.

Proposition 1.1. *[35] Let X be a nonempty set such that p is a partial and d is a $b-$metric with coefficient $s > 1$ on X. Then the function $b : X \times X \to \mathbb{R}^+$ defined by $b(x, y) = p(x, y) + d(x, y)$ for all $x, y \in X$ is a partial b-metric on X, that is, (X, b) is a partial $b-$metric space.*

Definition 1.4. *[35] Let (X, b) be be a partial $b-$metric space with coefficient s. Let $\{x_n\}$ be any sequence in X and $x \in X$. Then:*

- *(i) A sequence $\{x_n\} \subseteq X$ converges to a point $x \in X$ if $\lim_{n\to\infty} b(x_n, x) = b(x, x)$,*
- *(ii) A sequence $\{x_n\} \subseteq X$ is said to be a Cauchy sequence in (X, b) if, for every given $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $\lim_{n,m\to\infty} b(x_n, x_m)$ exists and is finite for all $m, n \geq n(\epsilon)$,*
- *(iii) (X, b) is said to be complete partial $b-$metric space if Cauchy sequence $\{x_n\} \subseteq X$ there exists $x \in X$ such that*

$$\lim_{n,m\to\infty} b(x_n, x_m) = \lim_{n\to\infty} b(x_n, x) = b(x, x).$$

Note that in a partial $b-$metric space the limit of convergent sequence may not be unique.

Samet el al. [31] introduced the notion of $\alpha-$admossible mapping and studied many fixed point theorems. After that several authors used the notion of α-admissible to prove and construct many fixed and common fixed point theorems (see [14]- [1]).

Samet et al. [31] presented the notion of α-admissible mapping as follows:

Definition 1.5. *[31] Let $f : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then f is called α-admissible if $\forall x, y \in X$ with $\alpha(x, y) \geq 1$ implies $\alpha(fx, fy) \geq 1$.*

Definition 1.6. *[17] Let $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$. Then T is called a triangular α-admissible mapping if*

1. *T is α-admissible;*
2. *$\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$ imply $\alpha(x, y) \geq 1$.*

Sintunavarat [32] presented the notion of weak α-admissible mappings as follows:

Definition 1.7. *[32] Let X be a nonempty set and let $\alpha : X \times X \to [0, \infty)$ be a given mapping. A mapping $f : X \to X$ is said to be a weak α-admissible mappings if the following condition holds:*

$$x \in X \text{ with } \alpha(x, fx) \geq 1 \Rightarrow \alpha(fx, f^2x) \geq 1.$$

Remark 1.3. *[32] It is customary to write $\mathcal{A}(X,\alpha)$ and $\mathcal{WA}(X,\alpha)$ to denote the collection of all α-admissible mappings on X and the collection of all weak α-admissible mappings on X. One can verify that $\mathcal{A}(X,\alpha) \subseteq \mathcal{WA}(X,\alpha)$.*

Qawaqneh et al. [23] presented the notion of α-admissible with respect to another function η for the pair of self-mappings S and T on a set X as follows:

Definition 1.8 ([23]). *Let $S, T : X \to X$ be two mappings and $\alpha : X \times X \to [0, +\infty)$ be a function such that the following conditions hold:*

(1) *if $\alpha(x, y) \geq 1$, then $\alpha(Sx, Ty) \geq 1$ and $\alpha(TSx, STy) \geq 1$;*
(2) *if $\alpha(x, z) \geq 1$ and $\alpha(z, y) \geq 1$, then $\alpha(x, y) \geq 1$.*

Then we say that the pair (S, T) is triangular α-admissible.

Definition 1.9 ([23]). *Let $S, T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions such that the following conditions hold:*

(1) *if $\alpha(x, y) \geq \eta(x, y)$, then $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$ and $\alpha(TSx, STy) \geq \eta(TSx, STy)$;*
(2) *if $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(z, y) \geq \eta(z, y)$, then $\alpha(x, y) \geq \eta(x, y)$.*

Then we say that the pair (S, T) is triangular α-admissible with respect to η.

Lemma 1.1 ([23]). *Let $S, T : X \to X$ be two mappings and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions such that the pair (S, T) is triangular α-admissible with respect to η. Assume that there exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$. Define a sequence $\{x_n\}$ in X by $Sx_{2n} = x_{2n+1}$ and $Tx_{2n+1} = x_{2n+2}$. Then $\alpha(x_n, x_m) \geq \eta(x_n, Sx_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

In 2014, Ansari [4] defined the concept of C-class function as the following:

Definition 1.10. *[4] A mapping $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is called a C-class function if it is continuous and for $s, t \in [0, \infty)$, F satisfies the following two conditions:*

(1) *$F(s, t) \leq s$; and*
(2) *$F(s, t) = s$ implies that either $s = 0$ or $t = 0$.*

The family of all $C-$class functions is denoted by $\mathcal{C}$.

Example 1.4. *[4] The following functions $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ are elements in $\mathcal{C}$.*

(1) *$F(s, t) = s - t$ for all $s, t \in [0, \infty)$.*
(2) *$F(s, t) = ks$ for all $s, t \in [0, \infty)$, where $0 < k < 1$.*
(3) *$F(s, t) = \frac{s}{(1+t)^r}$ for all $s, t \in [0, \infty)$, where $r \in [0, \infty)$.*
(4) *$F(s, t) = (s + l)^{(1/(1+t)^r)} - l$ for all $s, t \in [0, \infty)$, where $r \in (0, \infty)$, $l > 1$.*

(5) $F(s,t) = s\log_{t+a} a$ *for all* $s,t \in [0,\infty)$*, where* $a > 1$.

(6) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$ *for all* $s,t \in [0,\infty)$.

(7) $F(s,t) = s\beta(s)$ *for all* $s,t \in [0,\infty)$*, where* $\beta : [0,\infty) \to [0,1)$ *is continuous.*

(8) $F(s,t) = s - \varphi(s)$ *for all* $s,t \in [0,\infty)$*, where* $\varphi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\varphi(t) = 0$ *if and only if* $t = 0$.

(9) $F(s,t) = sh(s,t)$ *for all* $s,t \in [0,\infty)$*, where* $h : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $h(s,t) < 1$ *for all* $s,t \in [0,\infty)$.

(10) $F(s,t) = s - (\frac{2+t}{1+t})t$ *for all* $s,t \in [0,\infty)$.

(11) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$ *for all* $s,t \in [0,\infty)$.

In 2016, Ansari and Kaewcharoen [6] gave the definition of a generalized $\alpha-\eta-\psi-\varphi-F$-contraction type mapping and proved same fixed point theorems for such mappings in complete metric spaces.

Definition 1.11 ([6])**.** *Let* (X,d) *be a metric space and* $\alpha,\eta : X \times X \to [0,\infty)$ *be two functions. A mapping* $T : X \to X$ *is said to be a generalized* $\alpha-\eta-\psi-\varphi-F$*-contraction type mapping if* $\alpha(x,y) \geq \eta(x,y)$ *implies*

$$\psi(d(Tx,Ty)) \leq F(\psi(M(x,y)),\varphi(M(x,y))),$$

where

$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty)\}.$$

Hussain et al. [15] introduced the concepts of $\alpha-\eta$-complete metric spaces and $\alpha-\eta$-continuous functions.

Definition 1.12 ([15])**.** *Let* (X,d) *be a metric space and* $\alpha,\eta : X \times X \to [0,\infty)$ *be two functions. Then* X *is said to be an* α,η*-complete metric space if every Cauchy sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ *for all* $n \in \mathbb{N}$ *converges in* X.

Definition 1.13 ([15])**.** *Let* (X,d) *be a metric space and* $\alpha,\eta : X \times X \to [0,\infty)$ *be two functions. A mapping* $T : X \to X$ *is said to be an* α,η*-continuous mapping if each sequence* $\{x_n\}$ *in* X *with* $x_n \to x$ *as* $n \to \infty$ *and* $\alpha(x_n,x_{n+1}) \geq \eta(x_n,x_{n+1})$ *for all* $n \in \mathbb{N}$ *implies* $Tx_n \to Tx$ *as* $n \to \infty$.

Theorem 1.1 ([6])**.** *Let* (X,d) *be a metric space. Assume that* $\alpha,\eta : X \times X \to [0,\infty)$ *are two functions and* $T : X \to X$ *is a mapping. Suppose that the following conditions are satisfied:*

(1) (X,d) *is an* α,η*-complete metric space;*

(2) T *is generalized* $\alpha-\eta-\psi-\varphi-F$*-contraction type mapping;*

(3) T *is triangular* α*-orbital admissible mapping with respect to* η*;*

(4) *there exists* $x_1 \in X$ *such that* $\alpha(x_1,Tx_1) \geq \eta(x_1,Tx_1)$*;*

(5) T *is an* α,η*-continuous mapping.*

Then T has a fixed point $x^* \in X$.

Khan et al. [20] introduced the notion of an altering distance function as follows:

Definition 1.14. *[20] A mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is called an altering distance function if the following properties are satisfied:*

(1) *ψ is monotone and nondecreasing;*

(2) *$\psi(t) = 0$ if an only if $t = 0$.*

The set of all altering distance functions is denoted by Ψ.

In the rest of this paper, we let ϕ be the set of all functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(1) φ is continuous.

(2) $\varphi(t) = 0$ if and only if $t = 0$.

2. Main result

In this section, we introduce the concept of generalized $C-$class functions for Geraghty contraction type mappings on a set X and we prove fixed point results for self mappings on $\alpha, \eta-$ partial $b-$metric space.

Now, we present the notion of triangular weak α-admissible with respect to another function η for the self-mapping S on a set X.

Definition 2.1. *Let $S : X \to X$ be a mapping and $\alpha, \eta : X \times X \to [0, +\infty)$ be two functions such that the following conditions hold:*

(1) *if $\alpha(x, S^n x) \geq \eta(x, S^n x)$, then $\alpha(S^n x, S^{n+1}x) \geq \eta(S^n x, S^{n+1}x)$,*

(2) *if $\alpha(x, z) \geq \eta(x, z)$ and $\alpha(z, y) \geq \eta(z, y)$, then $\alpha(x, y) \geq \eta(x, y)$,*

for all $n \in \mathbb{N}$. Then we say that S is triangular weak α-admissible with respect to η.

Now, we introduce the following example to illustrate our new definition.

Example 2.1. *Let $X = [0, +\infty)$. Define $S : X \to X$ by $Sx = x^2$. Also, define the functions $\alpha, \eta : X \times X \to [0, +\infty)$ by $\alpha(x, y) = e^{x+y}$ and $\eta(x, y) = e^{y-x}$. Then S is triangular weak α-admissible with respect to η.*

Proof. If $\alpha(x, Sx) \geq \eta(x, Sx)$, then $e^{x+x^2} \geq e^{x^2-x}$. So $x + x^2 \geq x^2 - x$. So $2x \geq 0$. Hence $x \geq 0$. Since $x \geq -x$, then $x + x^4 \geq x^4 - x$. So $e^{x+^4} \geq e^{^4-x}$. Hence $\alpha(x,^4) \geq \eta(x,^4)$. So $\alpha(Sx, Ty) \geq \eta(Sx, Ty)$. Also, since $x^2 \geq -x^2$, then $x^2 + y^2 \geq y^2 - x^2$. So $e^{x^2+y^2} \geq e^{y^2-x^2}$. Hence $\alpha(x^2, y^2) \geq \eta(x^2, y^2)$. So $\alpha(Sx, S^2x) \geq \eta(Sx, S^2x)$. Also, if $\alpha(x, z) \geq \eta(x, z)$, and $\alpha(z, y) \geq \eta(z, y)$, then $x+z \geq z-x$ and $z+y \geq y-z$. So $x \geq -x$ and hence $x + x^2 \geq x^2 - x$. Therefore $e^{x+y} \geq e^{y-x}$. Therefore $\alpha(x, Sx) \geq \eta(x, Sx)$. □

By taking a special case of Lemma 1.1and generalize with is triangular weak $\alpha-$admissible with respect to η, we present a lemma that will be helpful for us to achieve our main result.

Lemma 2.1. *Let $S : X \to X$ be a mappings and $\alpha, \eta : X \times X \to \mathbb{R}$ are a functions such that S is triangular weak $\alpha-$admissible with respect to η. Assume that there exist $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$. Define a sequence $\{x_n\}$ in X by $Sx_n = x_{n+1}$. Then $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$ and S is weak $\alpha-$admissible, We get

$$\begin{cases} \alpha(x_0, x_1) = \alpha(x_0, Sx_0) \geq \eta(x_0, x_1), then \\ \alpha(Sx_0, Sx_1) = \alpha(Sx_0, S^2x_0) = \alpha(x_1, x_2) \geq \eta(x_1, x_2). \end{cases}$$

By triangular $\alpha-$admissibility, we get

$$\begin{cases} \alpha(Sx_0, Sx_1) = \alpha(x_1, x_2) \geq \eta(x_1, x_2), then \\ \alpha(S^2x_0, S^2x_1) = \alpha(x_2, x_3) \geq \eta(x_2, x_3) \end{cases}$$

and

$$\alpha(S^2x_1, S^2x_2) = \alpha(x_3, x_4) \geq \eta(x_3, x_4).$$

Again, since $\alpha(x_3, x_4) \geq \eta(x_3, x_4)$, then

$$\alpha(S^2x_3, S^2x_4) = \alpha(x_4, x_5) \geq \eta(x_4, x_5)$$

and

$$\alpha(S^2x_4, S^2x_5) = \alpha(x_5, x_6) \geq \eta(x_5, x_6).$$

By continuing the above process, we conclude that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we prove that

$\alpha(x_n, x_m) \geq 1$, $\forall m, n \in \mathbb{N}$ with $n < m$.

Given $m, n \in \mathbb{N}$ with $n < m$. Since

$$\begin{cases} \alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1}), \\ \alpha(Sx_n, S^2x_n) = \alpha(x_{n+1}, x_{n+2}) \geq \eta(x_{n+1}, x_{n+2}), \end{cases}$$

then, we have

$$\alpha(x_n, x_{n+2}) \geq \eta(x_n, x_{n+2}).$$

Again, since

$$\begin{cases} \alpha(x_n, x_{n+2}) \geq \eta(x_n, x_{n+2}) \\ \alpha(Sx_{n+1}, S^2x_{n+1}) = \alpha(x_{n+2}, x_{n+3}) \geq \eta(x_{n+2}, x_{n+3}), \end{cases}$$

we deduce that

$$\alpha(x_n, x_{n+3}) \geq \eta(x_n, x_{n+3}).$$

By continuing this process, we have

$$\alpha(x_n, x_m) \geq \eta(x_n, x_m)$$

for all $n \in \mathbb{N}$ with $m > n$.

□

In order to facilitate our subsequent arguments, we introduce the notion of generalized $C-$class functions for self mappings on a set X.

Definition 2.2. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$, $S : X \to X$ be a Geraghty contraction type mapping and $\alpha, \eta : X \times X \to \mathbb{R}$ be a function. Let $F \in \mathcal{C}$, $\psi \in \Psi$ and $\varphi \in \Phi$. Then S is called generalized $C-$class function with $\alpha(x, y) \geq \eta(x, y)$, then*

$$\psi(b(Sx, Sy)) \leq \lambda F(\psi(M(x, y)), \varphi(M(x, y))), \tag{2.1}$$

where

$$M(x, y) = \max\{b(x, y), b(x, Sx), b(y, Sy), \frac{b(x, Sy) + b(y, Sx)}{2}\} \tag{2.2}$$

and $\lambda \in [0, \frac{1}{s})$.

Theorem 2.1. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be Geraghty contraction type mapping on X. Assume that $\alpha, \eta : X \times X \to [0, +\infty)$ are a functions. Suppose that the following conditions hold:*

(1) *S is generalized $C-$class function.*
(2) *S is a triangular weak α-admissible.*
(3) *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*
(4) *S is $\alpha, \eta-$continuous mappings.*

Then S has a unique fixed point.

Proof. We divide the proof to three steps:

Step 1. Let $x_0 \in X$ be such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$. Define a sequence $\{x_n\}$ in X such that $x_{n+1} = Sx_n$ for all $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then it is very easy to show that S has a fixed point. Now, since the pair S is $\alpha-$admissible, then

$$\alpha(x_1, x_2) = \alpha(Sx_0, S^2x_0) \geq \eta(x_1, x_2)$$

and

$$\alpha(x_2, x_3) = \alpha(Sx_1, S^2x_1) \geq \ \eta(x_2, x_3).$$

Again, by using the property of weak $\alpha-$admissible and repeating the above process for n-times, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ and $\alpha(x_{n+1}, x_n) \geq \eta(x_{n+1}, x_n)$.

Using the property of triangular weak $\alpha-$admissible, we can deduce that for any $n, m \in \mathbb{N}$ with $m > n$, we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ and $\alpha(x_m, x_n) \geq \eta(x_m, x_n)$.

Suppose $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, by Lemma 2.1, we have $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Since S is a generalized $C-$class function, we have

$$\begin{aligned} \psi(b(x_{n+1}, x_n)) &= \psi(b(Sx_n, Sx_{n-1})) \\ &\leq \lambda F(\psi(M(x_n, x_{n-1})), \varphi(M(x_n, x_{n-1}))) \\ &\leq \lambda \psi(M(x_n, x_{n-1})), \end{aligned} \tag{2.3}$$

for all $n \in \mathbb{N}$, where

$$\begin{aligned} M(x_n, x_{n-1}) &= \max\{b(x_n, x_{n-1}), b(x_n, Sx_n), b(x_{n-1}, Sx_{n-1}), \frac{b(x_n, Sx_{n-1}) + b(x_{n-1}, Sx_n)}{2}\} \\ &= \max\{b(x_n, x_{n-1}), b(x_n, x_{n+1}), b(x_{n-1}, x_n), \frac{b(x_n, x_n) + b(x_{n-1}, x_{n+1})}{2}\} \\ &= \max\{b(x_n, x_{n-1}), b(x_n, x_{n+1})\}. \end{aligned} \tag{2.4}$$

If $M(x_n, x_{n-1}) = b(x_n, x_{n+1})$, then

$$\begin{aligned} \psi(b(x_{n+1}, x_n)) &\leq \lambda F(\psi(M(x_n, x_{n-1}), \varphi(M(x_n, x_{n-1}))) \\ &\leq \lambda \psi(M(x_n, x_{n-1})) \\ &= \lambda \psi(b(x_{n+1}, x_n)), \\ &< \psi(b(x_{n+1}, x_n)). \end{aligned} \tag{2.5}$$

Which is contraction. Thus we conclude that $M(x_n, x_{n-1}) = b(x_n, x_{n-1})$. By (2.2), we get that

$$\psi(b(x_{n+1}, x_n)) \leq \lambda \psi(b(x_n, x_{n-1}))$$

for all $n \in \mathbb{N}$.

On repeating this process, we obtain

$$\psi(b(x_{n+1}, x_n)) \leq \lambda^n \psi(b(x_1, x_0)) \tag{2.6}$$

for all $n > 0$.

Since ψ is nondecreasing, we have $b(x_{n+1}, x_{n+2}) \leq b(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.

Similarly, we can show that $b(x_n, x_{n+1}) \leq b(x_{n-1}, x_n)$.

for all $n \in \mathbb{N} \cup \{0\}$.

It follow that the sequence $\{b(x_n, x_{n+1})\}$ is nonincreasing for all $n \in \mathbb{N}$. Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} b(x_n, x_{n+1}) = r$. We claim that $r = 0$.

Now, we have

$$\psi(b(x_{n+1}, x_{n+2})) \leq \lambda F(\psi(b(x_n, x_{n+1})), \varphi(b(x_n, x_{n+1}))) < F(\psi(b(x_n, x_{n+1})), \varphi(b(x_n, x_{n+1}))).$$

Taking $n \to \infty$, we obtain that

$$\psi(r) \leq \lambda F(\psi(r), \varphi(r)) < F(\psi(r), \varphi(r)).$$

This implies that $\psi(r) = 0$ or $\varphi(r) = 0$ which yields

$$\lim_{n\to\infty} b(x_n, x_{n+1}) = 0. \tag{2.7}$$

Step 2. To prove that $\{x_n\}$ is a Cauchy sequence, there exist $\epsilon > 0$ and two subsequences $\{x_{m(k)}\}$ and $\{x_{n(k)}\}$ of $\{x_n\}$ with $m_k > n_k > k$ such that:

$$d(x_{n(k)}, x_{m(k)}) \geq \epsilon, d(x_{n(k)}, x_{m(k)-1}) < \epsilon.$$

Then, using the triangular inequality we get

$$\begin{aligned} b(x_n, x_{m(k)}) &\leq s[b(x_{n(k)}, x_{n(k)+1}) + b(x_{n(k)+1}, x_{m(k)})] - b(x_{n(k)+1}, x_{n(k)+1}) \\ &\leq sb(x_{n(k)}, x_{n(k)+1}) + s^2[b(x_{n(k)+1}, x_{n(k)+2}) + b(x_{n(k)+2}, x_{m(k)}) - sb(x_{n(k)+2}, x_{n(k)+2}) \\ &\leq sb(x_{n(k)}, x_{n(k)+1}) + s^2 b(x_{n(k)+1}, x_{n(k)+2}) + s^3 b(x_{n(k)+2}, x_{n(k)+2}) + ... + s^{m-n} b(x_{m(k)-1}, x_{m(k)}). \end{aligned}$$

Using (2.6) in the above inequality

$$\begin{aligned} b(x_n, x_{m(k)}) &\leq s\lambda^n b(x_1, x_0) + s^2\lambda^{n+1} b(x_1, x_0) + s^3\lambda^{n+3} b(x_1, x_0) + ... + s^{m-n}\lambda^{m-1} b(x_1, x_0) \\ &\leq s\lambda^n[1 + s\lambda + (s\lambda)^2 + ...]b(x_1, x_0) \\ &= \frac{s\lambda^n}{1 - s\lambda} b(x_1, x_0). \end{aligned}$$

As $\lambda \in [0, \frac{1}{s})$ and $s > 1$, it follows from the above inequality that

$$\lim_{n,m\to\infty} b(x_n, x_m) = 0.$$

Therefore, $\{x_n\}$ is a Cauchy sequence in the complete $b-$partial metric space X

Step3. We now prove that S has a fixed point.

Since $\{x_n\}$ is a Cauchy sequence in the complete $b-$partial metric space X and by completeness of X, then there exists $x^* \in X$ such that

$$\lim_{n,m\to\infty} b(x_n, x^*) = \lim_{n,m\to\infty} b(x_n, x_m) = b(x^*, x^*). \tag{2.8}$$

We will show that x^* is a fixed point of S. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} b(x^*, Sx^*) &\leq s[b(x^*, x_{n+1}) + b(x_{n+1}, Sx^*)] - b(x_{n+1}, x_{n+1})] \\ &\leq s[b(x^*, x_{n+1}) + b(Sx_n, Sx^*)] \\ &\leq sb(x^*, x_{n+1}) + s\lambda b(x_n, x^*). \end{aligned}$$

Using (2.8) in the above inequality, we obtain $b(x^*, Sx^*) = 0$, that is, $Sx^* = x^*$. Thus, x^* is a fixed point of S.

Step4. Let us show that the fixed point of S is unique.

Let $u, v \in X$ be two distinct fixed points of S, that is, $Su = u$ and $Sv = v$. It follows from (2.2) that

$$\begin{aligned} \psi(b(u,v)) &= \psi(b(Su, Sv)) \\ &\leq \lambda F(\psi(\max\{b(u,v), b(u,Su), b(v,Sv), \frac{b(u,Sv)+b(v,Su)}{2}\}), \varphi(\max\{b(u,v), b(u,Su), b(v,Sv), \\ &\quad \frac{b(u,Sv)+b(v,Su)}{2}\})) \\ &\leq \lambda\psi(\max\{b(u,v), b(u,Su), b(v,Sv), \frac{b(u,Sv)+b(v,Su)}{2}\}) \\ &= \lambda\psi(\max\{b(u,v), b(u,u), b(v,v), \frac{b(u,v)+b(v,u)}{2}\}) \\ &= \lambda\psi(b(u,v)), \\ &< \psi(b(u,v)). \end{aligned}$$

Which is contraction. Therefore, we must have $b(u, v) = 0$, that is, $u = v$. Thus, the fixed point of S is unique.

□

The continuity of S in Theorem 2.1 can be dropped.

Theorem 2.2. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be Geraghty contraction type mapping on X. Assume that $\alpha, \eta : X \times X \to [0, +\infty)$ are a functions. Suppose that the following conditions hold:*

(1) *S is $C-$class function.*
(2) *S is triangular weak α-admissible.*
(3) *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$.*
(4) *If $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$ and $x_n \to x^* \in X$ as $n \to \infty$, then there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.*

Then S has a unique fixed point.

Proof. Following the same proof as in Theorem 2.1, we construct the sequence $\{x_n\}$ be defining $x_{n+1} = Sx_n$ for all $n \in \mathbb{N}$ converging to $x^* \in X$ such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. By condition (5), there exist a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ for all $k \in \mathbb{N}$.

Therefore,

$$\begin{aligned} \psi(b(x_{n(k)+1}, Tx^*)) &= \psi(d(Sx_{n(k)}, Tx^*)), \\ &\leq \lambda F(\psi(M(x_{n(k)}, x^*), \varphi(M(x_{n(k)}, x^*))), \\ &\leq F(\psi(M(x_{n(k)}, x^*))), \quad (2.9) \end{aligned}$$

for all $n \in \mathbb{N}$.

Now,

$$\begin{aligned} M(x_{n(k)}, x^*) &= \max\{b(x_n, x^*), b(x_{n(k)}, Sx_{n(k)}), b(x^*, Sx^*), \quad (2.10) \\ &\quad \frac{b(x_{n(k)}, Sx^*) + b(x^*, Sx_{n(k)})}{2}\}, \\ &= \max\{b(x_{n(k)}, x^*), b(x_{n(k)}, x_{n(k)+1}), b(x^*, x^*), \quad (2.11) \\ &\quad \frac{b(x_{n(k)}, x^*) + b(x^*, Sx_{n(k)})}{2}\}, \\ &= \max\{d(x_{n(k)}, x^*), d(x_{n(k)}, x_{n(k)+1}))\}. \quad (2.12) \end{aligned}$$

By taking $n \to \infty$ in (2.9) and using (2.7), we obtain

$$\psi(b(x^*, Sx^*)) \leq \lambda F(\psi(b(x^*, Sx^*)), \phi(b(x^*, Sx^*))),$$

which implies that $b(x^*, Sx^*) = 0$, that is, $Sx^* = x^*$. □

Now, we use Theorem 2.1 and Theorem 2.2 to present many fixed point results:

Corollary 2.1. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be mapping on X. Assume that $\alpha : X \times X \to [0, +\infty)$ is a function. Also, suppose that the following conditions hold:*

(1) *For all $x, y \in X$ with $\alpha(x, y) \geq 1)$, we have $\psi(b(Sx, Sy)) \leq \lambda F(\psi(b(x, y)), \varphi(b(x, y))$.*

(2) *S is generalized $C-$class function.*

(3) *S is a triangular weak α-admissible.*

(4) *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

(5) *S is $\alpha, \eta-$continuous mappings.*

Then S has a unique fixed point.

Proof. Follows the same proof of the Theorem 2.1 by defining $\eta : X \times X \to \mathbb{R}$ via $\eta(x, y) = 1$. □

Corollary 2.2. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be mapping on X. Assume that $\alpha : X \times X \to [0, +\infty)$ is a function. Also, suppose that the following conditions hold:*

(1) *For all* $x, y \in X$ *with* $\alpha(x, y) \geq 1$, *we have* $\psi(b(Sx, Sy)) \leq \lambda F(\psi(b(x, y)), \varphi(b(x, y))$.

(2) S *is generalized* $C-$*class function.*

(3) S *is a triangular* α*-admissible.*

(4) *There exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$.

(5) *If* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x^* \in X$ *as* $n \to \infty$, *then there exist a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x^*) \geq 1$ *for all* $k \in \mathbb{N}$.

Then S *has a unique fixed point.*

Proof. Follows the same proof of the Theorem 2.2 by defining $\eta : X \times X \to \mathbb{R}$ via $\eta(x, y) = 1$. □

Let $\beta : [0, +\infty) \to [0, 1)$ be a continuous function. Define $S : [0, \infty) \times [0, \infty) \to [0, \infty)$ via $F(s, t) = s\beta(t)$. Then $F \in \mathcal{C}$. By Theorem 2.1 and Theorem 2.2, we have the following results:

Corollary 2.3. *Let* (X, b) *be a complete* $b-$*partial metric space with coefficient* $s \geq 1$ *and* S *be mapping on* X. *Assume that* $\alpha, \eta : X \times X \to [0, +\infty)$ *are a functions. Suppose there exist* $\psi \in \Psi$ *and a continuous function* $\beta : [0, +\infty) \to [0, 1)$ *such that for all* $x, y \in X$ *with* $\alpha(x, y) \geq \eta(x, y)$, *we have*

$$\psi(b(Sx, Sy)) \leq \lambda F(\beta(\psi(b(x, y))), \varphi(b(x, y)). \tag{2.13}$$

Also, suppose that the following conditions hold:

(1) S *is generalized* $C-$*class function.*

(2) S *is a triangular weak* α*-admissible.*

(3) *There exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq 1$.

(4) S *is* $\alpha, \eta-$*continuous mappings.*

Then S *has a unique fixed point.*

Corollary 2.4. *Let* (X, b) *be a complete* $b-$*partial metric space with coefficient* $s \geq 1$ *and* S *be mapping on* X. *Assume that* $\alpha, \eta : X \times X \to [0, +\infty)$ *are a functions. Suppose there exist* $\psi \in \Psi$ *and a continuous function* $\beta : [0, +\infty) \to [0, 1)$ *such that for all* $x, y \in X$ *with* $\alpha(x, y) \geq \eta(x, y)$, *we have*

$$\psi(b(Sx, Sy)) \leq \lambda F(\beta(\psi(b(x, y))), \varphi(b(x, y)). \tag{2.14}$$

Also, suppose that the following conditions hold:

(1) S *is generalized* $C-$*class function.*

(2) S *is a triangular weak* α*-admissible.*

(3) *There exists* $x_0 \in X$ *such that* $\alpha(x_0, Sx_0) \geq \eta(x_0, Sx_0)$.

(4) *If* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x^* \in X$ *as* $n \to \infty$, *then there exist a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ *for all* $k \in \mathbb{N}$.

Then S has a unique fixed point.

Example 2.2. *Let $X=[0,1]$ and $b: X\times X\to\mathbb{R}$ define by $b(x,y)=|x-y|^2$ for all $x,y\in X$. Define $\psi,\phi:[0,\infty)\to[0,\infty)$ by $\psi(t)=t$ and $\phi(t)=\frac{4}{25}t$. Define the mapping $S:\mathbb{R}\to\mathbb{R}$ by $Sx=\frac{\ln x}{5}$. Also, we define the functions$\alpha,\eta: X\times X\to[0,\infty)$ by*

$$\alpha(x,y)=\begin{cases} e^{x+y} & \text{if } x,y\in[0,1],\\ 0 & \text{if otherwise,}\end{cases}\qquad \eta(x,y)=\begin{cases} 1 & \text{if } x,y\in[0,1],\\ 0 & \text{if otherwise.}\end{cases}$$

and $F(r,t)=r-t$ for all $r,t,x,y\in X$.

Firstly, It is easy to see that (X,b) is a complete partial $b-$metric space with $s=3$.

Then S is a triangular weak α-admissible with respect to η. Indeed, if $\alpha(x,Sx)\geq\eta(x,Sx)$, then $\alpha(Sx,S^2x)\geq\eta(Sx,S^2x)$, So $\alpha(x,\ln x+1)=e^{x+\ln x}>1=\eta(x,\ln x)$,then $\alpha(\ln x,\ln(\ln x))=e^{\ln x+\ln(\ln x)}\geq e=\eta(\ln x,\ln(\ln x))$.So $x\geq 0$ and hence $Sx\leq 0$. Therefore, $\alpha(x,Sx)\geq\eta(x,Sx)$.

We will prove that S is a generalized $C-$class function. Since $\alpha(x,Sx)\geq\eta(x,Sx)$. Then we have $x,y\in[0,1]$ and then

$$\begin{aligned}\psi(d(Sx,Sy)) &= \left|\frac{\ln x}{5}-\frac{\ln y}{5}\right|^2\\ &= \frac{1}{25}|\ln x-\ln y|^2\\ &= \frac{1}{25}b(x,y)\\ &\leq \frac{1}{25}M(x,y)\\ &= M(x,y)-\frac{24}{25}M(x,y)\\ &= \psi(M(x,y))-\phi(M(x,y))\\ &= F(\psi(M(x,y)),\phi(M(x,y))).\end{aligned}$$

Then S is a generalized $C-$class function and all assumptions of Corollary 2.1 are satisfied. Hence S has a unique fixed point.

3. Applications

In this section, we apply our results to construct an application on Lebesgue-integrable. Denote by Γ the set of all functions $\gamma:\mathbb{R}^+\to\mathbb{R}^+$ satisfying the following conditions:

(1) γ is Lebesgue-integrable on each compact of $\mathbb{R}^+$;

(2) For each $\epsilon > 0$, we have

$$\int_0^{\epsilon} \gamma(z)dz > 0$$

.

Theorem 3.1. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be Geraghty contraction type mappings on X. Also, let $F \in \mathcal{C}$ and $\gamma_1, \gamma_2 \in \Gamma$. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we have*

$$\begin{aligned} \int_0^{d(Sx,Ty)} \gamma_1(z))dz \quad &\leq \quad F\Big(\int_0^{\max\{d(x,y),d(x,Sx),d(Tx,Ty),\frac{b(x,Sy)+b(y,Sx)}{2}\}} \gamma_1(z)dz, \\ &\int_0^{\max\{d(x,y),d(x,Sx),d(Tx,Ty),\frac{b(x,Sy)+b(y,Sx)}{2}\}} \gamma_2(z)dz \Big). \end{aligned}$$

Also, suppose the following hypotheses:

(1) *S is generalized $C-$class function.*

(2) *S is a triangular weak α-admissible.*

(3) *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

(4) *S is $\alpha, \eta-$continuous mappings.*

Then S has a unique fixed point.

Proof. Define the functions $\psi, \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ via $\psi(t) = \int_0^t \gamma_1(z))dz$ and $\varphi(t) = \int_0^t \gamma_2(z))dz$. Noting that ψ is an altering distance function and $\varphi \in \Phi$. So S is triangular weak $\alpha-$admissible with respect to η. So S satisfies all the hypotheses of theorem 2.1. Therefore S has a fixed point. □

Theorem 3.2. *Let (X, b) be a complete $b-$partial metric space with coefficient $s \geq 1$ and S be Geraghty contraction type mappings on X. Also, let $F \in \mathcal{C}$ and $\gamma_1, \gamma_2 \in \Gamma$. Assume that $\alpha, \eta : X \times X \to [0, \infty)$ be two functions such for all $x, y \in X$ with $\alpha(x, y) \geq \eta(x, y)$, we have*

$$\begin{aligned} \int_0^{d(Sx,Ty)} \gamma_1(z))dz \quad &\leq \quad F\Big(\int_0^{\max\{d(x,y),d(x,Sx),d(Tx,Ty),\frac{b(x,Sy)+b(y,Sx)}{2}\}} \gamma_1(z)dz, \\ &\int_0^{\max\{d(x,y),d(x,Sx),d(Tx,Ty),\frac{b(x,Sy)+b(y,Sx)}{2}\}} \gamma_2(z)dz \Big). \end{aligned}$$

Also, suppose the following hypotheses:

(1) *S is generalized $C-$class function.*

(2) *S is a triangular weak α-admissible.*

(3) *There exists $x_0 \in X$ such that $\alpha(x_0, Sx_0) \geq 1$.*

(4) *If* $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x^* \in X$ *as* $n \to \infty$*, then there exist a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, x^*) \geq \eta(x_{n(k)}, x^*)$ *for all* $k \in \mathbb{N}$.

Then S *has a unique fixed point.*

Proof. Follow from Theorem 2.2 by defining $\psi(t) = \int_o^t \gamma_1(z))dz$ and $\varphi(t) = \int_o^t \gamma_2(z))dz$. Noting that the mapping S satisfies all the hypotheses of theorem 2.2. □

4. Acknowledgement

The authors would like to acknowledge the grant: UKM Grant DIP-2014-034 and Ministry of Education, Malaysia grant FRGS/1/2014/ST06/UKM/01/1 for financial support.

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